Regularity of the maximal function and Poincaré inequalities

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Plan

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Part I: Introduction

Lebesgue differentiation theorem

The Lebesgue differentiation theorem asserts that if $f \in L^1_{loc}(\mathbb{R}^n)$, then the averages $f_{B(x,r)}$ converge as $r \to 0$ for n-Lebesgue almost every x.

If f is continuous, the convergence takes place *everywhere*.

Let $p \in (1, n)$. If f has p-integrable distributional derivatives, then the exceptional set (where the averages do not converge) exists, but it is small.

Sharper estimate

For compact $K \subset \mathbb{R}^n$, define

$$\mathrm{cap}_p(K) = \inf \int_{\mathbb{R}^n} |\nabla \varphi(x)|^p dx$$

with infimum over smooth and compactly supported $\varphi \geq 1_K$. Capacity of more general sets is defined through approximation.

Denote by $W^{1,p}(\mathbb{R}^n)$ the (Sobolev) space of all f with $f, |\nabla f| \in L^p(\mathbb{R}^n)$.

Let $f \in W^{1,p}(\mathbb{R}^n)$. Let E be a set where $f_{B(x,r)}$ do not converge. Then $\mathrm{cap}_p(E) = 0$ and consequently $\mathcal{H}^s(E) = 0$ for all $s > n - p$.

Reason

The sharpened Lebesgue differentiation theorem follows from the capacitary weak type estimate of the **Hardy–Littlewood maximal function** $Mf(x) = \sup_{r>0} |f|_{B(x,r)}$

$$\mathrm{cap}_p(\{Mf > \lambda\}) \lesssim \frac{1}{\lambda^p} \int |\nabla f|^p dx$$

valid for all $\lambda > 0$.

The capacitary weak type estimate follows, in turn, from the strong type gradient bound $\|\nabla Mf\|_{L^p} \lesssim \|\nabla f\|_{L^p}$.

Similar method can be used to sharpen Lebesgue's differentiation theorem for other function spaces.

History

- The centred Hardy–Littlewood maximal function on $W^{1,p}$ with $p > 1$ (on $\mathbb{R}^n$ Kinnunen 1997, on a domain Kinnunen–Lindqvist 1998).
- Fractional maximal function (Kinnunen–Saksman 2003).
- Non-centred Hardy–Littlewood on $W^{1,1}(\mathbb{R})$ (Tanaka 2002).
- Non-centred Hardy–Littlewood on $BV(\mathbb{R})$ (Aldaz and Pérez-Lazaro 2007).
- Centred Hardy–Littlewood on $BV(\mathbb{R})$ (Kurka 2015)
- Non-centred Hardy–Littlewood on radial functions in $W^{1,1}(\mathbb{R}^n)$ (Luiro 2017)
- Maximal functions for some other convolution kernels (Carneiro–Svaiter 2013, Carneiro–Finder–Sousa).
- Sharp estimates for the operator norms (Carneiro–Madrid 2016)
- More results on various related questions by Bober, Hajłasz, Heikkinen, Hughes, Korry, Luiro, Malý, Onninen, Pierce etc.

A major open problem

Question 1: Is the operator $u \mapsto |\nabla Mu|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for $n > 1$?

(P. Hajłasz and J. Onninen:
On boundedness of maximal functions in Sobolev spaces,
Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 167–176)

This operator is well understood only in dimension one (recall the results in the previous slide).

The aim of the talk

Many spaces of “smooth” functions can be defined without explicit reference to derivatives. Studying mean oscillations instead of derivatives gives results not weaker than the ones got through direct differentiation. In the case of the Hardy–Littlewood maximal function, this approach has turned out to be particularly efficient. The rest of the talk discusses this observation. At the end, we obtain an invariance result at the level of abstract Poincaré inequalities. This unifies many known results and hopefully helps to understand **Question 1**.

Part II: Sobolev and Poincaré

Derivative

Given a locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$, we define its distributional derivative in i-coordinate as the distribution

$$\partial_i f(\varphi) = -\int f \partial_i \varphi dx$$

acting on $\varphi \in C_0^\infty(\mathbb{R}^n)$.

If $\partial_i f$ is a locally integrable function, it is called a weak derivative.

Given a distribution, it is difficult, in general, to prove that its derivative is a locally integrable function.

Classical Poincaré and Sobolev–Poincaré inequalities

For any function $u \in W^{1,p}_{loc}(\mathbb{R}^n)$, $1 \leq p < n$, it is known that

$$\left(\fint_B |u - u_B|^p dx \right)^{1/p} \leq \left(\fint_B |u - u_B|^{p^*} dx \right)^{1/p^*} \lesssim r(B) \left(\fint_B |\nabla u|^p dx \right)^{1/p}$$

holds with a constant independent of u and the choice of the ball B. Here $p^* = np/(n-p)$.

Conversely: If $u \in L^1_{loc}$ and the same uniform Poincaré inequality

$$\fint_B |u - u_B| dx \lesssim r(B) \fint_{10B} g dx$$

holds for some $g \in L^1_{loc}$, then u is weakly differentiable and $|\nabla u| \lesssim g$. (Hajłasz 2003)

Abstract Poincaré inequalities

Since the validity of a Poincaré inequality characterizes weak differentiability, it makes sense to define function spaces in terms of similar conditions on mean oscillation. By choosing functions $u \in L^1_{loc}$ with

$$\fint_B |u - u_B| dx \lesssim a(B)$$

for $a : \{B(x,r) : x \in \mathbb{R}^n, r > 0\} \to [0, \infty)$, we can recover

- BMO with $a = 1$,
- Hölder–Lipschitz spaces with $a(B) = r(B)^\alpha$, $\alpha \in (0,1)$,
- $W^{1,p}$, $p > 1$ with $a(B) = r(B) \fint_B g$ and $g \in L^p$,
- BV with $a(B) = r(B)\mu(B)/|B|$ and μ a Radon measure,
- and much more less well-known function spaces.

Functionals a

All the functionals a of the previous slide are examples of so called fractional averages $a(B) = r(B)^\alpha \mu(B)/|B|$ where μ is a Radon measure and $\alpha \in [0,1]$.

The fractional averages are a subclass of the functionals a satisfying the condition D_q: For all balls B and pairwise disjoint collections $\{B' : B' \subset B\}$ it holds

$$\sum_{B'} |B'| a(B')^q \lesssim |B| a(B)^q.$$

Here $q > 1$. For fractional averages $q = n/(n-\alpha)$.

Franchi–Pérez–Wheeden self-improvement

Lemma (Franchi–Pérez–Wheeden 1998)

Let a satisfy D_q with $q > 1$ and let $u \in L^1_{loc}$. If

$$\fint_Q |u - u_Q| dx \lesssim a(Q)$$

then

$$\left\| \frac{1_Q |u - u_Q|}{|Q|} \right\|_{L^{q,\infty}(\mathbb{R}^n)} \lesssim a(Q).$$

Part III: Hardy–Littlewood maximal function

BMO

- $Mu \in BMO$ for $u \in BMO$ provided that $Mu \in L^1_{loc}$ (Bennett, DeVore and Sharpley 1981).
- The proof can be simplified using Muckenhoupt's A_1 weights (Chiarenza–Frasca 1987)
- With a one more slight change, the use of A_1 can be avoided.

The core of the proofs is the **John–Nirenberg theorem**: For u in BMO and all $p > 1$, it holds

$$\left(\fint_Q |u - u_Q|^p dx \right)^{1/p} \leq C(p) \|u\|_{BMO}.$$

Key observation

The Sobolev–Poincaré inequality has the same effect for weakly differentiable functions as what John–Nirenberg theorem has for BMO. More generally, the use of John–Nirenberg theorem can be replaced by the Franchi–Pérez–Wheeden lemma in the context of functions satisfying a generalized Poincaré inequality.

So if one wants to estimate

$$\fint_B |Mu - (Mu)_B| dx$$

for such functions, the **local contribution** part

$$1_{\{Mu = M(1_{3B} u)\}} (Mu - (Mu)_B)$$

is easy to estimate by $a(3B)$.

In general, the remaining part might not have a meaningful bound. However, in the case of fractional averages, it is possible to build a valid argument. We obtain:

The main theorem

Theorem

Let $u \in L^1_{loc}(\mathbb{R}^n)$ be a positive function such that $Mu \in L^1_{loc}(\mathbb{R}^n)$. Suppose that

$$\fint_Q |u - u_Q| \, \mathrm{d}x \leq C \mathrm{diam}(Q)^\alpha \frac{\mu(Q)}{|Q|} \tag{1}$$

where we freeze α and μ to one of the following two alternatives. Either

- $\alpha = 0$ *and μ equals the Lebesgue measure, or*
- $\alpha \in (0, 1]$ *and μ is a locally finite positive Borel measure.*

Then

$$\fint_Q |Mu - (Mu)_Q| \, \mathrm{d}x \leq C \mathrm{diam}(Q)^\alpha \inf_{z \in Q} M\mu(z)$$

for all Q.

Corollaries

- The theorem contains boundedness in $W^{1,p}$ with $p > 1$, Hölder spaces, and BMO.
- A slight modification allows to get similar result for the fractional maximal function.
- With some care, distributional ∇Mu can be identified with a function in $L^{1,\infty}$ outside an exceptional set for $u \in BV$ or $u \in W^{1,1}$. However, there is no bound for the size of the exceptional in general.
- To proceed further with $W^{1,1}$, it is necessary to improve the right hand side in the conclusion of the theorem.

References

All the references can be found in

- O. Saari, *Poincaré inequalities for the maximal function*, arXiv:1605.05176.

Thank you for your attention!