Department of Mathematics and Systems Analysis

Weights arising from parabolic partial differential equations

Olli Saari

Aalto University

DOCTORAL DISSERTATIONS

Olli Saari Weights arising from parabolic partial differential equations

Aalto University

2016

Aalto University publication series
DOCTORAL DISSERTATIONS 37/2016

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Olli Saari

A doctoral dissertation completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Science, at a public examination held at the lecture hall M1 of the school on 13 April 2016 at 12.

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Aalto University publication series
DOCTORAL DISSERTATIONS 37/2016

ISBN 978-952-60-6677-6 (printed)
ISBN 978-952-60-6678-3 (pdf)
ISSN-L 1799-4934
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)
http://urn.fi/URN:ISBN:978-952-60-6678-3

Unigrafia Oy
Helsinki 2016

Finland

441 697
Printed matter

Aalto University

Abstract

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Author
Olli Saari

Name of the doctoral dissertation
Weights arising from parabolic partial differential equations

Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series DOCTORAL DISSERTATIONS 37/2016

Field of research Mathematics

Manuscript submitted 11 November 2015 **Date of the defence** 13 April 2016

Permission to publish granted (date) 16 February 2016 **Language** English

☐ **Monograph** ☒ **Article dissertation** ☐ **Essay dissertation**

Abstract

This thesis is devoted to the study of one-sided weights and parabolic partial differential equations in the Euclidean n-space. We define a tailored maximal operator, whose weighted theory has the ideal connection to the regularity theory of parabolic partial differential equations. It can also be regarded as the multidimensional version of the one-sided maximal function.

We give a Muckenhoupt type characterization for the good weights of the parabolic forward-in-time maximal operator. This applies to both weak and strong type weighted norm inequalities. Moreover, we combine the characterization with the classical Rubio de Francia algorithm to prove a factorization result.

There is a related class of functions, those with parabolic bounded mean oscillation (parabolic BMO). We prove local-to-global results for their definition and John-Nirenberg inequality. As an application, global integrability of supersolutions to a wide class of parabolic partial differential equations is established. In addition, we give a characterization of the parabolic BMO through maximal functions of Borel measures.

Finally, we study related topics in the context of functions of bounded mean oscillation, in the classical sense as defined by John and Nirenberg. We study the relation between quasiconformal and BMO-preserving coordinate changes in the Heisenberg group. In addition, we generalize local-to-global results for a wide scale of BMO type spaces in the context of general metric measure spaces.

Keywords parabolic partial differential equation, heat equation, weighted norm inequality, maximal operator, Muckenhoupt weight, BMO, John-Nirenberg inequality, metric space

ISBN (printed) 978-952-60-6677-6 **ISBN (pdf)** 978-952-60-6678-3

ISSN-L 1799-4934 **ISSN (printed)** 1799-4934 **ISSN (pdf)** 1799-4942

Location of publisher Helsinki **Location of printing** Helsinki **Year** 2016

Pages 143 **urn** http://urn.fi/URN:ISBN:978-952-60-6678-3

A! Aalto-yliopisto

Tiivistelmä

Aalto-yliopisto, PL 11000, 00076 Aalto www.aalto.fi

Tekijä
Olli Saari

Väitöskirjan nimi
Parabolisten osittaisdifferentiaaliyhtälöiden painoteoria

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

Sarja Aalto University publication series DOCTORAL DISSERTATIONS 37/2016

Tutkimusala Matematiikka

Käsikirjoituksen pvm 11.11.2015 **Väitöspäivä** 13.04.2016

Julkaisuluvan myöntämispäivä 16.02.2016 **Kieli** Englanti

☐ **Monografia** ☒ **Artikkeliväitöskirja** ☐ **Esseeväitöskirja**

Tiivistelmä

Väitöskirjassa tutkitaan yksipuolisia painoja ja parabolisia osittaisdifferentiaaliyhtälöitä n-ulotteisessa euklidisessa avaruudessa. Työssä määritellään erityinen maksimaalioperaattori, jonka painotettu teoria liittyy toivotulla tavalla parabolisten osittaisdifferentiaaliyhtälöiden säännöllisyysteoriaan. Sitä voidaan myös pitää korkeaulotteisena yleistyksenä yksipuolisesta maksimaalifunktiosta.

Parabolisen maksimaalioperaattorin kannalta hyvät painofunktiot karakterisoidaan Muckenhoupt-tyyppisillä ehdoilla. Karakterisaatio pätee sekä heikon että vahvan tyypin painotettuihin normiepäyhtälöihin. Tulosta käytetään edelleen yhdessä klassisen Rubio de Francian algoritmin kanssa parabolisten painojen tekijöihinjakotuloksen todistamiseen.

Lisäksi tutkitaan parabolista rajoitetun keskivärähtelyn luokkaa (parabolinen BMO). Siihen kuuluville funktioille todistetaan lokaalista globaaliin tuloksia liittyen sekä määritelmään että John-Nirenberg epäyhtälöön. Tulosten sovelluksena onnistutaan varmistamaan globaali integroituvuus monien parabolisten osittaisdifferentiaaliyhtälöiden superratkaisuille. Parabolinen BMO karakterisoidaan myös Borel-mittojen maksimaalifunktioiden avulla.

Viimeisenä aiheena tutkitaan kysymyksiä klassisiin Johnin ja Nirenbergin BMO-funktioihin liittyen. Erityisesti selvitetään kvasikonformisten ja BMO-funktiot säilyttävien koordinaattimuunnosten yhteyttä Heisenbergin ryhmässä. Lisäksi joukko lokaalista globaaliin tuloksia laajalle skaalalle BMO tyyppisiä avaruuksia yleistetään metrisiin mitta-avaruuksiin.

Avainsanat parabolinen osittaisdifferentiaaliyhtälö, lämpöyhtälö, painotettu normiepäyhtälö, maksimaalioperaattori, Muckenhouptin paino, BMO, John-Nirenberg epäyhtälö, metrinen avaruus

ISBN (painettu) 978-952-60-6677-6 **ISBN (pdf)** 978-952-60-6678-3

ISSN-L 1799-4934 **ISSN (painettu)** 1799-4934 **ISSN (pdf)** 1799-4942

Julkaisupaikka Helsinki **Painopaikka** Helsinki **Vuosi** 2016

Sivumäärä 143 **urn** http://urn.fi/URN:ISBN:978-952-60-6678-3

Preface

The research project leading to this thesis started at the end of the summer 2013. Since then, and even earlier, I have had the privilege to benefit from the advice and guidance of my supervisor professor Juha Kinnunen. I would like to thank him. In addition, I am grateful to my other coauthors Riikka Korte and Niko Marola for the opportunity to work together while preparing the research articles that are now part of this thesis.

One more person to be mentioned is my friend and teacher Ioannis Parissis, whose lectures in the spring 2013 were my first introduction to harmonic analysis. I would like to thank him for his interest in my research. In particular, I am really grateful to him and Carlos Pérez for inviting me to stay at the University of the Basque Country (UPV/EHU), Bilbao, the fall semester 2015.

For financial support, I am indebted to the Finnish Academy of Science and Letters; the Vilho, Yrjö and Kalle Väisälä foundation; and the Department of Mathematics and Systems Analysis of Aalto University. In addition, I am grateful to Sheldy Ombrosi and Nicola Garofalo for carrying out the preliminary examination of this thesis.

Finally, I want to thank my family and friends.

Espoo, February 22, 2016,

Olli Saari

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Riikka Korte, Niko Marola and Olli Saari. Homeomorphisms of the Heisenberg group preserving BMO. *Archiv der Mathematik*, Volume 106, Issue 2, pp 175–182, February 2016.

II Niko Marola and Olli Saari. Local to global results for spaces of BMO type. *Mathematische Zeitschrift*, Volume 282, Issue 1, pp 473–484, February 2016.

III Olli Saari. Parabolic BMO and global integrability of supersolutions to doubly nonlinear parabolic equations. Accepted for publication in *Revista Matemática Iberoamericana*, July 2015.

IV Juha Kinnunen and Olli Saari. Parabolic weighted norm inequalities for partial differential equations. *arXiv:1410.1396v3*, November 2015.

V Juha Kinnunen and Olli Saari. On weights satisfying parabolic Muckenhoupt conditions. *Nonlinear Analysis*, Volume 131, pp 289–299, January 2016.

Author's Contribution

Publication I: "Homeomorphisms of the Heisenberg group preserving BMO"

The author has performed a substantial part of the research.

Publication II: "Local to global results for spaces of BMO type"

The author has performed a substantial part of the research.

Publication III: "Parabolic BMO and global integrability of supersolutions to doubly nonlinear parabolic equations"

The article results from the author's independent research.

Publication IV: "Parabolic weighted norm inequalities for partial differential equations"

The author has performed a substantial part of the research.

Publication V: "On weights satisfying parabolic Muckenhoupt conditions"

The author has performed a substantial part of the research.

1. Introduction

Elliptic partial differential equations can be thought as of equations describing steady states or equilibria. A particularly interesting class of such equations is the one associated with the p-Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty,$$

which is the Euler-Lagrange equation for the minimization problem related to the integral functional

$$\int |\nabla u(x)|^p \, \mathrm{d}x.$$

In many cases the exact form of the equation is not important, but the characteristic behaviour of the solutions stems from some of its structural key properties. For instance, the theory of the p-Laplacian mostly carries over to equations of the form

$$\operatorname{div} A(x, \nabla u) = 0 \tag{1.0.1}$$

where A is a suitable function subject to the growth condition

$$A(x, \nabla u) \cdot \nabla u \approx |\nabla u|^p.$$

The ellipticity, or the fact that there are no preferred directions is manifested in the symmetries enjoyed by the equations (1.0.1). Both translations and dilations of the coordinate space leave the class of solutions invariant. As a consequence, regularity estimates derived in the unit ball carry over to all balls in a scale and location invariant fashion. In addition to these obvious symmetries, a certain subclass of quasiconformal coordinate changes also respects the structure (1.0.1) (see [35]).

The positive solutions to elliptic equations constitute an important subclass of Muckenhoupt weights that will be the second motivation for the theory whose study we are going to embark on. We will hence give a short

overview of A_q classes and their relation to partial differential equations. The classical paper [71] of Benjamin Muckenhoupt from the 1970s characterized the good weights for the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{Q \ni x} \fint_Q |f|$$

as the ones satisfying the A_q condition

$$\sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-q'} \right)^{q-1} < \infty, \quad q \in (1, \infty). \tag{1.0.2}$$

More precisely, $M : L^q(w) \to L^q(w)$ boundedly if and only if $w \in A_q$. The original proof consisted of first proving the weak type boundedness

$$M : L^q(w) \longrightarrow L^{q,\infty}(w)$$

and then using a reverse Hölder property to get

$$M : L^{q-\epsilon}(w) \longrightarrow L^{q-\epsilon,\infty}(w)$$

whence the result followed by Marcinkiewicz interpolation.

Muckenhoupt's original paper was motivated by applications to Fourier series, but the result drew plenty of interest also from other areas, and A_p weights have become a standard part of the theory of Calderón-Zygmund operators and that of partial differential equations. In addition to the qualitative one-weight norm inequalities, the study of sharp quantitative estimates as well as attempts to extend the results to the two weight case have been very active areas of research in the recent years (see for example [40, 41, 51]). We will not review the classical theory, but we refer to the classical books [22, 27, 32, 33, 84] instead.

The class of Muckenhoupt weights is also connected to the regularity theory of elliptic equations. It plays a crucial role in Moser's proof of the Harnack inequality [68], namely, if u is a positive solution, then there is $\varepsilon > 0$ such that

$$\operatorname{ess\,sup}_{x \in Q} u(x) \lesssim \left(\frac{1}{|Q|} \int_Q u^{\varepsilon} \right)^{1/\varepsilon} \lesssim \left(\frac{1}{|Q|} \int_Q u^{-\varepsilon} \right)^{-1/\varepsilon} \lesssim \operatorname{ess\,inf}_{x \in Q} u(x).$$

The inequality in the middle reads $u^{\epsilon} \in A_2$, and it is reached by showing that $\log u$ is a function of bounded mean oscillation whence the actual A_2 property follows by the John–Nirenberg inequality.

There are also other instances of the theory of differential equations where Muckenhoupt weights arise naturally. For instance, they are used in the theory of L^p solvability of boundary value problems in rough domains (see [46]), and elliptic operators with degeneracy governed by a

suitable Muckenhoupt weight have also been studied. Namely, the additional degeneracy of the ellipticity of a linear differential operator of divergence form does not deteriorate the regularity of solutions to the corresponding differential equation if it satisfies a certain Muckenhoupt condition (see [24]).

The theory of elliptic equations and Muckenhoupt weights is well established by now, but when it comes to nonlinear parabolic (or time dependent) equations, much less has been done. The aim of this thesis is to investigate systematically how the similarities of elliptic and parabolic equations carry over to weights. Our focus is on the parabolic analogues of the functions of bounded mean oscillation together with related weighted norm inequalities. In particular, we are interested in a point of view that is compatible with the standard parabolic regularity theory. With respect to this kind of parabolic Muckenhoupt theory, the appended publications III, IV and V make almost all of what is known about the subject.

In what follows, we will give an account of what are the weight classes generated by the most common parabolic equations, and what is the time-dependent theory of weighted norm inequalities that degenerates to the classical theory of Muckenhoupt weights as we move to the time independent context. Along the way, we will point out how our theory solves the long standing open problem of generalizing one-sided weights to n-space with $n \geq 2$.

2. Theory of parabolic Muckenhoupt weights

2.1 Trudinger's equation

In the 1960s Jürgen Moser [69, 70] proved an inequality of Harnack type for linear parabolic differential equations. At the level of methods, the linearity was inessential, and the result was generalized to nonlinear equations of divergence form by Trudinger [85] just a few years later. Finally, the class of equations for which Moser's parabolic scale and location invariant Harnack inequality holds true includes so called Trudinger's equation

$$\partial_t(|u|^{p-2}u) - \operatorname{div}(|\nabla u|^{p-2}\cdot\nabla u) = 0, \quad 1 < p < \infty, \tag{2.1.1}$$

together with equations structurally similar to it.

Trudinger's equation is sometimes called the doubly nonlinear equation, and the slight generalization to equations with similar structure means studying equations of the form

$$\partial_t(|u|^{p-2}u) - \operatorname{div} A(x,t,u,\nabla u) = 0,$$

where A is a measurable function subject to the growth conditions

$$\begin{aligned} A(x,t,u,\nabla u)\cdot\nabla u &\geq C_0|\nabla u|^p, \\ |A(x,t,u,\nabla u)| &\leq C_1|\nabla u|^{p-1}, \end{aligned}$$

where $p > 1$ is fixed.

Due to the nonlinearity in the time derivative, the model equation enjoys certain homogeneity. That is, if u is a solution, so is αu whenever α is a real number. Moreover, if this property is destroyed by replacing $|u|^{p-2}u$ by u in the equation, there will be no scale and location invariant Harnack inequality (see [21]). On the other hand, as a drawback of the doubly

nonlinear structure, constants cannot be added to solutions, which makes the regularity theory of Trudinger's equation different from that of its p-parabolic cousin.

After all, having weighted norm inequalities in mind, we are lead to regard the doubly nonlinear equation as a natural starting point for studying weight properties of solutions to parabolic equations, since in that context multiplication by positive constants should not have effect on the character of weights whereas adding a constant to a weight function is not so important an operation to be worried about. Note that even if there are already studies on degeneracies governed by weights (see [13]), those problems have very little to do with the questions we are going to study.

The proof of the parabolic Harnack inequality follows the same scheme as Moser's earlier work on the elliptic Harnack inequality [68], that is, one first proves two endpoint reverse Hölder inequalities and then glues them together with a condition similar to Muckenhoupt's A_2. More briefly, for a space time cylinder R^- together with its suitable forward-in-time translate R^+, one may establish the string of inequalities

$$\operatorname{ess\,sup}_{z\in R^-} u(z) \lesssim \left(\frac{1}{|2R^-|}\int_{2R^-} u^{\varepsilon}\right)^{1/\varepsilon} \lesssim \left(\frac{1}{|2R^+|}\int_{2R^+} u^{-\varepsilon}\right)^{-1/\varepsilon} \lesssim \operatorname{ess\,inf}_{z\in R^+} u(z).$$

There is a striking difference between the Harnack inequalities that the solutions to stationary and evolutionary problems satisfy. Instead of the full comparability familiar from the theory of elliptic equations, the solutions to parabolic equations satisfy a much weaker condition. Namely, the values that a positive solution can attain in a cube are controlled by the values the solution has in the same cube after a waiting time has passed. This *time lag* is completely invisible in the stationary case, and it is a real phenomenon showing up already in the behaviour of the fundamental solution of the heat equation with constant coefficients.

A careful study of Moser's proof reveals that the origin of the time lag can be traced back to a condition playing the role that BMO had in the corresponding elliptic proof. Because of this analogue, the related class of functions is called parabolic BMO. In addition to parabolic BMO, one can define parabolic weights, parabolic maximal operators and so forth. The main difference between them and their elliptic counterparts can be summarized as that all the inequalities defining classical concepts such as Muckenhoupt weight or reverse Hölder inequality have a reverse that is

universally true: Jensen's inequality. In the context of partial differential equations, this is a manifestation of the fact that elliptic equations describe equilibria of evolutions associated with parabolic ones, and consequently they are invariant under reversion of the direction of time. Once this symmetry is lost, a horde of completely new phenomena appears.

Whereas the classical BMO of John and Nirenberg [45] has been studied extensively in the context of harmonic analysis, the knowledge of parabolic BMO prior to this work (i.e. III and IV) has been limited to the very few core properties necessary for running the parabolic Moser iteration, that is, one only knows that a weak oscillation bound improves itself to a bound of exponential type. This property has many proofs. In addition to the original paper by Moser [70], there is a simplified proof for the quadratic growth case [23] and a general approach valid in spaces of homogeneous type [2]. Before writing down the definition of parabolic BMO, we will recall the definition of weak solutions to parabolic differential equations.

2.2 Solutions

Equation (2.1.1) can be studied in various subdomains $D \subset \mathbb{R}^{n+1}$ of the space-time, but for us the cases of interest will be the cylinders $\Omega_T = \Omega \times (0,T)$ with $\Omega \subset \mathbb{R}^n$ a domain. In these cases $u \in L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega))$ is a local solution if it satisfies the equality (2.1.1) after multiplication by a compactly supported and non-negative test function $\varphi \in C_0^\infty(\Omega_T)$ and a formal integration by parts, that is, if it satisfies it in the sense of distributions. In case the inequality

$$\iint_{\Omega_T} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi - |u|^{p-2}u\partial_t \varphi \right) \geq 0$$

holds instead of an equality, then u is called a supersolution.

Recall that $u \in L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega))$ if $x \mapsto u(x,t)$ is in $W^{1,p}_{loc}(\Omega)$ for almost every $t \in (0,T)$, $t \mapsto \|u(\cdot,t)\|_{W^{1,p}(\Omega')}$ is measurable, and

$$\int_I \|u(\cdot,t)\|_{W^{1,p}(\Omega')} \,\mathrm{d}t < \infty$$

for every Ω' that is the interior of some compact subset of Ω and for every closed $I \subset (0,T)$.

2.3 Geometry

The first n coordinates of $\mathbb{R}^{n+1}$ will constitute the space and the last one is for time. With this convention, it is important to note that the equation (2.1.1) is not invariant under the standard scaling $(x,t) \mapsto (ax, at)$ associated with Euclidean cubes, but the right scaling is $(x,t) \mapsto (ax, a^p t)$. In addition to this parabolic scaling, also translations of the time-space leave the class of solutions invariant. It follows that the class of Euclidean cubes must be replaced by the class of sets generated by the transformations described previously in all regularity estimates from the parabolic Caccioppoli estimate to the ultimate Harnack inequality

$$\operatorname{ess\,sup}_{z \in R^-(\gamma)} u(z) \leq C(n,p,\gamma) \operatorname{ess\,inf}_{z \in R^+(\gamma)} u(z). \tag{2.3.1}$$

Here the parabolic rectangles

$$R = Q \times (t - l(Q)^p, t + l(Q)^p) \subset \mathbb{R}^{n+1}$$

are based on cubes $Q \subset \mathbb{R}^n$ with sides parallel to coordinate axes and sidelengths $l(Q)$. The parameter $\gamma \in (0,1)$ quantifies the time lag and $R^-(\gamma) = Q \times (t - l(Q)^p, t - \gamma l(Q)^p)$.

Among the properties of parabolic rectangles we use, the most important one is probably the following. Parabolic rectangles are metric balls with respect to the translation invariant metric

$$d((x,t),(0,0)) = \max\{\|x\|_\infty, \frac{1}{2}|t|^{1/p}\},$$

which gives the time axis dimension $p > 1$. This is one of the phenomena behind the infinite speed of propagation that the heat equation exhibits, and it will be crucial for the validity of certain self improving properties we study.

2.4 Parabolic BMO

Recall that a real valued $u \in L^1_{loc}(\mathbb{R}^n)$ is said to be a function of bounded mean oscillation if its oscillation

$$\fint_Q |u - u_Q| := \frac{1}{|Q|} \int_Q \left| u - \frac{1}{|Q|} \int_Q u \right|$$

is uniformly bounded as a function of Euclidean cubes $Q \subset \mathbb{R}^n$. This means that whatever cube we take, there exists a constant to approximate u with respect to L^1 norm in that cube with error proportional to the scale.

In the parabolic case, we still require good approximation by constants, but instead of cubes, we use parabolic rectangles, and instead of the absolute value, we measure upper and lower deviations separately in different locations: the parabolic oscillation is

$$\inf_{a_R\in\mathbb{R}}\left(\fint_{R^+(\gamma)}(u-a_R)^+ + \fint_{R^-(\gamma)}(u-a_R)^-\right), \tag{2.4.1}$$

and if this quantity is bounded uniformly in R, we say that u is of parabolic bounded mean oscillation, abbreviated PBMO^+. Note that the sets $R^{\pm}(\gamma)$ appearing in the definition are same as the ones in the Harnack inequality (2.3.1).

The original definition used in Moser's work [69] is

$$\sup_R \fint_{R^+(0)}\fint_{R^-(0)} \sqrt{(u(z)-u(w))^+}\,\mathrm{d}w\,\mathrm{d}z < \infty.$$

However, the results in III proved that the definition through (2.4.1) leads to a more clean picture, and in that sense (2.4.1) is the correct definition. To improve Moser's $L^{1/2}$ type estimate to L^1, one has to accept that a *time lag* appears. That is, functions satisfying Moser's condition are in PBMO^+. The latter condition, in turn, can be used to get exponential decay for the integrands, that is,

$$\sup_R\left(\fint_{R^+(\gamma)}\exp(\epsilon(u-a_R)^+) + \fint_{R^-(\gamma)}\exp(\epsilon(u-a_R)^-)\right) < \infty$$

for some $\epsilon > 0$. At the level of exponential integrability, the counterexample by Moser can be used to see that the time lag is necessary. Some references for these facts are [69], [23], [2] and [48].

The previous reasoning reveals that one can win integrability at the cost of inducing a lag, and the dimensional difference between space and time coordinates makes this lag a qualitative phenomenon, which was exploited in III and IV in order to prove various self-improving properties. In particular, by incorporating the time lag in the very definition of parabolic BMO, one can make sure that the function classes defined through exponential integrability and larger time gap coincide with the ones defined by the boundedness of (2.4.1). If the lag is not present in the definition, it is not clear whether all functions satisfying the corresponding John-Nirenberg inequality also belong to parabolic BMO or not.

2.5 The direction of time

We will next look at the role of the time variable in the definition of PBMO^+. Parabolic BMO is obviously not a vector space since multiplication by a negative constant does not leave the parabolic oscillation unchanged. Instead of that, it reverses the direction of time. Indeed, as the condition (2.4.1) on u measures its upper deviation in future, the same condition written for $-u$ measures the upper deviation of u in the past. This motivates the definition of the reflected class $\mathrm{PBMO}^- = -\mathrm{PBMO}^+$, which could also be defined as $\mathrm{PBMO}^- = \{u(x,-t) : u(x,t) \in \mathrm{PBMO}^+\}$.

For each property of PBMO^+ there is a corresponding statement about PBMO^-, the only difference being that the direction of time is reversed. It is also clear that the intersection $\mathrm{PBMO}^+ \cap \mathrm{PBMO}^-$ equals a BMO space of classical type provided that we replace the family of cubes in the definition by the family of parabolic rectangles. Moreover, it is easy to see that if $u(x,t) = u(x) \in \mathrm{PBMO}^+$ then $u \in \mathrm{BMO}(\mathbb{R}^{n+1} \times \{\tau\})$ so that PBMO^+ is a consistent generalization of the classical BMO space. This fits in the picture where we associate the classical BMO with elliptic partial differential equations.

Parabolic BMO shares many properties with the classical one. For instance, in space time cylinders the local PBMO^+ (testing over rectangles whose dilates are compactly contained in the domain of definition) coincides with the global one (testing over all rectangles). This is an extension of a theorem usually attributed to Reimann and Rychener [75], and it was one of the main results in III. On the other hand, the most remarkable difference between BMO and PBMO^+ that is immediate from the definition is that functions in the parabolic class can decay arbitrarily fast in the positive time direction, and fast decay can actually compensate the violations of the oscillation bounds of stationary type. What is more, the final result of IV tells that, roughly speaking, this is actually the principal difference between the two spaces.

From a PDE point of view the result makes sense. Since the evolution is driven by the deviation from a stationary solution, a function far from the properties of a stationary solutions must have high rate of change in the time direction.

2.6 Parabolic A_q^+ classes

In analogy with the classical theory, the good weights of a certain maximal operator are tightly connected to parabolic BMO. Indeed, a simple application of the parabolic John-Nirenberg theorem reveals that

$$\mathrm{PBMO}^+ = \{-\alpha \log w : w \in A_q^+, \alpha \geq 0\}$$

where the A_q^+ class is defined through the forward in time Muckenhoupt condition

$$\sup_R \left(\fint_{R^-(\gamma)} w \right) \left(\fint_{R^+(\gamma)} w^{1-q'} \right)^{q-1} < \infty, \quad q \in (1, \infty).$$

The importance of this relation, whose classical analogue was found in [16], lies in the fact that the path to an explicit representation of parabolic BMO goes through factorization of A_q^+. This, in turn, is a deep property difficult to prove in any way elementary enough to be a reasonable alternative in the parabolic setting (compare to [43]), and for succeeding in finding a short-cut, some powerful tools from weighted norm inequalities are needed. In other words, once the celebrated Rubio de Francia algorithm is available, the factorization property will follow from few lines by clever arguments originally due to Coifman, Jones and Rubio de Francia [15]. But in order to use Rubio de Francia's method, one has to prove that A_q^+ is sufficient for the $L^q(w)$-boundedness of the related maximal operator.

There is also another motivation to study A_q^+. Apart from the obvious fact that the weights $w \in A_q^+$ with no dependence on *time* are exactly the classical Muckenhoupt weights on $\mathbb{R}^n$, the weights with no dependence on *space* can be identified with the one-sided weights on real line. This is not as straightforward to see as the previous case (due to the time lag), but consulting [62] and [63] it is easy to convince oneself that this is indeed the case. Hence the class A_q^+ arising from Trudinger's equation contains both one-sided and classical Muckenhoupt weights as special cases. Many properties shared by those simpler weights come directly from A_q^+, but there are also questions that remain open. For instance, is there a satisfactory theory of A_∞^+ in the parabolic multidimensional case? Some details of this problem are discussed in V.

As it is easy to guess from the properties of PBMO^+, A_q^+ also has its reflected counterpart A_q^-, defined in an obvious way. The class A_q^- has the same relation to PBMO^- as A_q^+ has to PBMO^+. Moreover, the standard

duality $(A_p)^{1-p'} = A_{p'}$ takes the form

$$(A_q^+)^{1-q'} = A_{q'}^-$$

in the parabolic setting. The reflected classes have an important and nontrivial role in the theory, and when it comes to problems concerning conditions of A_∞ type, some very interesting questions are tightly related to their essence.

2.7 The results

The main part of this dissertation consists of a systematic study of the field described previously, and the most important achievements of the appended research articles can be summarized in the following items:

i. The definition and local-to-global properties of parabolic BMO together with applications to parabolic partial differential equations are treated in III. Once parabolic BMO is defined correctly, it makes sense to ask whether its definition, which is only based on what happens inside a domain, also tells something about the behaviour up to the boundary. It is proved that the answer is affirmative and that functions in local parabolic BMO are globally exponentially integrable. Consequently, positive supersolutions to doubly nonlinear equation are globally integrable to some small power.

ii. Parabolic BMO is characterized in the spirit of Coifman and Rochberg in IV. Each function satisfying the parabolic BMO condition can be written as a sum of a forward-in-time maximal function of a Borel measure, a backward-in-time maximal function of another measure, and a bounded function. Conversely, all functions defined through a similar formula are in parabolic BMO. Compared to the classical case, the maximal operators appearing in this formula are rather tame, but the precise assumptions made about the Borel measures allow rather rough objects to be generated.

iii. The theory of one-sided weights is finally extended to n-space with $n \geq 2$ in IV. That is, the characterization of parabolic weights through a strong type norm inequality in IV is the first fully successful attempt to define weights in $\mathbb{R}^n$ so that the case $n = 1$ reduces to Sawyer's original

one-sided weights, first introduced in 1986 [77].

These three facts together establish a nontrivial theory of weighted norm inequalities with natural connections to the regularity theory of parabolic partial differential equations. In the following chapters we will discuss both the history of the problems as well as the ideas leading to techniques which are powerful enough to establish the results.

3. One-sided analysis

3.1 Differentiation bases

The fact that the Hardy-Littlewood maximal function and A_q condition are written in terms of cubes is not essential, and the boundedness of maximal functions has also been studied in a setting where the basis consisting of cubes with sides parallel to the coordinate axes are replaced by more general open sets. For more about this kind of results, see for instance [42].

For our purposes, theorems about weights for differentiation bases have one severe drawback. A basis is usually defined to be a collection of open sets $\mathcal{B}$ so that every point $x \in \mathbb{R}^n$ is associated with sets $x \in B \in \mathcal{B}$. As a consequence, all the Muckenhoupt conditions related to these bases look like (1.0.2) with cubes replaced by some sets $B \in \mathcal{B}$. In the context of parabolic PDE, this is too restrictive since we usually index the halves of parabolic rectangles according to their exterior points. Consequently the parabolic Muckenhoupt weights are not included in the general results for differentiation bases, and in fact, their theory is different.

3.2 One-sided weights on $\mathbb{R}$

As one renounces the definition of a basis discussed previously, completely new classes of weights and maximal functions appear. One may associate to a point x a collection of sets so that x is an exterior point (as we will do with parabolic weights) or that x is a boundary point. The latter case was studied first.

In his influential paper from 1986 Eric Sawyer [77] studied the boundedness of the one-sided maximal function on weighted L^p spaces on the

real line. The one-sided maximal function is defined as

$$M^+ f(x) := \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|.$$

Here each $x \in \mathbb{R}$ has its collection of intervals $\{(x, x+h) : h > 0\}$ so that x is a boundary point. Sawyer managed to characterize the good weights as the ones satisfying

$$\sup_{\substack{x\in\mathbb{R}\\ h>0}} \left(\frac{1}{h}\int_{x-h}^{x} w\right)\left(\frac{1}{h}\int_x^{x+h} w^{1-q'}\right)^{q-1} < \infty, \quad q \in (1,\infty).$$

A short explanation for the difference between this and the previous condition (1.0.2) is that the values of f on $(x, x+h)$ are seen by $M^+ f$ only from the half-line $(-\infty, x)$.

The original motivation to study one-sided maximal operator came from ergodic maximal functions

$$f^*(x) := \sup_{k>0} \frac{1}{k} \sum_{i=1}^{k} f(T^i x),$$

where $f : X \to \mathbb{R}$, X is a probability space and $T : X \to X$ is an ergodic transformation, but the theory of one-sided weights has also some intrinsic interest.

After Sawyer's result [77] plenty of papers were written by Francisco Martín Reyes and many other authors in order to extend the results related to weighted norm inequalities to the one-sided setting. The extensive theory includes one-sided maximal and minimal operators [8, 18, 19, 20, 54, 59, 60, 61, 63, 65, 66], one-sided BMO^+ and A_∞^+ [3, 62, 64, 17], singular integrals with one-sided kernel [4, 79, 73, 76], and commutators [55, 56]. The results are based on effective use of one-dimensional covering arguments as well as the simple geometry of the real line, and the methods do not work in higher dimensions.

3.3 One-sided analysis in higher dimensions

The first attempt to generalize one-sided weights to $\mathbb{R}^n$ was in 2005 by Sheldy Ombrosi [72]. The latter paper defines a dyadic model for a coordinatewise analogue of the one-sided maximal function and characterizes the weighted weak type (q, q) norm inequality for it.

For a dyadic cube $Q = (x_1 - h, x_1) \times \cdots \times (x_n - h, x_n)$ one defines $Q^+ = (x_1, x_1 + h) \times \cdots \times (x_n, x_n + h)$, and the related maximal function is given by

$$x \mapsto \sup_{Q \ni x} \fint_{Q^+} |f|.$$

The good weights are the obvious ones:

$$\sup_Q \left(\fint_Q w \right) \left(\fint_{Q^+} w^{1-q'} \right)^{q-1} < \infty. \tag{3.3.1}$$

The techniques were pushed further in [25], where the one-sided weights (3.3.1) without restriction to dyadic cubes were shown to be exactly the ones that support the weighted weak type (q,q) inequality for the one-sided maximal function in $\mathbb{R}^2$. Later on, it was proved in [52] and [7] that (3.3.1) is sufficient for weighted strong type (q,q) for all maximal operators $\{N^{r+}\}_{r\in(0,1)}$ which look at collections formed by

$$\{(x_1 + rh, x_1 + h) \times \cdots \times (x_n + rh, x_n + h) : h > 0\}.$$

The bounds are not uniform in r, and it is not known, whether the limiting case $r = 0$ is included or not.

The multidimensional one-sided setting introduced by Ombrosi is slightly different from what we prefer to work in, and the results cannot just be transferred from one point of view to another. Our approach has to deal with many phenomena that are absent from the earlier considerations, but once the right tools to handle them are constructed, the results we get are stronger in comparison. Moreover, the parabolic setting is justified by its tight connection to partial differential equations. On the other hand, both theories are extensions of the same one-dimensional case, and many problems together with their solutions have similar flavour. In the next sections, we will discuss in detail some of the ideas that are useful in the parabolic setting. In first of them, which is devoted to what we call Ombrosi's covering technique, we will describe a modification of the key argument developed in [72] and [25] whereas the remaining two sections introduce the cornerstones of the parabolic theory whose discovery in III and IV resulted in the breakthrough, the proof of the open ended property of the parabolic weights.

3.4 Ombrosi's covering technique

The characterization of weighted weak type (q,q) inequality in [72] and [25] was based on a clever covering argument, whose adaptation to the parabolic setting we will next briefly sketch next. The maximal operator related to parabolic weights is

$$M^{\gamma+} f(z) := \sup_R \fint_{R^+(\gamma)} |f|,$$

where the supremum is over parabolic rectangles R centred at z, and we call it *parabolic forward in time maximal operator*. For expository purposes, we will concentrate on the geometrically simple (but theoretically inessential) case $p = 1$ and $\gamma = 0$ (the time scales as the space does and there is no time lag).

In order to prove a weak type estimate for a maximal operator acting on $L^q(w)$, that is, in order to prove an inequality of the form

$$w(\{x \in \mathbb{R}^n : M^{0+} f > \lambda\}) \leq \frac{C}{\lambda^q} \int_{\mathbb{R}^n} |f|^q w,$$

one usually takes advantage of a covering argument. By this we mean covering the level set on the left hand side by rectangles that have good properties in terms of the corresponding means of f on these sets and in terms of their overlap.

In comparison with more classical cases, the one-sided setting is distinguished by two main difficulties. Firstly, the maximal operator is very far from a centred one. More precisely, arguments such as Besicovitch covering lemma are not applicable to coverings obtained by naïve use of the definition of the superlevel set. Secondly, the collection used to cover the level set must preserve its bounded overlap properties when each of its members is shifted forward in time by its own side length, which will happen as one applies a one-sided Muckenhoupt condition. Especially the latter problem is serious. Indeed, it is easy to construct an infinite collection of pairwise disjoint dyadic cubes so that the intersection of the shifted family is nonempty.

The scheme that allows us to deal with these problems consists of two parts. First, in order to live with the fact that the maximal function produces cubes that cannot be used as a cover, one covers with dilates of them, keeping the absolute continuity of the integral in mind. Moreover, since it suffices to consider

$$\{x \in \mathbb{R}^n : 2\lambda \geq M^+ f > \lambda\},$$

we have a two-sided bound

$$⨍_{R^+(0)} |f| \eqsim \lambda \tag{3.4.1}$$

for averages of f. This property is crucial in extracting a subcollection of the closed, nondilated cubes $\{R_i^-(0)\}$ such that the forward-in-time translates have subsets with bounded overlap but still accommodate a considerable amount of mass of f.

For the rest of the argument, the idea is as follows. It suffices to estimate the weighted measure of compact subsets of the level set. For such a compact set, one may get a finite number of cubes, and for each cube a small ball centred at the center of the face with largest time coordinate (evaluation point) so that the *balls* cover the level set. A selection algorithm yields *a minimal subcollection of cubes* to cover the corresponding evaluation points.

Using the comparability (3.4.1), one may fix a suitable number of allowed overlaps and design another algorithm to remove parts of the forward in time translates of the cubes in the subcollection that violate this bound so that the resulting sets still give averages high enough. We will call them reduced translates. Now certain dilates of the cubes in the minimal subcollection cover the balls, and the balls cover the level set. By the absolute continuity of the integral, it is possible to control the weighted measure of the dilated cubes by that of non-dilated ones. Moreover, the reduced translates give high averages and they have bounded overlap. Now the reader familiar with the classical theory easily deduces that there will be no additional problems in the rest of the proof. The detailed argument can be found in IV.

3.5 Covering through projection

Once the sufficiency of A_q^+ for the weak type inequality is known, the proof of the strong type inequality boils down to proving a self-improving property. In order to improve the A_q^+ condition to $A_{q-\epsilon}^+$ one has to use a reverse Hölder inequality, which in the context of parabolic weights looks like

$$\left(⨍_{R^-(0)} w^{1+\delta}\right)^{1/(1+\delta)} \lesssim ⨍_{R^+(0)} w.$$

The proof of the reverse Hölder estimate reduces to proving a level set estimate of the form

$$w(R^-(0) \cap \{w > \lambda\}) \lesssim \lambda |R \cap \{w > \beta\lambda\}|, \quad \beta \in (0,1).$$

Strictly speaking this estimate is not useful, but one needs a very special pair of sets to replace the naive choice $(R^-(0), R)$ in the display above. However, the idea is to perform a Calderón-Zygmund covering for the left hand side, to apply the A_q^+ condition (inducing a forward-in-time shift to the collection), and to collect the pieces together to get the quantity on the right hand side.

To succeed in this, one has to design a suitable covering of Calderón-Zygmund type very carefully. The problems with this part are related to two main issues: it is easy to find a collection of sets with bounded overlap, but preserving this property in a time shift is difficult. Moreover, there is no obvious way to use dyadic stopping time arguments. The way to get around these obstacles is to use properties of the spatial and temporal dimensions of the space-time separately. That is, one has to factor the space-time into the space, which can be equipped with a good dyadic structure, and into the time, which is one-dimensional.

We will briefly sketch the idea of how to do this. Recall that $R^-(0) = Q \times (t - l^p, t)$. Take the dyadic subcubes of Q, and extend them into backwards-halves of parabolic rectangles. This forms a basis of metric balls. The corresponding non-centred maximal function $\mathcal{M}w$ controls w almost everywhere. Applying Calderón-Zygmund decomposition at each time-slice

$$\{(x,t) \in Q \times \{\tau\} \cap \{\mathcal{M}w > \lambda\}\},$$

we get a family pairwise disjoint collections of metric balls. Putting the collections corresponding to each of the uncountably many slices together, separating the scales, applying one-dimensional covering argument at each scale, and running an additional selection process, we get a cover that can be used to complete the proof. Again, the full proof is more delicate, and the details are written in IV.

3.6 The lag

In the previous two sections we have not paid much attention to the time lag. Up to this point, the lag has only been an annoying additional technicality, but once we try to compose a theorem characterizing the good weights, it becomes important. More precisely, we can push the non-lagged theory to the point where *weak type* norm inequalities are characterized by the non-lagged conditions of Muckenhoupt type. However, in order to prove that weak type inequalities imply the validity of strong type norm inequalities, some self-improving property is needed. This amounts to establishing that membership to a parabolic Muckenhoupt class implies that to a more restrictive one. It is exactly this point which proves that the way of thinking we have chosen is the correct one.

Roughly speaking, the phenomenon ruining the validity of the elliptic Harnack inequality in the parabolic case is also present in the context

of weights. Techniques that usually lead to gain in integrability cause a certain temporal shift in the geometry of estimates, which often precludes further applications. The new approach of IV takes the creation of time lag into account from the very beginning, and a careful study of the properties of the lagged weight classes shows that in the case relevant to parabolic differential equations, all self-worsening properties turn out to be delusive.

A first hint of the time-lag phenomenon for weights was visible already in the paper [52] of Lerner and Ombrosi: the weights that do not have a lag in their definition are good for maximal operators that do have. Moreover, the proof of a generalization of their result given by Berkovits in [7] reformulated this as follows: if a weight satisfies a one-sided A_q condition, then the same weight has to belong to a one-sided $A_{q-\epsilon}$ class with a lag. Careful investigation of the proof in [7] also reveals that the weight classes defined with respect to a given lag are good for maximal operators defined with respect to an even longer waiting time. However, since these results were considered in a geometry without parabolic scaling, it was not apparent how to address the problems caused by the lag.

As mentioned earlier, the appearance of the lag is a qualitative phenomenon in the parabolic theory, and exactly this allows us to prove a Muckenhoupt theorem for parabolic weights. A detailed discussion of this last ingredient of the proof is postponed until the next chapter, which is devoted to geometric aspects of the theory. In fact, the lag is handled by the same tools that are used in studying local-to-global properties for spaces of BMO type.

4. Geometric aspects

4.1 Metric spaces

Even if analysis in metric spaces has been a motivation and a source of inspiration for many of the results discussed in what follows, we will stick to Euclidean notation for the rest of the chapter. We hope that this will clarify some points in the exposition. We will, however, devote this first section to describing the set up of metric spaces, since especially I and II are about extending results known in $\mathbb{R}^n$ to more general spaces by composing new proofs that do not use the special structure of the Euclidean space. Moreover, the generality at which results can be proved usually hints at what is their true nature.

A triple (X, d, μ) consisting of a set X, a metric d and a Borel regular doubling measure μ is called a metric measure space with doubling measure. The doubling condition means that for all metric balls $B(x, r) := \{y \in X : d(x, y) < r\}$, $x \in X$ and $r \in (0, \infty)$ it holds

$$0 < \mu(B(x, 2r)) \leq c_\mu \mu(B(x, r)) < \infty$$

for an absolute constant $c_\mu > 1$.

It is quite standard to add some more assumptions to those mentioned above, such as completeness, local compactness and validity of a Poincaré inequality. Other ways to restrict the class of spaces studied are assumptions on existence of paths realizing the distance as their length (geodesic spaces) or requirements on the shape of balls (chain conditions to be discussed). In all cases one usually tries to keep some model examples like Euclidean spaces, Heisenberg groups, and Carnot groups included. As general references on the subject we mention [34] and [9].

4.2 Local-to-global theorems

To describe the class of geometric phenomena leading to the theorems we prove, we return to the Euclidean setting. We will start by giving a review of local-to-global theorems. The basic question they usually address is simple: given an open and connected set, i.e. a domain, $\Omega \subset \mathbb{R}^n$ and a function $u : \Omega \to \mathbb{R}$ that is assumed to satisfy some local property in Ω, for instance $u \in L^1_{loc}(\Omega)$, is it true that the same property holds globally? In general this kind of claims are highly nontrivial, the simplest example being the failure of global integrability of x^{-1} on $(0, 1)$. However, especially in the context of BMO functions, local-to-global phenomena are ubiquitous. Apart from the fact that this kind of results are interesting as such, the techniques used in their proofs have proved to be invaluable in the study of parabolic weights.

Recall that if $\Omega \subset \mathbb{R}^n$ is a domain, we say that $u \in L^1_{loc}(\Omega)$ is in local BMO if

$$\|u\|_{\mathrm{BMO}} := \sup_{2Q \subset \Omega} \fint_Q |u - u_Q| < \infty.$$

A theorem of Reimann, Rychener and Staples ([75] and [81]) says that this implies a global BMO condition, that is, if the quantity in the display above is finite, then the corresponding quantity with $2Q$ replaced by Q is finite too. Moreover, a theorem of Smith, Stegenga and Hurri-Syrjänen ([80] and [38]) tells that if Ω satisfies an additional quasihyperbolic boundary condition to be discussed later, then we have a global John-Nirenberg inequality

$$|\Omega \cap \{|u - u_\Omega| > \lambda\}| \leq C_1 |\Omega| e^{-\frac{C_2}{\|u\|_{\mathrm{BMO}}}\lambda}.$$

These results were applied by Lindqvist [53] to prove that positive supersolutions to

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty$$

raised to some small power $\epsilon > 0$, are globally integrable. This means that even if the boundary values are rough and create singularities, the property of being a local solution ensures that the singularities are not worse that power-like up to the boundary. Before Lindqvist's nonlinear result, the global integrability of superharmonic functions had also been an active area of research, see [1, 5, 30, 57, 58, 67, 82, 83].

Another class of local-to-global results arises from a less known space JN_p defined by John and Nirenberg in their celebrated paper [45]. We say

that $u \in JN_p(\Omega)$ with $1 < p < \infty$ if

$$K^p(\Omega) := \sup_{\mathcal{W}} \sum_{Q \in \mathcal{W}} |Q| \left(\fint_Q |u - u_Q| \right)^p < \infty,$$

where $\mathcal{W}$ is a pairwise disjoint collection of cubes $2Q \subset \Omega$ so that the collection $\{2Q : Q \in \mathcal{W}\}$ has bounded overlap. Instead of exponential integrability, this condition implies embedding into $L^{p,\infty}$ at every subcube with $2Q \subset \Omega$.

A result by Hurri-Syrjänen et al. [39] tells that also in this case the requirement $2Q \subset \Omega$ can be dropped and that the embedding into weak L^p holds globally in John domains Ω. By suitable definitions and use of clever arguments from [26] and [14], the local-to-global results for JN_p were generalized to a wide class of metric spaces in II.

4.3 Classical chaining techniques

The proofs that local spaces of BMO type coincide with the global ones usually need no assumption on the domain. This is due to the fact that one may proceed via reducing the estimation to a cube, which makes the actual domain invisible. When it comes to proving global inequalities of John-Nirenberg type, the domain becomes more visible again. This will be the subject of this section, and as general references for the classical techniques we mention [81] and [10].

A proof of a global inequality is usually implemented as follows: take a domain Ω, choose a reference cube $Q_0 \subset \Omega$, form a suitable decomposition of Whitney type, connect each Whitney cube to the reference cube by a curve avoiding the boundary, form a chain of cubes along it, and estimate

$$|Q \cap \{|u - c_{Q_0}| > \lambda\}| \leq |Q \cap \{|u - c_Q| > \frac{\lambda}{2}\}| + |Q \cap \{|c_Q - c_{Q_0}| > \frac{\lambda}{2}\}| \quad (4.3.1)$$

by using the known local inequality for the first term and by using the properties of the chain, the function, and the domain for the second term. That is, the result is achieved by patching the local estimates together by studying how difficult it is to access the reference cube from each one of the Whitney cubes.

We will next clarify the meaning of this. Given a starting point $x_1 \in \Omega$ and an endpoint $x_* \in \Omega$, we look at the line integral

$$\int_{\gamma_{x_1 x_*}} \frac{\mathrm{d}l(y)}{d(y, \Omega^c)}$$

along a curve $\gamma_{x_1 x_*}$ connecting the points. A curve to minimize the integral above is called a quasihyperbolic geodesic, and the value of the integral along a minimizing curve is the quasihyperbolic distance

$$k_\Omega(x_1, x_*) := \inf_{\gamma_{x_1 x_*}} \int_{\gamma_{x_1 x_*}} \frac{\mathrm{d}l(y)}{d(y, \Omega^c)}.$$

Let $\beta \in (0, 1)$. If we define

$$r(x) := \frac{2}{\sqrt{n}} \beta d(x, \Omega^c)$$

and denote by $Q(x, l)$ the cube with sidelength l and center x, we may construct a collection of cubes $\{Q_i\}_{i=1}^k$ by taking $Q_1 = Q(x_1, r(x_1))$, choosing a quasihyperbolic geodesic connecting x_1 to x_*, and defining Q_{i+1} to be the cube centred at the point x_{i+1} where the quasihyperbolic geodesic exits Q_i (or x_* in case $x_* \in Q_i$) with sidelength $r(x_{i+1})$. This chain will always have the properties

(i) $Q_1 \ni x_1$ and $Q_k \ni x_*$

(ii) $\frac{1}{\beta} Q_i \subset \Omega$ for all $i \in \{1, \dots, k\}$

(iii) $|Q_i \cap Q_{i+1}| \gtrsim \max\{|Q_i|, |Q_{i+1}|\}$ for all $i \in \{1, \dots, k-1\}$

(iv) $k \lesssim k_\Omega(x_1, x_*)$.

It immediately follows from the properties above that

$$|u_{Q_1} - u_{Q_k}| \leq \sum_{i=1}^{k-1} |u_{Q_i} - u_{Q_{i+1}}| \lesssim (k-1) \|u\|_{\mathrm{BMO}}.$$

This inequality controls the second term in (4.3.1) by the distribution of the quasihyperbolic metric k_Ω, which makes it possible to use estimates coming from the geometry of the domain Ω. The paper [81] was among the first ones to realize the importance of this phenomenon.

An interesting feature of the behaviour of BMO functions on a domain is also revealed: it can always be controlled by the behaviour of the quasihyperbolic metric. More specifically, if the quasihyperbolic distance to a fixed point is exponentially integrable, so are the functions in BMO. Moreover, this is a characterization. Namely, it is not very difficult to prove that $k_\Omega(\cdot, x_*) \in \mathrm{BMO}(\Omega)$.

The domains in which quasihyperbolic metric happens to be exponen-

tially integrable are exactly those with

$$k_\Omega(x, x_*) \leq K \left(\log \frac{1}{d(x, \Omega^c)} + 1 \right)$$

for some constant K. This is called the quasihyperbolic boundary condition and the domains satisfying it are sometimes called Hölder domains, see [28] and [80]. We remark that the class of domains satisfying the quasihyperbolic boundary condition is rather large. It includes all the domains in the classes John, Uniform, NTA and Lipschitz.

The chaining technique using quasihyperbolic metric is very convenient when dealing with BMO, but in the case of JN_p, the situation is different. No simple connection between JN_p and the quasihyperbolic metric is known, and consequently the current results on local-to-global results of JN_p are proved under different assumptions. At the moment the most general class of domains for which JN_p is known to embed into $L^{p,\infty}$ is the class of John domains.

John domains Ω can be characterized [11] through the existence of a collection of balls $\mathcal{F}$ and constants $C_2 > C_1 > 1, \lambda > 1$ such that

(a) $\Omega = \cup_{B \in \mathcal{F}} C_1 B = \cup_{B \in \mathcal{F}} C_2 B$.

(b) $\{C_2 B : B \in \mathcal{F}\}$ has bounded overlap.

(c) Each ball $C_1 B_1$ with $B_1 \in \mathcal{F}$ can be connected to the prescribed central ball $B_* \in \mathcal{F}$ by a chain similar to that described in (i), (ii) and (iii) so that cubes are replaced by balls $C_1 B$ where $B \in \mathcal{F}$.

(d) If V is in the chain connecting B to center, then $\lambda V \supset B$.

This characterization is usually called the Boman chain condition (see [12]), and the property that distinguishes between John domains and Hölder domains is (d). And it is exactly this property that makes Boman chains so easy to use when dealing with JN_p or some more general spaces. One may control all the chains simultaneously instead of estimating them one-by-one using the quasihyperbolic metric; see II.

4.4 Parabolic chains

The classical chaining techniques do not work as such for parabolic BMO and designing a substitute for them was the main achievement of III. We will briefly discuss the parabolic chains, and this will also be the final ingredient needed for the completion of the parabolic theory of weights reported in IV.

The first thing one has to understand when dealing with parabolic BMO is that once a reference point is chosen, a lot of information is lost about a certain half-space. That is, if we take $(x,t) \in \Omega\times\mathbb{R}$ and $u \in \mathrm{PBMO}^+(\Omega\times\mathbb{R})$, we should only try to control the *positive* part of u in $\Omega \times (t,\infty)$ and the *negative* part in $\Omega \times (-\infty, t)$.

Once this is clear, a hint about the actual form of the chains to be used has already been given: the chains come from one time direction and end at (x,t). That is, recalling the definition

$$\sup_R \left(\fint_{R^+(\gamma)} (u-a_R)^+ + \fint_{R^-(\gamma)} (u-a_R)^- \right) < \infty,$$

we see that given a chain $\{R_i\}_{i=1}^k$ (and the constants $a_{R_i} = a_i$), the best we can do is (for some sets C_i)

$$\begin{aligned}(u-a_k)^+ &\le (u-a_1)^+ + \sum_{i=1}^{k-1} (a_i - a_{i+1})^+ \\ &= (u-a_1)^+ + \sum_{i=1}^{k-1} \fint_{C_i} (a_i - a_{i+1})^+ \\ &\le (u-a_1)^+ + \sum_{i=1}^{k-1} \fint_{C_i} ((a_i-u)^+ + (u-a_{i+1})^+),\end{aligned}$$

and the choice suggested by the definition of parabolic BMO is

$$C_i \subset R_i^-(\gamma) \cap R_{i+1}^+(\gamma).$$

Hence the rough idea is to first construct a classical chain in Ω, as described in the previous section, and then choose the correct time coordinates to make it a chain in space time respecting the requirement $C_i \subset R_i^-(\gamma) \cap R_{i+1}^+(\gamma)$.

The construction sketched above gives each chain constructed in space a length in time. It is, however, very important to be able to construct arbitrarily long chains in space with controlled length in time. This final problem is solved by using the fact that parabolic rectangles have the scaling (l, l^p). Hence, forcing an upper bound $\alpha \ll 1$ on the sidelength of

the rectangles used in a chain, we can impose a flat shape on them. This upper bound $l \leq \alpha$ appears as $l^p \leq \alpha^p$ in the time axis, and hence the time needed to travel along a chain can be controlled by as small a number as we wish.

A variant of this chaining technique was used together with the forward in time doubling property of parabolic weights in IV in order to prove that if $w \in A_q^+(\gamma)$ with some $\gamma \in (0,1)$, then $w \in A_q^+(\gamma')$ for all $(0,1)$, which opened the doors for using a lagged reverse Hölder inequality to improve the index q of parabolic Muckenhoupt conditions.

The forward in time doubling of parabolic weights $w \in A_q^+$ means that if R is a parabolic rectangle and $E \subset R^+(\gamma)$ a measurable set, then

$$w(R^-(\gamma)) \lesssim \left(\frac{|R^+(\gamma)|}{|E|}\right)^q w(E).$$

A similar backwards in time doubling property holds for $w^{1-q'} \in A_q^-$. These inequalities allow us to control quantities like $w(R^-(\gamma))$ by $w(E)$ where E is any set that is connected to $R^-(\gamma)$ by a forward in time chain. The proof of the qualitative nature of the time lag boils down to cutting a rectangle under study into small pieces and transporting the pieces face to face. Due to the scaling difference of time and space the small pieces will have larger relative Euclidean distance than the original ones, which completes the proof.

4.5 Quasiconformal mappings

The last topic we study is the invariance of BMO under coordinate changes or composition operators. The meaning and applications of this category of results are still very unclear in the parabolic case. Moreover, there is a rather famous open question already in the context of classical BMO. The class of quasiconformal mappings will take a decisive role in these considerations. In addition to the fact that they preserve solutions to equations of type (1.0.1) with $p = n$; see again [35]), their good properties are also visible in the realm of harmonic analysis related to rougher objects.

Given a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, we would like to know when the operator

$$C_f u = u \circ f$$

is bounded on BMO. On the real line the answer is simple. The operator C_f preserves BMO if and only if f is an increasing function with $f' \in A_\infty$. This is due to Peter Jones [44]. There is also a corresponding result in

Euclidean spaces with higher dimension. Reimann [74] proved that if f is quasiconformal, then C_f is bounded. Conversely, if C_f is bounded, and f is differentiable as well as absolutely continuous on lines; then f is quasiconformal. A suitable localization procedure due to Astala [6] allows a characterization through quasiconformality without additional assumptions. However, the question whether

$$\|u\|_{\mathrm{BMO}} \approx \|C_f u\|_{\mathrm{BMO}}$$

implies quasiconformality remains open.

Before going further, we recall some definitions. A homeomorphism is quasiconformal if the ratio of maximal stretching and contracting remains bounded at infinitesimal scales, that is, there is a finite $H \geq 1$ such that

$$\limsup_{r \to 0} \frac{\max\{|f(x) - f(y)| : y \in \overline{B}(x, r)\}}{\min\{|f(x) - f(y)| : y \in \mathbb{R}^n \setminus B(x, r)\}} \leq H$$

for all $x \in \mathbb{R}^n$. This is the metric definition. It is equivalent to the so called geometric definition through conformal modulus of families of curves, and to the analytic definition given through the Jacobian and the differential. The standard reference for quasiconformal mappings on Euclidean spaces is [88].

The theory of quasiconformal mappings can be established in very general metric spaces; see [36, 37, 49], and the Publication I gives a new proof of Reimann's theorem that applies to so called Carnot groups, of which the Euclidean space and the Heisenberg groups are the most well-known examples. The proof is based on Gotoh's measure density characterization of BMO self maps [31], which was extended to metric spaces in [47]. This result of Reimann type complements the earlier results of [87], which generalized Astala's theorem to metric spaces.

In order to explain how the Carnot group structure is used in the proof, we will briefly outline the procedure in the first Heisenberg group. Recall that the first Heisenberg group can be realized as $\mathbb{R}^3$ endowed with the group operation

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + yx' - xy').$$

In addition to translations defined through action of group elements, it is possible to define dilations $(x, y, t) \mapsto (\delta x, \delta y, \delta^2 t), \delta > 0$, that are compatible with the standard metric structure. A homomorphism that commutes with both translations and dilations is said to be homogeneous. In the Euclidean space $\mathbb{R}^n$ the notion of homogeneous homomorphism coincides

with that of a linear mapping, and in more general contexts it is used whenever differentiability is needed.

The fact that quasiconformal mappings preserve BMO is fairly easy to prove once we know that the pull-back of the volume measure under quasisymmetric maps $|f(\cdot)|$ is A_∞ related to the original volume. The converse direction is, however, more difficult and it is not known whether it can be proved without any group structure on the underlying space o. In case the underlying space is a Carnot group and the homeomorphism is assumed to be differentiable, one may approximate it by a homogeneous homomorphism and verify that the measure density characterization of BMO self maps implies quasiconformality in its metric form. The use of the measure density characterization was the key novelty in Publication I.

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Aalto-DD 37/2016

ISBN 978-952-60-6677-6 (printed)
ISBN 978-952-60-6678-3 (pdf)
ISSN-L 1799-4934
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)

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