NON-LOCAL GEHRING LEMMAS

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Abstract. We prove a self-improving property for reverse Hölder inequalities with non-local right hand side. We attempt to cover all the most important situations that one encounters when studying elliptic and parabolic partial differential equations as well as certain fractional equations. We also consider non-local extensions of A_∞ weights. We write our results in spaces of homogeneous type.

Contents

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1. Introduction

Gehring's lemma [10] establishes the open-ended property of reverse Hölder classes. If

$$\left(\frac{1}{|B|}\int_B u^q\,dx\right)^{1/q} \lesssim \frac{1}{|B|}\int_B u\,dx \tag{1.1}$$

with $q > 1$ and all Euclidean balls $B \subset \mathbb{R}^n$, then

$$\left(\frac{1}{|B|}\int_B u^{q+\epsilon}\,dx\right)^{1/(q+\epsilon)} \lesssim_q \frac{1}{|B|}\int_B u\,dx$$

for a certain $\epsilon > 0$ and all Euclidean balls. This self-improving property has proved to be an important tool when studying elliptic [8, 11] and parabolic [12] partial differential equations as well as quasiconformal mappings [18]. In this case, one has to enlarge the ball in the right hand side. We come back to this.

In this work, we are concerned with reverse Hölder inequalities when the right hand side is non-local. Understanding an analogue of Gehring's lemma in this generality turned out to be crucial in [3], where we prove Hölder continuity in time for solutions of parabolic systems. The non-local nature arises from the use of half-order time derivatives. The ambient space being quasi-metric instead of Euclidean is also an assumption natural from the point of view of parabolic partial differential equations. Hence, we shall explore these non-local Gehring lemmas in spaces of homogeneous type.

It is well known that Gehring's lemma holds for the so called weak reverse Hölder inequality where the right hand side of (1.1) is an average over a dilated ball $2B$. We replace the single dilate by a significantly weaker non-local tail such as

$$\sum_{k=0}^{\infty} 2^{-k}\frac{1}{|2^k B|}\int_{2^k B} u\,dx$$

and certain averages over additional functions f and h that have a special meaning in applications. The main result of this paper is Theorem 3.2 asserting that a variant of Gehring's lemma, and in particular the local higher integrability of u still holds in this setting. We present a core version of the theorem already in the next section. It comes with the introduction of some necessary notation but we tried to keep things simple to give the reader a first flavor of our results. Once the strategy is in place, we discuss various consequences (Section 5), ways to generalize it (Sections 4 and 7) as well as self-improving properties for the right-hand side of the reverse Hölder inequality with tail (Section 6). We aim at covering all the aspects that usually arise from applications. We also illustrate our main result by an application to regularity of solutions of a fractional elliptic equation different from the ones treated in [5, 19, 25] in Section 9.

The context of our work is the following. Gehring's lemma in a metric space endowed with a doubling measure was proved in [29]. See also the book [7]. By [21], every quasi-metric space carries a compatible metric structure so that Gehring's lemma also holds in that setting. However, in the case of homogeneous reverse Hölder inequalities, a very clean argument using self-improving properties of A_∞ weights was used in [2] to give an intrinsically quasi-metric proof (see also the very closely related work [16]). We do not attempt to review the literature in the Euclidean n-space, but we refer to the excellent survey in [17] instead. In addition, we want to point out the recent paper on Gehring's lemma for fractional Sobolev spaces [19]. That paper studies fractional equations, whose solutions are self-improving in terms of both integrability and differentiability. Such phenomena are different from what we encounter here, but we found the technical part of [19] very inspiring.

Among generalizations, we mention that the tails may be replaced by some supremum of averages taken over balls larger than the original ball on the left hand side and/or that one may work on open subsets. In this way, our methods can also be applied to obtain a generalization of A_∞ weights: In [2], a larger class of *weak A_∞ weights*, generalizing the one considered in [9, 28] was defined and their higher (than one) integrability was proved (in spaces of homogeneous type). This class of weights, larger than the usual A_∞ Muckenhoupt class, is defined by allowing a uniform dilation of the ball in the right hand side compared to the one on the left hand side. Here, we show that, in fact, the dilation may be arbitrary (depending on the ball) provided it is finite. Another family of weights covered by our methods is the C_p class studied in [22, 24]. Precise definitions are given in Section 8.

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2. Metric spaces

A *space of homogeneous type* (X, d, μ) is a triple consisting of a set X, a function $d : X \times X \to [0, \infty)$ satisfying the quasi-distance axioms

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$, and
(iii) $d(x, z) \leq K(d(x, y) + d(y, z))$ for a certain $K \geq 1$ and all $x, y, z \in X$;

and a Borel measure μ that is doubling in the sense that

$$0 < \mu(B(x, 2r)) \leq C_d\, \mu(B(x, r)) < \infty$$

holds for a certain C_d and all radii $r > 0$ and centers $x \in X$. If the constant K appearing in the triangle inequality (iii) equals 1, we call (X, d, μ) a *metric space* with doubling measure. The topology is understood to be the one generated by the quasi-metric balls. For simplicity, we impose the additional assumption that all quasi-metric balls are Borel measurable. In general, they can even fail to be open.

The doubling condition implies there is $C > 0$ so that for some $D > 0$,

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C \left(\frac{R}{r}\right)^D \tag{2.1}$$

for all $x \in X$ and $R \geq r > 0$. We can always take $D = \log_2 C_d$. In the following we call this number the *homogeneous dimension* (although there might be smaller positive numbers D than $\log_2 C_d$ for which this inequality holds: our proofs work with any such D). For all these basic facts on analysis in metric spaces, we refer to the book [7].

The following theorem is concerned with the special case of metric spaces, but it has an analogue in the general case of quasi-metric spaces, see Theorem 3.2 below.

Theorem 2.2. *Let (X, d, μ) be a metric space with doubling measure. Let $s, \beta > 0$ and $q > 1$ be such that $s < q$ and $\beta \geq D(1/s - 1/q)$ where D is any number satisfying (2.1). Let $N > 1$ and let $(\alpha_k)_{k\geq 0}$ be a non-increasing sequence of positive numbers with $\alpha := \sum_k \alpha_k < \infty$, and define*

$$a_u(B) := \sum_{k=0}^{\infty} \alpha_k \fint_{N^k B} u \, d\mu \tag{2.3}$$

for $u \geq 0$ locally integrable and B a metric ball.

Suppose that $u, f, h \geq 0$ with $u^q, f^q, h^s \in L^1_{loc}(X, d\mu)$ and $A \geq 0$ is a constant such that for every ball $B = B(x, R)$,

$$\left(\fint_B u^q \, d\mu \right)^{1/q} \leq A a_u(B) + (a_{f^q}(B))^{1/q} + R^{\beta}(a_{h^s}(B))^{1/s}. \tag{2.4}$$

Then there exists $p > q$ depending on $\alpha_0, \alpha, A, q, s, N$ and C_d such that for all balls B,

$$\begin{aligned} \left(\fint_B u^p \, d\mu \right)^{1/p} &\lesssim a_u(NB) + (a_{f^q}(NB))^{1/q} + R^{\beta}(a_{h^s}(NB))^{1/s} \\ &\quad + \left(\fint_{NB} f^p \, d\mu \right)^{1/p} + R^{\beta} \left(\fint_{NB} h^{ps/q} \, d\mu \right)^{q/sp}, \end{aligned} \tag{2.5}$$

with implicit constant depending on $\alpha_0, \alpha, A, q, s, \beta, N$ and C_d.

Remark 2.6. If one assumes the sequence $(\alpha_k)_{k\geq 0}$ is finite, the functional is comparable to one single average on $N^{k_0}B$ for some k_0. This gives a proof of the classical Gehring lemma with dilated balls. Note the shift from $N^{k_0}B$ to $N^{k_0+1}B$ in the conclusion. But well-known additional covering arguments show that the dilation factor N^{k_0+1} can be changed to any number larger than 1. If one assumes

$$\exists C < \infty : \forall k \geq 0 \quad \alpha_k \leq C\alpha_{k+1}, \tag{2.7}$$

then it follows that $a_u(NB) \leq C a_u(B)$ for all $u \geq 0$ and all balls B. In that case, one can replace NB by B in the right hand side of (2.5). Geometric sequences, which are typical in application, do satisfy this condition but this rules out finite sequences. Finally, note that the higher integrability of u on B depends only on the higher integrability of f and h on the first dilated ball NB.

Proof. We prove (2.5) for $B = B(x_0, R)$ with $x_0 \in X$ and $R > 0$. Throughout, we reserve the symbol C for a constant that depends at most on $\alpha_0, \alpha, A, q, s, \beta, N$ and C_d but that may vary from line to line.

Step 1. Preparation. Having fixed B, we set $g^q := A_R^q h^s \mathbb{1}_{NB}$ with A_R a constant so that for any ball B_r with radius r contained in NB, we have

$$r^\beta \left(\fint_{B_r} h^s \, d\mu \right)^{1/s} \leq \left(\fint_{B_r} g^q \, d\mu \right)^{1/q} \tag{2.8}$$

and

$$\left(\fint_{NB} g^q \, d\mu \right)^{1/q} \leq C_1 (NR)^\beta \left(\fint_{NB} h^s \, d\mu \right)^{1/s} \tag{2.9}$$

for some C_1 depending only on the doubling condition, s and q. Indeed, write $B_r = B(x, r)$. As $x \in NB$, we have $NB = B(x_0, NR) \subset B(x, 2NR)$, hence

$$\frac{\mu(NB)}{\mu(B_r)} \leq \frac{\mu(B(x, 2NR))}{\mu(B(x, r))} \leq C_0 \left(\frac{2NR}{r} \right)^D \leq C_0 2^D \left(\frac{NR}{r} \right)^{\beta(1/s - 1/q)^{-1}}$$

where C_0 depends only on the doubling condition. Unraveling this inequality and setting $C_1 = (C_0 2^D)^{1/q - 1/s}$ yield

$$r^\beta \mu(B_r)^{1/q - 1/s} \leq C_1 (NR)^\beta \mu(NB)^{1/q - 1/s}.$$

Hence, as $q > s$,

$$r^\beta \mu(B_r)^{1/q-1/s} \left(\int_{B_r} h^s \, d\mu \right)^{1/s - 1/q} \leq C_1 (NR)^\beta \mu(NB)^{1/q-1/s} \left(\int_{NB} h^s \, d\mu \right)^{1/s-1/q}$$

so that

$$\begin{aligned} r^\beta \left(\fint_{B_r} h^s \, d\mu \right)^{1/s} &= r^\beta \left(\fint_{B_r} h^s \, d\mu \right)^{1/s-1/q} \left(\fint_{B_r} h^s \, d\mu \right)^{1/q} \\ &\leq C_1 (NR)^\beta \left(\fint_{NB} h^s \, d\mu \right)^{1/s-1/q} \left(\fint_{B_r} h^s \, d\mu \right)^{1/q}. \end{aligned}$$

Thus, we set

$$A_R := C_1 (NR)^\beta \left(\fint_{NB} h^s \, d\mu \right)^{1/s-1/q} \tag{2.10}$$

and (2.8) is proved. Observing that if $B_r = NB$ we have equalities with constant 1 in the inequalities above, the constant C_1 works for (2.9).

Step 2. Local setup. For $\ell \in \mathbb{N}$, fix r_0 and ρ_0 real numbers satisfying $R \leq r_0 < \rho_0 \leq NR$ with $N^\ell(\rho_0 - r_0) = R$. For $x \in B(x_0, r_0)$, we have that $N^k N^\ell (\rho_0 - r_0) = N^k R$ for $k \geq 0$ so

$$B(x, N^k(\rho_0 - r_0)) \subset B(x_0, N^{k+1} R) \subset B(x, N^{k+\ell+j}(\rho_0 - r_0)),$$

where $j = 2$ when $N \geq 2$ and $j = \lceil (\log_2 N)^{-1} + 1 \rceil$ when $1 < N < 2$, and consequently for any positive μ-measurable function v,

(2.11)
$$\begin{aligned}\fint_{B(x,N^k(\rho_0-r_0))} v\,d\mu &= \frac{1}{\mu(B(x,N^k(\rho_0-r_0)))}\int_{B(x,N^k(\rho_0-r_0))} v\,d\mu \\ &\leq \frac{\mu(B(x,N^{k+\ell+j}(\rho_0-r_0)))}{\mu(B(x,N^k(\rho_0-r_0)))}\frac{1}{B(x_0,N^{k+1}R)}\int_{B(x_0,N^{k+1}R)} v\,d\mu \\ &\leq C_N^{\ell+j}\fint_{B(x_0,N^{k+1}R)} v\,d\mu,\end{aligned}$$

where we used (2.1) in the last line. The constant $C_N \geq 1$ only depends on the doubling constant C_d and the number N.

Step 3. Beginning of the estimate. For $m > 0$, set $u_m := \min\{u,m\}$, $B_{r_0} := B(x_0,r_0)$ and $B_{\rho_0} := B(x_0,\rho_0)$. Using the Lebesgue-Stieltjes formulation of the integral we have

$$\begin{aligned}\int_{B_{r_0}} u_m^{p-q}u^q\,d\mu &= (p-q)\int_0^m \lambda^{p-q-1}u^q(B_{r_0}\cap\{u>\lambda\})\,d\lambda \\ &= (p-q)\int_0^{\lambda_0} \lambda^{p-q-1}u^q(B_{r_0}\cap\{u>\lambda\})\,d\lambda \\ &\quad + (p-q)\int_{\lambda_0}^m \lambda^{p-q-1}u^q(B_{r_0}\cap\{u>\lambda\})\,d\lambda \\ &\leq \lambda_0^{p-q}u^q(B_{r_0}) + (p-q)\int_{\lambda_0}^m \lambda^{p-q-1}u^q(B_{r_0}\cap\{u>\lambda\})\,d\lambda \\ &=: I + II,\end{aligned} \tag{2.12}$$

where $u^q(\mathcal{A}) = \int_{\mathcal{A}} u^q\,d\mu$ for any measurable set $\mathcal{A} \subseteq X$ and λ_0 is a constant chosen below.

Step 4. Choice of the threshold λ_0. We define three functions

$$U(x,r) := \fint_{B(x,r)} u\,d\mu,\quad F(x,r) := \left(\fint_{B(x,r)} f^q\,d\mu\right)^{1/q},\quad G(x,r) := \left(\fint_{B(x,r)} g^q\,d\mu\right)^{1/q}$$

and for $\lambda > \lambda_0$, we denote the relevant level sets by

$$U_\lambda := B_{r_0}\cap\{u>\lambda\},\qquad F_\lambda := B_{r_0}\cap\{f>\lambda\},\qquad G_\lambda := B_{r_0}\cap\{g>\lambda\}.$$

It follows from (2.11) with $k = 0$ that for $x \in B_{r_0}$,

$$U(x,\rho_0-r_0) = \fint_{B(x,\rho_0-r_0)} u\,d\mu \leq \frac{C_N^{\ell+j}}{\alpha_0}a_u(NB)$$

and one has the same observation for F with f^q. For the last term, we use (2.11) in conjunction with (2.9) to obtain

$$\fint_{B(x,\rho_0-r_0)} g^q\,d\mu \leq C_N^{\ell+j}\fint_{NB} g^q\,d\mu \leq C_N^{\ell+j}C_1^q(NR)^{\beta q}\left(\fint_{NB} h^s\,d\mu\right)^{q/s}$$

$$\leq \frac{C_N^{(\ell+j)q/s}}{\alpha_0^{q/s}} C_1^q (NR)^{\beta q} a_{h^s}(NB)^{q/s},$$

where we used $C_N \geq 1$ and $q/s > 1$. Consequently, we choose

$$\lambda_0 := \frac{C_N^{\ell+j}}{\alpha_0} a_u(NB) + \left(\frac{C_N^{\ell+j}}{\alpha_0} a_{f^q}(NB)\right)^{1/q} + C_1(NR)^\beta \left(\frac{C_N^{\ell+j}}{\alpha_0} a_{h^s}(NB)\right)^{1/s}.$$

Finally, set

$$\Omega_\lambda := \Big\{x \in U_\lambda \cup F_\lambda \cup G_\lambda : x \text{ is a Lebesgue point for } u, f^q \text{ and } g^q\Big\}.$$

Step 5. Estimate of the measure of U_λ. We begin to estimate II in (2.12) so we assume $\lambda > \lambda_0$. For $x \in B_{r_0}$,

$$U(x, \rho_0 - r_0) + F(x, \rho_0 - r_0) + G(x, \rho_0 - r_0) \leq \lambda_0 < \lambda. \tag{2.13}$$

On the other hand, by definition of U_λ, F_λ and G_λ, if $x \in \Omega_\lambda$ then

$$\lim_{r\to 0} U(x, r) + F(x, r) + G(x, r) > \lambda$$

Thus for $x \in \Omega_\lambda$ we can define the stopping time radius

$$r_x := \sup \Big\{N^{-m}(\rho_0 - r_0) : m \in \mathbb{N} \\ \text{and } U(x, N^{-m}(\rho_0 - r_0)) + F(x, N^{-m}(\rho_0 - r_0)) + G(x, N^{-m}(\rho_0 - r_0)) > \lambda\Big\}.$$

We remark that (2.13) implies that $r_x < \rho_0 - r_0$. Of course $\Omega_\lambda \subset \cup_{x\in\Omega_\lambda} B(x, r_x/5)$. By the Vitali Covering Lemma (5r-Covering Lemma) there exists a countable collection of balls $\{B(x_i, r_i)\} = \{B_i\}$ with $r_i = r_{x_i}$ such that $\{\frac{1}{5}B_i\}$ are pairwise disjoint and $\Omega_\lambda \subset \cup_i B_i$. Let $m_i \geq 1$ such that $N^{m_i} r_i = \rho_0 - r_0$.

We make three observations:

(i) For each i, either $\fint_{B_i} u\, d\mu > \frac{\lambda}{3}$, $(\fint_{B_i} f^q\, d\mu)^{1/q} > \frac{\lambda}{3}$, or $(\fint_{B_i} g^q\, d\mu)^{1/q} > \frac{\lambda}{3}$.

(ii) The radius of each B_i is less than $\rho_0 - r_0$ and $x_i \in B_{r_0}$ so $B_i \subset B_{\rho_0}$.

(iii) For $0 \leq k < m_i$, $N^k r_i = N^{-(m_i-k)}(\rho_0 - r_0) < \rho_0 - r_0$, so $N^k r_i$ is 'above' or at the stopping time and

$$\fint_{N^k B_i} u\, d\mu + \left(\fint_{N^k B_i} f^q\, d\mu\right)^{1/q} + \left(\fint_{N^k B_i} g^q\, d\mu\right)^{1/q} \leq C\lambda,$$

where C shows up since we have used doubling once in the case $k = 0$.

Using that $\mu((U_\lambda \cup F_\lambda \cup G_\lambda) \setminus \Omega_\lambda) = 0$, $\Omega_\lambda \subset \cup_i B_i$ and (2.4), we obtain

$$\begin{aligned} u^q(U_\lambda) &\leq u^q(U_\lambda \cup F_\lambda \cup G_\lambda) \leq \sum_i u^q(B_i) \\ &\leq \sum_i \mu(B_i)[A a_u(B_i) + (a_{f^q}(B_i))^{1/q} + r_i^\beta (a_{h^s}(B_i))^{1/s}]^q. \end{aligned} \tag{2.14}$$

Then using $m_i \geq 1$, $\sum_k \alpha_k = \alpha$ and observation (iii) we obtain

$$\begin{aligned} a_{f^q}(B_i) &= \sum_{k=0}^{\infty} \alpha_k \fint_{N^k B_i} f^q \, d\mu = \sum_{k=0}^{m_i-1} \alpha_k \fint_{N^k B_i} f^q \, d\mu + \sum_{k=m_i}^{\infty} \alpha_k \fint_{N^k B_i} f^q \, d\mu \\ &\leq C^q \alpha \lambda^q + \sum_{k=0}^{\infty} \alpha_{k+m_i} \fint_{B(x_i, N^k(\rho_0 - r_0))} f^q \, d\mu, \end{aligned} \tag{2.15}$$

where we simply re-indexed the second sum and used that $N^{m_i} r_i = \rho_0 - r_0$. Now we use (2.11) and that $\alpha_{k+m_i} \leq \alpha_k$ to deduce

$$\begin{aligned} a_{f^q}(B_i) &\leq C^q \alpha \lambda^q + C_N^{\ell+j} \sum_{k=0}^{\infty} \alpha_k \fint_{N^{k+1} B} f^q \, d\mu \\ &\leq C^q \alpha \lambda^q + C_N^{\ell+j} a_{f^q}(NB) \leq C^q \alpha \lambda^q + \alpha_0 \lambda_0^q \\ &\leq C \lambda^q, \end{aligned}$$

where we used the definition of λ_0 in Step 4 and $\lambda > \lambda_0$. Similarly, $a_u(B_i) \leq C\lambda$. For h^s, using $m_i \geq 1$, we obtain

$$\begin{aligned} &(r_i)^{\beta s} a_{h^s}(B_i) \\ &\quad = \sum_{k=0}^{\infty} \alpha_k (r_i)^{\beta s} \fint_{N^k B_i} h^s \, d\mu \\ &\quad = \sum_{k=0}^{m_i-1} \alpha_k (r_i)^{\beta s} \fint_{N^k B_i} h^s \, d\mu + \sum_{k=m_i}^{\infty} \alpha_k (r_i)^{\beta s} \fint_{N^k B_i} h^s \, d\mu \\ &\quad \leq \sum_{k=0}^{m_i-1} \alpha_k \left(\fint_{N^k B_i} g^q \, d\mu \right)^{s/q} + (r_i)^{\beta s} \sum_{k=0}^{\infty} \alpha_{k+m_i} \fint_{B(x_i, N^k(\rho_0 - r_0))} h^s \, d\mu, \end{aligned}$$

where we used (2.8) and $N^k B_i \subset NB$ when $k < m_i$ for the first sum, re-indexed the second sum and used that $N^{m_i} r_i = \rho_0 - r_0$. With $\sum_k \alpha_k = \alpha$ and observation (iii) for the first sum and $\alpha_{k+m_i} \leq \alpha_k$ along with (2.11) for the second one, we deduce

$$\begin{aligned} (r_i)^{\beta s} a_{h^s}(B_i) &\leq C^s \alpha \lambda^s + C_N^{\ell+j} (NR)^{\beta s} \sum_{k=0}^{\infty} \alpha_k \fint_{N^{k+1} B} h^s \, d\mu \\ &\leq C^s \alpha \lambda^s + C_N^{\ell+j} (NR)^{\beta s} a_{h^s}(NB) \leq C^s \alpha \lambda^s + \alpha_0 \lambda_0^s C_1^{-s} \\ &\leq C \lambda^s, \end{aligned}$$

where we used $\lambda > \lambda_0$. Combining the above estimates with (2.14) we obtain

$$u^q(U_\lambda) \leq C \lambda^q \sum_i \mu(B_i) \leq C C_d^3 \lambda^q \sum_i \mu\left(\tfrac{1}{5} B_i\right) \leq C C_d^3 \lambda^q \mu\left(\cup_i B_i\right) \tag{2.16}$$

where we used that $\{\frac{1}{5} B_i\}$ are pairwise disjoint. Now we use (i) and (ii) to conclude that

$$\cup_i B_i \subset \{M(u \mathbb{1}_{B_{\rho_0}}) > \lambda/3\} \cup \{M(f^q \mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^q\} \cup \{M(g^q \mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^q\}, \tag{2.17}$$

where M is the uncentered maximal function.

Step 6. Estimate of II and I. Plugging (2.16) and (2.17) into II we obtain

$$\begin{aligned}
II &= (p-q)\int_{\lambda_0}^{m} \lambda^{p-q-1} u^q(U_\lambda)\, d\lambda \\
&\le C(p-q)\int_0^m \lambda^{p-1}\mu(\{M(u\mathbb{1}_{B_{\rho_0}}) > \lambda/3\})\, d\lambda \\
&\quad + C(p-q)\int_0^m \lambda^{p-1}\mu(\{M(f^q\mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^q\})\, d\lambda \\
&\quad + C(p-q)\int_0^m \lambda^{p-1}\mu(\{M(g^q\mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^q\})\, d\lambda \\
&=: II_1 + II_2 + II_3.
\end{aligned}$$

We handle II_2 and II_3 in the same way. Using the Hardy-Littlewood maximal theorem for spaces of homogeneous type and recalling that the $L^{p/q} \to L^{p/q}$ operator norm of the maximal function is bounded by $C\frac{p/q}{(p/q)-1}$, we obtain

$$\begin{aligned}
II_3 &= C(p-q)\int_0^m \lambda^{p-1}\mu(\{M(g^q\mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^q)\})\, d\lambda \\
&\le C\frac{p-q}{p}\int_X (M(g^q\mathbb{1}_{B_{\rho_0}}))^{p/q}\, d\mu \\
&\le C\Big(\frac{p}{p-q}\Big)^{p/q-1}\int_{NB} g^p\, d\mu,
\end{aligned}$$

where we used $B_{\rho_0} \subset NB$ in the last step. Similarly we have that

$$II_2 \le C\big(\frac{p}{p-q}\big)^{p/q-1}\int_{NB} f^p\, d\mu.$$

To handle II_1 we notice that

$$\{M(u\mathbb{1}_{B_{\rho_0}}) > \lambda/3\} \subset \{M(u\mathbb{1}_{B_{\rho_0}\cap\{u>\lambda/6\}}) > \lambda/6\}.$$

From this estimate and the weak type (1,1) bound for the Hardy-Littlewood maximal function for spaces of homogeneous type we have

$$\mu(\{M(u\mathbb{1}_{B_{\rho_0}}) > \lambda/3\}) \le \frac{C}{\lambda}\int_{B_{\rho_0}\cap\{u>\lambda/6\}} u\, d\mu.$$

Using this bound in II_1 yields

$$\begin{aligned}
II_1 &\le C(p-q)\int_0^m \lambda^{p-2}\int_{B_{\rho_0}\cap\{u>\lambda/6\}} u\, d\mu\, d\lambda \\
&= C(p-q)\int_{B_{\rho_0}} u\int_0^{\max\{m,6u\}} \lambda^{p-2}\, d\lambda\, d\mu \\
&= C6^{p-1}\frac{p-q}{p-1}\int_{B_{\rho_0}} u_{m/6}^{p-1} u\, d\mu \\
&\le C6^{p-1}\frac{p-q}{p-1}\int_{B_{\rho_0}} u_m^{p-q} u^q\, d\mu,
\end{aligned} \tag{2.18}$$

and we note that we can make the constant in front of the integral arbitrarily small by choice of $p > q$. Combining our estimates for II_1, II_2 and II_3 we obtain for any $p \in (q, 2q)$,

$$II \leq \epsilon_p \int_{B_{\rho_0}} u_m^{p-q} u^q \, d\mu + C\epsilon_p^{-1} \int_{NB} f^p \, d\mu + C\epsilon_p^{-1} \int_{NB} g^p \, d\mu, \tag{2.19}$$

where $\epsilon_p := C(p-q)$.

Now we bound I. Note that $B \subset B_{r_0} \subset NB$. By definition of λ_0 and using (2.4),

$$\begin{aligned} I &\leq \lambda_0^{p-q} u^q(NB) \\ &\leq \lambda_0^{p-q} \mu(NB) \big(A a_u(NB) + (a_{f^q}(NB))^{1/q} + (NR)^\beta (a_{h^s}(NB))^{1/s}\big)^q \\ &\leq C(\widetilde{C}_N^{\ell+j})^{p-q} \mu(NB) \alpha_p(NB), \end{aligned} \tag{2.20}$$

where we denoted $\alpha_p(NB) := \big(a_u(NB) + (a_{f^q}(NB))^{1/q} + (NR)^\beta (a_{h^s}(NB))^{1/s}\big)^p$ and put $\widetilde{C}_N := \max(C_N, C_N^{1/s}) \geq 1$ on recalling that we allow for $s < 1$.

Step 7. Conclusion. Setting

$$\varphi(t) := \int_{B(x_0,t)} u_m^{p-q} u^q \, d\mu$$

and combining estimates (2.12), (2.19) and (2.20), we may summarize our estimates as

$$\varphi(r_0) \leq C\mu(NB)\alpha_p(NB)\widetilde{C}_N^{(p-q)(\ell+j)} + \epsilon_p \varphi(\rho_0) + C\epsilon_p^{-1} \int_{NB} f^p \, d\mu + C\epsilon_p^{-1} \int_{NB} g^p \, d\mu,$$

whenever $R \leq r_0 < \rho_0 \leq NR$ and $N^\ell(\rho_0 - r_0) = R$ and where j depends at most on N, see Step 2. For notational convenience set

$$\begin{aligned} M_1 &:= C\mu(NB)\alpha_p(NB), \\ M_2 &:= C \int_{NB} f^p \, d\mu + C \int_{NB} g^p \, d\mu, \end{aligned}$$

so that

$$\varphi(r_0) \leq M_1 \widetilde{C}_N^{(p-q)\ell} + \epsilon_p \varphi(\rho_0) + \epsilon_p^{-1} M_2. \tag{2.21}$$

Now, we set up an iteration scheme to conclude: We fix $K \in \mathbb{N}$ large enough to guarantee $\sum_{\ell=0}^\infty N^{-K\ell} \leq N$, initiate with $t_0 := R$ and put $t_{\ell+1} := t_\ell + N^{-K(\ell+1)} R$ for $\ell = 0, 1, \ldots$. Then $R \leq t_\ell < t_{\ell+1} \leq NR$ and $N^{K(\ell+1)}(t_{\ell+1} - t_\ell) = R$ so that

$$\varphi(t_\ell) \leq M_1 \widetilde{C}_N^{(p-q)K(\ell+1)} + \epsilon_p^{-1} M_2 + \epsilon_p \varphi(t_{\ell+1}). \tag{2.22}$$

Iterating the above inequality we obtain for any $\ell_0 \in \mathbb{N}$

$$\begin{aligned} \varphi(t_0) &\leq M_1 \sum_{\ell=1}^{\ell_0} \widetilde{C}_N^{(p-q)K\ell} \cdot \epsilon_p^{\ell-1} + M_2 \sum_{\ell=1}^{\ell_0} \epsilon_p^{\ell-2} + \epsilon_p^{\ell_0} \varphi(t_{\ell_0}) \\ &\leq CM_1 + CM_2 + C\epsilon_p^{\ell_0} \varphi(NR) \end{aligned}$$

provided that $\epsilon_p \leq (2\widetilde{C}_N^{(p-q)K})^{-1} \leq 1/2$ by now fixing $p \in (q, 2q)$ with $p - q$ small enough.

Noting that $\varphi(NR) < \infty$ (by truncation of u) and $t_0 = R$, we may let $\ell_0 \to \infty$ above to conclude

$$\varphi(R) \leq CM_1 + CM_2.$$

Upon replacing M_1, M_2, $\alpha_p(NB)$ and $\varphi(R)$ we obtain

$$\begin{aligned}\int_B u_m^{p-q} u^q \, d\mu \leq C\mu(NB)\big(a_u(NB) + (a_{f^q}(NB))^{1/q} + (NR)^{\beta}(a_{h^s}(NB))^{1/s}\big)^p \\ + C\int_{NB} f^p \, d\mu + C\int_{NB} g^p \, d\mu.\end{aligned} \tag{2.23}$$

Dividing both sides of the inequality by $\mu(B)$, taking $p-$th roots and letting $m \to \infty$ we obtain (2.5) except for the presence of the term $(\fint_{NB} g^p \, d\mu)^{1/p}$. We handle this term using the definition of g in terms of h and the definition of A_R in (2.10), obtaining

$$\begin{aligned}\left(\fint_{NB} g^p \, d\mu\right)^{1/p} &= A_R \left(\fint_{NB} h^{ps/q} \, d\mu\right)^{1/p} \\ &= C_1 (NR)^{\beta} \left(\fint_{NB} h^s \, d\mu\right)^{1/s - 1/q} \left(\fint_{NB} h^{ps/q} \, d\mu\right)^{1/p} \\ &\leq C_1 (NR)^{\beta} \left(\fint_{NB} h^{ps/q} \, d\mu\right)^{q/ps - 1/p} \left(\fint_{NB} h^{ps/q} \, d\mu\right)^{1/p} \\ &= C_1 (NR)^{\beta} \left(\fint_{NB} h^{ps/q} \, d\mu\right)^{q/sp}.\end{aligned}$$

□

3. Quasi-metric spaces

In Theorem 2.2, it is possible to relax the structural assumption of (X, d, μ) being a metric space by allowing the constant K in the quasi triangle inequality to take values greater than one. The proof of Theorem 2.2 does not carry over as such (or rather it becomes very technical) but we can take advantage of the fact that every quasi-metric ($K > 1$) is equivalent to a power of a proper metric ($K = 1$). See [1, 21, 23]. The following proposition is from [23].

Proposition 3.1. *Let (X, ρ) be a quasi-metric space and let $0 < \delta \leq 1$ be given by $(2K)^{\delta} = 2$. Then there is another quasi-metric $\tilde{\rho}$ such that $\tilde{\rho}^{\delta}$ is a metric and for all $x, y \in X$,*

$$E^{-1}\rho(x, y) \leq \tilde{\rho}(x, y) \leq E\rho(x, y),$$

where $E \geq 1$ is a constant only depending on the quasi triangle inequality constant of ρ.

With Proposition 3.1 at hand, the following theorem is a straightforward consequence of its metric counterpart. For the reader's convenience and since akin reductions to the metric case will be used at other occasions in this paper, we present the full details here.

Theorem 3.2. *Let (X, ρ, μ) be a space of homogeneous type. Let $s, \beta > 0$ and $q > 1$ be such that $s < q$ and $\beta \geq D(1/s - 1/q)$ where D is any number satisfying*

(2.1). *Let* $N > 1$ *and* $(\alpha_k)_{k\geq 0}$ *be a non-increasing sequence of positive numbers with* $\alpha := \sum_k \alpha_k < \infty$. *Define*

$$a_u(B) := \sum_{k=0}^{\infty} \alpha_k \fint_{N^k B} u \, d\mu$$

for $u \geq 0$ *locally integrable and* B *a quasi-metric ball.*

Suppose that $u, f, h \geq 0$ *with* $u^q, f^q, h^s \in L^1_{loc}(X, d\mu)$ *and there exists a constant* A *such that for every ball* $B = B(x, R)$,

$$\left(\fint_B u^q \, d\mu \right)^{1/q} \leq A a_u(B) + (a_{f^q}(B))^{1/q} + R^\beta (a_{h^s}(B))^{1/s}. \tag{3.3}$$

Then there exists $p > q$ *depending on* $\alpha_0, \alpha, A, q, s, K, N$ *and* C_d *such that for all balls* B,

$$\begin{aligned} \left(\fint_B u^p \, d\mu \right)^{1/p} &\lesssim \tilde{a}_u(NB) + (\tilde{a}_{f^q}(NB))^{1/q} + R^\beta (\tilde{a}_{h^s}(NB))^{1/s} \\ &\quad + \left(\fint_{NE^{2/\delta}B} f^p \, d\mu \right)^{1/p} + R^\beta \left(\fint_{NE^{2/\delta}B} h^{ps/q} \, d\mu \right)^{q/sp}. \end{aligned} \tag{3.4}$$

Here, $\tilde{a}_u$ *is obtained from* a_u *by replacing the sequence* α_k *with* $\alpha_{\max(0, k-j_0-j_1)}$, *where* j_0 *and* j_1 *are the minimal integers with* $E^2 \leq N^{j_0}$ *and* $E^{2/\delta} \leq N^{j_1}$ *and* E, δ *are the constants from Proposition 3.1. The implicit constant depends on* $\alpha_0, \alpha, A, q, s, \beta, K, N$ *and* C_d.

Remark 3.5. The same remarks as after Theorem 2.2 apply.

Proof. Let d be the metric so that $d^{1/\delta}$ with $\delta \in (0, 1]$ is equivalent to ρ, provided by Proposition 3.1. Then there is a constant $E > 1$ only depending on the quasi-metric constant K of ρ so that

$$B^\rho(x, r) = \{z : \rho(z, x) < r\} \subseteq \{z : E^{-1} d^{1/\delta}(z, x) < r\} = B^d(x, (Er)^\delta)$$

and

$$B^\rho(x, r) = \{z : \rho(z, x) < r\} \supseteq \{z : E d^{1/\delta}(z, x) < r\} = B^d(x, (E^{-1} r)^\delta).$$

In total $B^d(x, (E^{-1}r)^\delta) \subseteq B^\rho(x, r) \subseteq B^d(x, (Er)^\delta)$. Consequently μ is doubling with respect to the metric d, and we also see that the hypothesis (3.3) implies that

$$\begin{aligned} \left(\fint_{B^d(x,(E^{-1}R)^\delta)} u^q \, d\mu \right)^{1/q} &\lesssim \sum_{k=0}^{\infty} \alpha_k \fint_{N^{\delta k} E^\delta B^d(x, R^\delta)} u \, d\mu \\ &\quad + \left(\sum_{k=0}^{\infty} \alpha_k \fint_{N^{\delta k} E^\delta B^d(x, R^\delta)} f^q \, d\mu \right)^{1/q} \\ &\quad + R^\beta \left(\sum_{k=0}^{\infty} \alpha_k \fint_{N^{\delta k} E^\delta B^d(x, R^\delta)} h^s \, d\mu \right)^{1/s} \end{aligned}$$

holds for all $x \in X$ and $R > 0$. Setting $R' := (E^{-1}R)^\delta$, we can rewrite this as

$$\left(\fint_{B^d(x,R')} u^q\,d\mu\right)^{1/q} \lesssim \sum_{k=0}^\infty \alpha_k \fint_{N^{\delta k}B^d(x,E^{2\delta}R')} u\,d\mu + \left(\sum_{k=0}^\infty \alpha_k \fint_{N^{\delta k}B^d(x,E^{2\delta}R')} f^q\,d\mu\right)^{1/q} + (R')^{\beta/\delta}\left(\sum_{k=0}^\infty \alpha_k \fint_{N^{\delta k}B^d(x,E^{2\delta}R')} h^s\,d\mu\right)^{1/s}.$$

We set $N' := N^\delta$. Then j_0 is the smallest positive integer so that $E^{2\delta} \le (N')^{j_0}$. We note that

$$\sum_{k=0}^\infty \alpha_k \fint_{(N')^k B^d(x,E^{2\delta}R')} u\,d\mu \lesssim \sum_{k=0}^\infty \alpha_k \fint_{(N')^{k+j_0}B^d(x,R')} u\,d\mu \le \sum_{k=0}^\infty \alpha'_k \fint_{(N')^k B^d(x,R')} u d\mu,$$

where $\alpha'_k := \alpha_{\max(k-j_0,0)}$. Analogous estimates hold with f^q and h^s in place of u. Finally, we set $\beta' := \beta/\delta$ so that (2.4) is satisfied in the metric space (X,d,μ) with $(\alpha'_k)_k, \beta', N'$ replacing $(\alpha_k)_k, \beta, N$ there. We also have control over the homogeneous dimension of (X,d,μ). Indeed, for $x \in X$ and $R > r$, we see that $ER > E^{-1}r$ and therefore

$$\frac{\mu(B^d(x,R^\delta))}{\mu(B^d(x,r^\delta))} \le \frac{\mu(B^\rho(x,ER))}{\mu(B^\rho(x,E^{-1}r))} \lesssim \left(\frac{E^2R}{r}\right)^D,$$

where D is a number satisfying (2.1) for (X,ρ,μ).

It follows that $D' = D\delta^{-1}$ satisfies (2.1) for (X,d,μ). As a consequence, $\beta' = \beta/\delta \ge D'(1/s - 1/q)$, and we can apply Theorem 2.2.

We obtain

$$\left(\fint_{B^d} u^p\,d\mu\right)^{1/p} \lesssim \sum_{k=0}^\infty \alpha'_k \fint_{(N')^{k+1}B^d} u\,d\mu + \left(\sum_{k=0}^\infty \alpha'_k \fint_{(N')^{k+1}B^d} f^q\,d\mu\right)^{1/q} + (R')^{\beta'}\left(\sum_{k=0}^\infty \alpha'_k \fint_{(N')^{k+1}B^d} h^s\,d\mu\right)^{1/s} + \left(\fint_{N'B^d} f^p\,d\mu\right)^{1/p} + (R')^{\beta'}\left(\fint_{N'B^d} h^{ps/q}\,d\mu\right)^{q/sp}$$

for all balls B^d with radius R'. Note that R' is arbitrary. Comparing the d-balls with ρ-balls once again, we see that $B^\rho(x,(E^{-1}r)^{1/\delta}) \subset B^d(x,r) \subset B^\rho(x,(Er)^{1/\delta})$. Arguing as in the beginning of the proof and denoting $R = (E^{-1}R')^{1/\delta}$, we can get back to an inequality in the quasi-metric space (X,ρ,μ): We only need to recall that $N' = N^\delta$, that j_1 is the smallest integer so that $E^{2/\delta} \le N^{j_1}$ and set $\alpha''_k := \alpha'_{\max(0,k-j_1)} = \alpha_{\max(0,k-j_0-j_1)}$. This together with the doubling condition implies that (3.4) holds. □

4. Variants

One might wonder whether one can use in the proof of Theorem 2.2 the fractional maximal operator $M^{\beta s}$ where

$$M^{\beta}v(x) := \sup_{B \ni x} r(B)^{\beta} \fint_B |v|, \qquad x \in X, \beta > 0,$$

to control the terms stemming from h^s more efficiently. (Here $r(B)$ denotes the radius of B.) However, this operator has no boundedness property in this generality and one has to assume *volume lower bound* in the following sense:

$$\exists Q > 0 \ : \ \forall \text{ balls } B, \quad \mu(B) \gtrsim r(B)^Q. \tag{4.1}$$

Lemma 4.2. *Let (X, ρ, μ) be a space of homogeneous type. Assume that the volume has a lower bound with exponent $Q > 0$. Then M^{β} is bounded from $L^p(X)$ to $L^{p^*}(X)$ when $1 < p$ and $0 < \beta < Q/p$ with $p^* = \frac{pQ}{Q - \beta p}$. For $p = 1$, it is weak type $(1, 1^*)$.*

Proof. See e.g. Section 2 in [15] for a simple proof on metric spaces with doubling measure that applies *verbatim* in the quasi-metric setting. In fact, the result follows from the inequality $M^{\beta}v(x) \lesssim Mv(x)^{1-\beta/Q} \|v\|_1^{\beta/Q}$ using the lower bound, the uncentered maximal function M and interpolation. □

We obtain the following variant in the presence of a volume lower bound.

Theorem 4.3. *Let (X, ρ, μ) be a space of homogeneous type having a volume lower bound with exponent Q. Let $s > 0$, $\beta \geq 0$ and $q > 1$ be such that $s < q$ and $\beta \leq Q(1/s - 1/q)$. Suppose that $u, f, h \geq 0$ with $u^q, f^q, h^s \in L^1_{loc}(X, d\mu)$ and (3.3) holds. Then there exists $p > q$ such that (3.4) holds with $R^{\beta}(\fint_{NE^{2/\delta}B} h^{ps/q} \, d\mu)^{q/sp}$ replaced by $\mu(B)^{\beta/Q}(\fint_{NE^{2/\delta}B} h^{p_*} \, d\mu)^{1/p_*}$ where $p_* = \frac{pQ}{Q + \beta p}$.*

Proof. It suffices to give a proof for (X, ρ, μ) a metric space with doubling measure. Then we can apply the general reduction argument from the previous section. In this regard, we note that if ρ is equivalent to $d^{1/\delta}$, then d has lower volume bound with exponent $Q' := Q/\delta$.

For any $p > q$ set $\sigma := \frac{pQ}{s(Q + \beta p)} = \frac{p_*}{s}$. Note the condition $\beta q s \leq Q(q - s)$ ensures for all $p > q$ the bound $\beta p s < Q(p - s)$. Hence $\sigma > 1$. Now we indicate the changes in the proof of Theorem 2.2.

One does not introduce the function g in Step 1 and the function G in Step 4 becomes $H(x, t) = r^{\beta}\big(\fint_{B(x,t)} h^s \, d\mu\big)^{1/s}$. The choice of λ_0 is similar and then we can follow the argument until we need to estimate II_3 in Step 6. Here we now have

$$\begin{aligned} II_3 &= C(p - q) \int_0^m \lambda^{p-1} \mu(\{M^{\beta s}(h^s \mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^s\}) \, d\lambda \\ &\lesssim \int_X (M^{\beta s}(h^s \mathbb{1}_{B_{\rho_0}}))^{p/s} \, d\mu \\ &\lesssim \left(\int_{NB} h^{s\sigma} \, d\mu\right)^{p/(s\sigma)} = \left(\int_{NB} h^{p_*} \, d\mu\right)^{p/p_*} \end{aligned}$$

by definition of σ and we used the $L^\sigma(X) \to L^{p/s}(X)$ boundedness of $M^{\beta s}$ from Lemma 4.2. Recall near then end of Theorem 2.2 we divide by $\mu(B)$ then take p-th roots. Thus, the power of $\mu(B)$ in front of $(\fint_{NB} h^{p_*}\,d\mu)^{1/p_*}$ comes from the equality

$$\mu(B)^{-1}\left(\int_B h^{p_*}\,d\mu\right)^{\frac{p}{p_*}} = \mu(B)^{\beta p/Q}\left(\fint_B h^{p_*}\,d\mu\right)^{\frac{p}{p_*}}.$$

□

Remark 4.4. Assume $\beta = 1$, $s = \frac{2n}{n+2}$ and $q = 2$ in the Euclidean space $\mathbb{R}^n$ with Lebesgue measure, which is typical of elliptic equations. Then the Lebesgue exponent for h in Theorem 3.2 is $\frac{ps}{q} = \frac{pn}{n+2}$ while above we get $p_* = \frac{pn}{n+p}$, which is smaller. If $\beta = 0$, then $p_* = p$. Of course the interest is to have β as large as possible so that p_* is as small as possible, but in applications to PDEs the value of β is usually not free to choose but determined by scaling arguments. Finally note that the admissible ranges for β in the two theorems are almost complementary in this example: Indeed, since $D = Q = n$, we have $\beta \geq n(1/s - 1/q)$ in Theorem 2.2 and $\beta \leq n(1/s - 1/q)$ in Theorem 3.2.

Another variant is to replace powers of the radius by powers of the volume already in the assumption and then no further hypothesis on the measure is required.

Theorem 4.5. *Let (X, ρ, μ) be a space of homogeneous type. Let $s > 0$, $\gamma \geq 0$ and $q > 1$ be such that $s < q$ and $\gamma \leq 1/s - 1/q$. Suppose that $u, f, h \geq 0$ with $u^q, f^q, h^s \in L^1_{loc}(X, d\mu)$ and (3.3) holds with $R^\beta(a_{h^s}(B))^{1/s}$ replaced by $\mu(B)^\gamma(a_{h^s}(B))^{1/s}$. Then there exists $p > q$ such that (3.4) holds with $R^\beta(\fint_{NE^{2/\delta}B} h^{ps/q}\,d\mu)^{q/sp}$ replaced with $\mu(B)^\gamma(\fint_{NE^{2/\delta}B} h^{s\sigma}\,d\mu)^{1/s\sigma}$, where $s\sigma = \frac{p}{1+\gamma p}$.*

Remark 4.6. Note that one can take $\gamma = 0$ in which case $s\sigma = p$. In accordance with Remark 4.4 we note that the higher γ the smaller the integrability needed on h.

Proof. Once again it suffices to treat the metric case. The modification to the proof of Theorem 2.2 are the same as in the above argument, except for now using instead of $M^{\beta s}$ the modified fractional maximal operator $\tilde{M}^{\gamma s}$, $0 \leq \gamma < 1$, where

$$\tilde{M}^{\gamma s}v(x) := \sup_{B \ni x}\,(\mu(B))^{\gamma s}\fint_B |v|, \qquad x \in X.$$

It maps $L^\sigma(X)$ into $L^{\frac{\sigma}{1-\gamma s\sigma}}(X)$ when $1 < \sigma$ and $\gamma s\sigma < 1$, see Remark 2.4 in [15]. □

5. Global integrability

A typical application of Gehring's lemma is to prove higher integrability locally and globally. To extract a conclusion at the level of global spaces $L^p(X)$, we need some further hypotheses. We say that the space of homogeneous type (X, ρ, μ) is *ϕ-regular* if it satisfies

$$\phi(r) \sim \mu(B(x, r))$$

for all $x \in X$ and $r > 0$, where $\phi : (0, \infty) \to (0, \infty)$ is a non-decreasing function with $\phi(r) > 0$ and $\phi(2r) \sim \phi(r)$ for $r > 0$. An important subclass of such spaces are the *Ahlfors–David regular* metric spaces where $\phi(r) = r^Q$ for some $Q > 0$. The case of local and global different dimensions which occur on connected nilpotent Lie groups (see [27]) is also covered with $\phi(r) \sim r^d$ for $r \leq 1$ and $\phi(r) \sim r^D$ for $r \geq 1$.

Theorem 5.1. *Let (X, ρ, μ) be a ϕ-regular space of homogeneous type. In addition to assumptions of Theorem 3.2, suppose that $u^q, f^q, h^s \in L^1(X, d\mu)$. Then*

$$\|u\|_{L^p(X)} \lesssim \|u\|_{L^q(X)} + \|f\|_{L^p(X)} + \|h\|_{L^{ps/q}(X)}$$

with the implicit constant depending on u, f, h only through the parameters quantified in the assumption.

Proof. For the sake of simplicity let us assume $N = 2$ in the statement of Theorem 3.2. We shall see in Section 6.2 below that upon changing the sequence $(\alpha_k)_k$ we can do so without loss of generality. Alternatively, we could also adapt the following argument to cover the general case.

Take any $R > 0$ and choose a maximal R separated set of points $\{x_i\}$, that is, $\rho(x_i, x_j) \geq R$ for all $i \neq j$ and for every $y \in X$ there exists x_i such that $\rho(y, x_i) < R$. Since we assume that X is doubling, such a collection necessarily has only finitely many members in any fixed ball, hence, it is countable. The balls $B_i := B(x_i, R)$ cover X, and there is C only depending on K and C_d such that

$$\sum_i \mathbb{1}_{B_i}(x) \leq C$$

for every $x \in X$. Also the balls $(2K)^{-1}B_i$ are disjoint. Further, we have

$$\sum_i \mathbb{1}_{2^k B_i}(x) \sim \frac{\phi(2^k R)}{\phi(R)} \tag{5.2}$$

for every $x \in X$ and every integer $k \geq 1$. Indeed, fix $k \geq 1$ and $x \in X$. We can assume that $2^{k-1} \geq K$, since otherwise we can just use that the left- and right-hand sides are comparable to constants depending only on K, C_d and ϕ. Let I_x be the set of i giving a non zero contribution, and N_x be the cardinal of I_x, that is, the value of the sum. Clearly, N_x is not exceeding the number of i for which $\rho(x, x_i) \leq 2^k R$. As the balls $(2K)^{-1}B_i$, $i \in I_x$, are disjoint and contained in $B(x, K(2^k + (2K)^{-1})R)$, we have

$$\phi(R/2K)N_x \lesssim \sum_{i \in I_x} \mu((2K)^{-1}B_i) \lesssim \mu(B(x, K(2^k + (2K)^{-1})R)) \lesssim \phi(K2^{k+1}R).$$

Also the balls B_i, $i \in I_x$, cover $B(x, (K^{-1}2^k - 1)R)$, hence

$$\begin{aligned}\phi(R/2K)N_x &\gtrsim \sum_{i \in I_x} \mu((2K)^{-1}B_i) \gtrsim \sum_{i \in I_x} \mu(B_i) \gtrsim \mu(B(x, (K^{-1}2^k - 1)R)) \\ &\gtrsim \phi((K^{-1}2^k - 1)R) \geq \phi(K^{-1}2^{k-1}R)\end{aligned}$$

by the assumption on k. The claim follows using the comparability $\phi(K2^{k+1}R) \sim \phi(K^{-1}2^{k-1}R) \sim \phi(2^k R)$ and $\phi(R/2K) \sim \phi(R)$.

Applying Theorem 3.2 in each B_i and denoting by $\tilde{\alpha}_k := \alpha_{\max(0,k-j_0-j_1)}$ the summable sequence appearing in the conclusion of that theorem, we see that

$$\phi(R)^{-1}\int_X u^p\,d\mu \lesssim \sum_i \fint_{B_i} u^p\,d\mu \lesssim \sum_i \left(\tilde{a}_u(2B_i)^p + \tilde{a}_{f^p}(2B_i) + R^\beta \tilde{a}_{h^{ps/q}}(2B_i)^{q/s}\right)$$
$$=: I + II + III,$$

where we used ϕ-regularity, Hölder's inequality $\tilde{a}_{f^q}(2B_i)^{1/q} \lesssim \tilde{a}_{f^p}(2B_i)^{1/p}$ and absorbed the term with f^p in (2.5) in $\tilde{a}_{f^p}(2B_i)$ and similarly for the terms with h. Let us treat III: From the continuous embedding $\ell^1(\mathbb{N}) \subset \ell^{q/s}(\mathbb{N})$ and (5.2) we obtain

$$\begin{aligned}
\sum_i \tilde{a}_{h^{ps/q}}(2B_i)^{q/s} &\le \sum_i \left(\sum_{k=0}^\infty \tilde{\alpha}_k \fint_{2^{k+1}B_i} h^{ps/q}\,d\mu\right)^{q/s} \\
&\le \left(\sum_i \sum_{k=0}^\infty \tilde{\alpha}_k \fint_{2^{k+1}B_i} h^{ps/q}\,d\mu\right)^{q/s} \\
&\le \left(\sum_{k=0}^\infty \tilde{\alpha}_k \sum_i \fint_{2^{k+1}B_i} h^{ps/q}\,d\mu\right)^{q/s} \\
&\lesssim \left(\sum_{k=0}^\infty \tilde{\alpha}_k\, \phi(2^{k+1}R)^{-1}\int_X \sum_i \mathbb{1}_{2^{k+1}B_i} h^{ps/q}\,d\mu\right)^{q/s} \\
&\lesssim \left(\tilde{\alpha}\phi(R)^{-1}\int_X h^{ps/q}\,d\mu\right)^{q/s},
\end{aligned}$$

where $\tilde{\alpha} := \sum_k \tilde{\alpha}_k$. Doing the same for I and II, implies

$$\|u\|_{L^p(X)} \lesssim \phi(R)^{1/p-1/q}\|u\|_{L^q(X)} + \|f\|_{L^p(X)} + R^\beta \phi(R)^{(1/p)-(q/sp)}\|h\|_{L^{ps/q}(X)}\,. \tag{5.3}$$

Note that since R is fixed ($R = 1$ for example), this concludes the proof. □

Remark 5.4. We note that the implicit constant in (5.3) does not depend on R. If $h = 0$, then we may let $R \to \infty$ as $p > q$ and obtain $\|u\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}$.

We have a global analogue of Theorem 4.3, which might be of independent interest as it can be proved directly in the quasi-metric setting without recursing to Section 3.

Theorem 5.5. *Let (X, d, μ) be a space of homogeneous type having volume lower bound with exponent Q. Let $s > 0$, $\beta \ge 0$ and $q > 1$ be such that $s < q$ and $\beta \le Q(1/s - 1/q)$. Suppose that $u^q, f^q, h^s \in L^1(X, d\mu)$ and that (3.3) holds. Then there exists $p > q$ such that*

$$\|u\|_{L^p(X)} \lesssim \|f\|_{L^p(X)} + \|h\|_{L^{p_*}(X)}\,,$$

where $p_* = \frac{pQ}{Q+\beta p}$. The implicit constant depends on u, f, h only through the parameters quantified in the assumption.

As for the choice of β, the same comments as in Remark 4.4 apply.

Proof. We indicate the modification to the argument of Theorem 2.2, which, as said, works directly in the quasi-metric setting for this result. This is basically the one in [3].

There is no need for the first and second steps and the proof begins as in Step 3 without the balls B_{r_0} and B_{ρ_0} and we have

$$\int_X u_m^{p-q} u^q \, d\mu = (p-q) \int_0^m \lambda^{p-q-1} u^q(\{u > \lambda\}) \, d\lambda.$$

There is also no need for a threshold λ_0 and we set for $x \in X$ and $r > 0$,

$$U(x,r) := a_u(B(x,r)), \quad F(x,r) := (a_{f^q}(B(x,r))^{1/q}, \quad H(x,r) := r^\beta (a_{h^s}(B(x,r)))^{1/s},$$

and for $\lambda > 0$, we denote $U_\lambda := \{u > \lambda\}$. Note that without loss of generality we may assume $\alpha_0 \geq 1$ right from the start as this only increases the right-hand side of our hypothesis (3.3). Thus,

$$\liminf_{r \to 0} \big(U(x,r) + F(x,r) + H(x,r) \big) \geq u(x)$$

for almost every x because already the first term in $a_u(B(x,r))$ tends to $\alpha_0 u(x) \geq u(x)$. We define $\tilde{U}_\lambda$ as the subset of U_λ where this holds. Note that

$$\lim_{r \to \infty} \big(U(x,r) + F(x,r) + H(x,r) \big) = 0$$

for all x using the global assumptions on u, f, h. For the term with h, this follows from $H(x,r)^s \lesssim r^{\beta s - Q} \int_X h^s \, d\mu$, provided $\beta s < Q$, which holds under our assumption. For $x \in \tilde{U}_\lambda$, we can define the stopping time radius

$$r_x := \sup\{r > 0 : U(x,r) + F(x,r) + H(x,r) > \lambda\}.$$

Remark that $\sup_{x \in \tilde{U}_\lambda} r_x < \infty$. Indeed, at $r = r_x$, $U(x,r) + F(x,r) + H(x,r) = \lambda$ and therefore either $U(x,r) \geq \lambda/3$ or $F(x,r) \geq \lambda/3$ or $H(x,r) \geq \lambda/3$. In the last case, we obtain $r^{Q-\beta s}(\lambda/3)^s \lesssim \int_X h^s \, d\mu < \infty$. The other cases also give us a bound on r. By the Vitali covering lemma, there exists a countable collection of balls $\{B(x_i, r_{x_i})\} = \{B_i\}$ such that $\frac{1}{V} B_i$ are pairwise disjoint and $\tilde{U}_\lambda \subset \cup_i B_i$. (Usually, $V = 5$ but our metric is only a quasi-metric in which case the Vitali covering lemma still holds but with a larger constant V depending on K, and we apply it to the covering $\tilde{U}_\lambda \subset \cup_{x \in \tilde{U}_\lambda} B(x, V^{-1} r_x)$. A direct way to see this is by the technique in Section 3.) Now, using the hypothesis for each B_i and pairwise disjointness of the balls $\frac{1}{V} B_i$,

$$\begin{aligned} u^q(\tilde{U}_\lambda) &\leq \sum_i u^q(B_i) \leq \sum_i \mu(B_i) \big(A a_u(B_i) + (a_{f^q}(B_i))^{1/q} + r_i^\beta (a_{h^s}(B_i))^{1/s} \big)^q \\ &= \sum_i \mu(B_i) \lambda^q \lesssim V^D \sum_i \mu(\tfrac{1}{V} B_i) \lambda^q \lesssim V^D \mu(\cup_i B_i) \, \lambda^q, \end{aligned}$$

where D is the homogeneous dimension. The stopping time implies

$$\cup_i B_i \subset \{Mu \geq \lambda/3\} \cup \{M(f^q) \geq (\lambda/3)^q\} \cup \{M^{\beta s}(h^s) \geq (\lambda/3)^s\}. \tag{5.6}$$

From there the estimates are as in Step 6 and we use Lemma 4.2 for boundedness of $M^{\beta s}$. We obtain

$$\int_X u_m^{p-q} u^q \, d\mu \leq C(p-q) \int_X u_m^{p-1} u \, d\mu + C_p \int_X f^p \, d\mu + C_p \left(\int_X h^{p_*} \, d\mu \right)^{p/p_*}, \tag{5.7}$$

with p_* as in the statement. As $\int_X u_m^{p-1} u \, d\mu \leq \int_X u_m^{p-q} u^q \, d\mu$ we can hide this term if $p - q > 0$ is small enough and then let $m \to \infty$. □

6. Self-improvement of the right hand side

We discuss here the change in the exponents in the tails on the right hand side and subsequently the change of the dilation parameter N. Both induce change in the sequence α. These remarks can be used to reduce some seemingly different properties to cases covered by Theorem 3.2.

6.1. **Exponent.** It is a direct consequence of the log-convexity of the L^p norms that if

$$\left(\fint_B u^p\,d\mu\right)^{1/p} \lesssim \left(\fint_B u^q\,d\mu\right)^{1/q}$$

with $p > q$, then for every $s \in (0,q)$ we can write $1/q = \theta/p + (1-\theta)/s$ for some $\theta \in (0,1)$ and consequently

$$\|u\|_{L^p(B,\nu)} \lesssim \|u\|_{L^q(B,\nu)} \le \|u\|^{(1-\theta)}_{L^s(B,\nu)} \|u\|^{\theta}_{L^p(B,\nu)}$$

so that $\|u\|_{L^p(B,\nu)} \lesssim \|u\|_{L^s(B,\nu)}$. Here $\nu = d\mu/\mu(B)$. The same self-improving property holds true for the weak reverse Hölder inequality [18] and even for the reverse Hölder inequality with tails as we now show. To prove the claim for the inequality with tails, we use a modification of the argument from [6], Appendix B.

Proposition 6.1. *Let (X,ρ,μ) be a space of homogeneous type. Let $q \in (0,p)$, $s_0, s_1, s_2 \in (0,q]$ with $f^{s_1}, h^{s_2} \in L^1_{loc}(X)$ and set $\tau := \min\left(\frac{s_0}{q}, \frac{s_1}{q}, \frac{s_2}{q}\right)$. Let $(\alpha_k)_{k\ge0}$ be a summable sequence of strictly positive numbers. Let $N > 1$ and $\beta \ge 0$. Let $(\tilde\alpha_k)_{k\ge0}, (\alpha^\sharp_k)_{k\ge0}$ be summable sequences of non negative numbers with $\tilde\alpha_0, \alpha^\sharp_0 > 0$ and assume*

$$\sum_{k=0}^{m} \tilde\alpha_k \alpha^{\tau}_{m-k} \lesssim \tilde\alpha_m, \qquad \sum_{k=0}^{m} \alpha^\sharp_k \alpha_{m-k} \lesssim \alpha^\sharp_m \tag{6.2}$$

and

$$\sum_{k=0}^{m} \tilde\alpha_k \alpha^{s_2/q}_{m-k} N^{(m-k)\beta s_2} \lesssim \tilde\alpha_m. \tag{6.3}$$

Define $a_u(B), \tilde a_u(B), a^\sharp_u(B)$ as in (2.3) in terms of the three respective sequences, for $u \ge 0$ locally integrable, $N > 1$ and B a quasi-metric ball.

Assume that

$$\left(\fint_B u^p\,d\mu\right)^{1/p} \lesssim (a_{u^q}(B))^{1/q} + b(B), \tag{6.4}$$

where the implicit constant does not depend on B, with

$$b(B) = (a_{f^{s_1}}(B))^{1/s_1} + r(B)^{\beta}(a_{h^{s_2}}(B))^{1/s_2}.$$

Then, for any ball B for which $a^\sharp_{u^q}(B) < \infty$, one has

$$\left(\fint_B u^p\,d\mu\right)^{1/p} \lesssim (\tilde a_{u^{s_0}}(B))^{1/s_0} + \tilde b(B), \tag{6.5}$$

where $\tilde b$ is obtained by replacing α by $\tilde\alpha$ in the definition of b.

Remark 6.6. Note that, in contrast with the improvement of integrability, we do not need the non-increasing assumption on the sequence α for this proposition. The condition (6.2) together with $\alpha_0^\sharp > 0$ implies $\alpha_m \lesssim \alpha_m^\sharp$ and similarly, since $\tau \le 1$, we have $\alpha_m \lesssim \tilde{\alpha}_m$. Hence we have to assume more than $a_{u^q}(B) < \infty$. For example, with τ as above, if $\alpha_k = N^{-\gamma k}$ for $\gamma > 0$, then $\tilde{\alpha}_k = N^{-\gamma' \tau k}$ and $\alpha_k^\sharp = N^{-\gamma' k}$ work in the theorem for any $0 < \gamma' < \gamma$ such that $\beta s_2 < \gamma s_2/q - \gamma' \tau$. In particular, decay $\gamma > \beta q$ is needed to obtain any improvement, which typically is hard to obtain in applications. On the other hand, if $\beta = 0$, we can improve the right-hand exponent by only paying an arbitrarily small amount of decay to replace γ by $\gamma' < \gamma$. The condition (6.3) takes into account the presence of $r(B)^\beta$ in (6.4). Finally, the strict positivity of α_k rules out in particular the case where the α_k form a finite sequence, but in that case, the argument in [6] already covers the situation.

Proof. Define

$$K(\delta, s_0) := \sup \frac{(a_{u^q}(B))^{1/q}}{(\tilde{a}_{u^{s_0}}(B))^{1/s_0} + \tilde{b}(B) + \delta(a^\sharp_{u^q}(B))^{1/q}},$$

where the supremum is taken on the set of balls B such that the denominator is finite. Indeed, there is nothing to prove if the right hand side of (6.5) is infinite, which is equivalent to the denominator being infinite since we assume $a^\sharp_{u^q}(B) < \infty$. As $\alpha_m \lesssim \alpha_m^\sharp$, we have $a_{u^q}(B) \lesssim a^\sharp_{u^q}(B)$ and the presence of $\delta > 0$ guarantees that $K(\delta, s_0) \lesssim \delta^{-1}$. We show a uniform bound in terms of δ. To this end we can of course assume $K(\delta, s_0) \ge 1$ since otherwise there is nothing to prove.

Fix a ball B with the above restriction. Let $\theta \in (0, 1)$ be such that

$$\frac{1}{q} = \frac{\theta}{s_0} + \frac{1-\theta}{p}.$$

We see that

$$\left(\fint_B u^q \, d\mu \right)^{1/q} \le \left(\fint_B u^{s_0} \, d\mu \right)^{\theta/s_0} \left(\fint_B u^p \, d\mu \right)^{(1-\theta)/p}.$$

Using (6.4), $b(B) \lesssim \tilde{b}(B)$ and $K(\delta, s_0) \ge 1$,

$$\begin{aligned}
\left(\fint_B u^q \, d\mu \right)^{1/q} &\lesssim \left(\fint_B u^{s_0} \, d\mu \right)^{\theta/s_0} \left((a_{u^q}(B))^{1/q} + b(B) \right)^{1-\theta} \\
&\lesssim \left(\fint_B u^{s_0} \, d\mu \right)^{\theta/s_0} K(\delta, s_0)^{1-\theta} \left((\tilde{a}_{u^{s_0}}(B))^{1/s_0} + \tilde{b}(B) + \delta(a^\sharp_{u^q}(B))^{1/q} \right)^{1-\theta} \\
&\lesssim K(\delta, s_0)^{1-\theta} \left((\tilde{a}_{u^{s_0}}(B))^{1/s_0} + \tilde{b}(B) + \delta(a^\sharp_{u^q}(B))^{1/q} \right).
\end{aligned}$$

We apply this last inequality to $N^k B$. This is possible provided $N^k B$ belongs to the same set of balls and this follows from the assumption on the sequences: For example, using (6.2),

$$a^\sharp_u(N^k B) = \sum_{j=k}^\infty \alpha^\sharp_{j-k} \fint_{N^j B} u^q \, d\mu \lesssim \alpha_k^{-1} \sum_{j=k}^\infty \alpha^\sharp_j \fint_{N^j B} u^q \, d\mu \le \alpha_k^{-1} a^\sharp_{u^q}(B) < \infty.$$

Similar calculations can be done for the other terms. Thus,

$$\begin{aligned} a_{u^q}(B)^{1/q} = \left(\sum_{k=0}^{\infty} \alpha_k \fint_{N^k B} u^q \, d\mu\right)^{1/q} &\lesssim K(\delta, s_0)^{(1-\theta)} \left(\sum_{k=0}^{\infty} \alpha_k \tilde{a}_{u^{s_0}}(N^k B)^{q/s_0}\right)^{1/q} \\ &\quad + K(\delta, s_0)^{(1-\theta)} \left(\sum_{k=0}^{\infty} \alpha_k \tilde{b}(N^k B)^q\right)^{1/q} \\ &\quad + K(\delta, s_0)^{(1-\theta)} \delta \left(\sum_{k=0}^{\infty} \alpha_k a^\sharp_{u^q}(N^k B)\right)^{1/q} \\ &=: I + II + III. \end{aligned}$$

Each of the three sums is estimated similarly so we restrict our attention to the first one. Using the continuous embedding $\ell^1(\mathbb{N}) \subset \ell^{q/s_0}(\mathbb{N})$ and the properties of α in (6.2), we compute

$$\begin{aligned} I^{s_0} \leq \sum_{k=0}^{\infty} \alpha_k^{s_0/q} \tilde{a}_{u^{s_0}}(N^k B) &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \alpha_k^{s_0/q} \tilde{\alpha}_{m-k}\right) \fint_{N^m B} u^{s_0} \, d\mu \\ &\lesssim \sum_{m=0}^{\infty} \tilde{\alpha}_m \fint_{N^m B} u^{s_0} \, d\mu = \tilde{a}_{u^{s_0}}(B). \end{aligned}$$

The same kind of argument applies to the remaining two terms so that

$$a_{u^q}(B)^{1/q} \lesssim K(\delta, s_0)^{1-\theta} \big((\tilde{a}_{u^{s_0}}(B))^{1/s_0} + \tilde{b}(B) + \delta (a^\sharp_{u^q}(B))^{1/q} \big). \tag{6.7}$$

We remark that it is the part of $\tilde{b}$ involving $r(B)^\beta$ that requires us to use the strong condition (6.3). As the right hand side is finite, we readily obtain $K(\delta, s_0) \lesssim K(\delta, s_0)^{1-\theta}$, therefore $K(\delta, s_0) \lesssim 1$. Now, all the bounds are independent of δ, so we may send $\delta \to 0$ in (6.7). Plugging this inequality into (6.4) concludes the proof of (6.5). □

6.2. **Dilation.** Another direction to which the reverse Hölder inequalities self-improve is the dilation parameter on the right hand side of

$$\left(\fint_B u^p \, d\mu\right)^{1/p} \lesssim \left(\fint_{NB} u^q \, d\mu\right)^{1/q}.$$

Indeed, if such an inequality holds in a space of homogeneous type, then the similar inequality

$$\left(\fint_B u^p \, d\mu\right)^{1/p} \lesssim \left(\fint_{CB} u^q \, d\mu\right)^{1/q}$$

holds for all balls with any $C > K$ where K is the quasi-metric constant. See for instance Theorem 3.15 in [2]. The proof of this fact is based on a covering of B by small balls whose N-dilates are still contained in CB and applying the weak reverse Hölder inequality in each small ball individually.

It is worth a remark that a change of geometry similar to the property just described can be carried out with the reverse Hölder inequality with tails. To formulate this technical remark, we introduce some notation. Given a sequence of positive numbers $\alpha = (\alpha_k)_{k\geq 0}$ and real numbers with $1 < n \leq m$, we define the (m,n)-*stretch* $S^{m,n}\alpha$ by

$$(S^{m,n}\alpha)_j := \alpha_k, \quad j \geq 0 : \ m^{k-1} < n^j \leq m^k.$$

We also define the (m,n)-*regrouping* by

$$(R^{m,n}\alpha)_k := \sum_{j: m^{k-1} < n^j \leq m^k} \alpha_j \ + \beta_k, \quad k \geq 0,$$

where β_k is a correction term. It makes each $(R^{m,n}\alpha)_k$ to be the sum of equally many terms and hence the regrouping of a non-increasing sequence remains non-increasing: The intervals

$$\left((k-1)\frac{\ln m}{\ln n}, k\frac{\ln m}{\ln n}\right]$$

contain ℓ or $\ell + 1$ integers when ℓ is the integer such that

$$\ell < \frac{\ln m}{\ln n} \leq \ell + 1.$$

We set $\beta_k = 0$ if $\sum_{j: m^{k-1} < n^j \leq m^k} 1 = \ell + 1$ and $\beta_k = \alpha_{\min\{j: n^j > m^{k-1}\}}$ otherwise.

For example, for $\gamma > 0$, the (m,n)-stretch of $(m^{-\gamma k})$ is (term-wise) comparable to $(n^{-\gamma k})$, and the (m,n)-regrouping of $(n^{-\gamma k})$ is (term-wise) comparable to $(m^{-\gamma k})$. More generally, if α is summable, so are its stretch and regrouping. For the latter, it is obvious and for the former, is follows from bounding the number of possible repetitions by $1 + \frac{\ln m}{\ln n}$. In addition, if α is non-increasing so are its (m,n)-stretch (m,n)-regrouping.

Proposition 6.8. *Let (X, ρ, μ) be a space of homogeneous type and let $(\alpha_k)_{k\geq 0}$ be a summable sequence of positive numbers. For $u \in L^1_{loc}(X)$, $u \geq 0$, $N > 1$, define*

$$a_u(B) := \sum_{k=0}^{\infty} \alpha_k \fint_{N^k B} u \, d\mu.$$

Then for any $M > 1$, one has

$$a_u(B) \lesssim \sum_{k=0}^{\infty} \beta_k \fint_{M^k B} u \, d\mu,$$

with, if $M > N$, $\beta = R^{M,N}\alpha$ and if $M < N$, $\beta = S^{M^\ell,M} R^{M^\ell,N}\alpha$ where ℓ is the least integer to satisfy $\ell \geq \ln N / \ln M$.

Proof. We start with $M > N$. Then,

$$\sum_{k=0}^{\infty} \alpha_k \fint_{N^k B} u \, d\mu = \sum_{k=0}^{\infty} \sum_{j: M^{k-1} < N^j \leq M^k} \alpha_j \fint_{N^j B} u \, d\mu \lesssim \sum_{k=0}^{\infty} \sum_{j: M^{k-1} < N^j \leq M^k} \alpha_j \fint_{M^k B} u \, d\mu$$
$$\leq \sum_{k=0}^{\infty} (R^{M,N}\alpha)_k \fint_{M^k B} u \, d\mu$$

as claimed, using the doubling condition.

Let now $M < N$. Assume first that there is an integer $\ell \geq 2$ such that $M^\ell = N$. Then we can write

$$\sum_{k=0}^{\infty} \alpha_k \fint_{N^k B} u\, d\mu = \sum_{k=0}^{\infty} \alpha_k \fint_{M^{\ell k} B} u\, d\mu \leq \sum_{j=0}^{\infty} (S^{M^\ell, M} \alpha)_j \fint_{M^j B} u\, d\mu.$$

In general, we can find an integer $\ell \geq 2$ so that $M^{\ell-1} < N \leq M^\ell$ so that by the previous case ($M > N$)

$$\sum_{k=0}^{\infty} \alpha_k \fint_{N^k B} u\, d\mu \lesssim \sum_{k=0}^{\infty} (R^{M^\ell, N} \alpha)_k \fint_{M^{\ell k} B} u\, d\mu \leq \sum_{k=0}^{\infty} (S^{M^\ell, M} R^{M^\ell, N} \alpha)_j \fint_{M^j B} u\, d\mu.$$

□

7. Extensions

There are several ways to further generalize the Gehring lemma with tails that follow by the argument used in the proof of Theorem 2.2. For the sake of clear exposition, we have not included them in the main theorem, but we briefly discuss some of them in this separate section. For simplicity we work in the metric situation (but quasi-metric works the same).

7.1. **Sequences.** We usually asked the sequence α_k in the definition of a_u to be non-increasing. Of course, this assumption can always be relaxed by asking the sequence to be non-increasing starting from a certain index k_0 and then replacing the terms α_k with $0 \leq k \leq k_0$ with $\alpha'_k := \max_{0 \leq k \leq k_0} \alpha_k$. The resulting sequence with $\alpha'_k := \alpha_k$ for $k > k_0$ is always non-increasing and summable.

7.2. **Maximal function.** The functional a_u can also take the form

$$m_u^{\Omega, loc}(B(x,t)) = \sup_{r \in [t, (1/2)\operatorname{dist}(x, \Omega^c))} \fint_{B(x,r)} u\, d\mu$$

where $\Omega \subset X$ is an open set. In other words, the supremum is over "large" balls B so that $2B \subset \Omega$. We also define

$$m_u^{\Omega}(B(x,t)) = \sup_{r \in [t, (3/4)\operatorname{dist}(x, \Omega^c))} \fint_{B(x,r)} u\, d\mu.$$

Corollary 7.1. *Let $\Omega \subset X$ be an open set in a metric space (X, d, μ) with doubling measure. Let $s, \beta > 0$ and $q > 1$ be such that $s < q$ and $\beta \geq D(1/s - 1/q)$ where D is any number satisfying (2.1). Suppose that $u, f, h \geq 0$ with $u^q, f^q, h^s \in L^1_{loc}(\Omega, d\mu)$ and $A \geq 0$ is a constant such that for every ball $B = B(x, R)$ with $2B \subset \Omega$*

$$\left(\fint_B u^q\, d\mu \right)^{1/q} \leq A m_u^{\Omega, loc}(B) + (m_{f^q}^{\Omega, loc}(B))^{1/q} + R^\beta (m_{h^s}^{\Omega, loc}(B))^{1/s}. \tag{7.2}$$

Then there exists $p > q$ such that for all balls B with $12B \subset \Omega$,

$$\begin{aligned} \left(\fint_B u^p\, d\mu \right)^{1/p} &\lesssim m_u^{\Omega}(B) + (m_{f^q}^{\Omega}(B))^{1/q} + R^\beta (m_{h^s}^{\Omega}(B))^{1/s} \\ &\quad + \left(\fint_{2B} f^p\, d\mu \right)^{1/p} + R^\beta \left(\fint_{2B} h^{ps/q}\, d\mu \right)^{q/sp}, \end{aligned} \tag{7.3}$$

where both p and the implicit constant depend on A, D, s, q, β.

Proof. We prove the claim for $B = B(x_0, R)$ with $x_0 \in X$, $R > 0$ and $12B \subset \Omega$. We point out the relevant changes to the proof of Theorem 2.2. Having fixed B, we repeat *Step 1* as before (we take $N = 2$) to define $g^q = A_R^q h^s \mathbb{1}_{2B}$ with A_R a constant so that for any ball B_r contained in $2B$ we have

$$r^\beta \left(\fint_{B_r} h^s \, d\mu \right)^{1/s} \lesssim \left(\fint_{B_r} g^q \, d\mu \right)^{1/q} \lesssim R^\beta \left(\fint_{2B} h^s \right)^{1/s}.$$

In *Step 2*, fix r_0 and ρ_0 real numbers satisfying $R \le r_0 < \rho_0 \le 2R$. For $x \in B_{r_0} := B(x_0, r_0)$, we have that

$$B(x, \rho_0 - r_0) \subset B(x_0, 2R) \subset B(x, 4R),$$

and consequently for any positive function v,

$$\begin{aligned} \fint_{B(x,(\rho_0 - r_0))} v \, d\mu &\le \frac{\mu(B(x_0, 2R))}{\mu(B(x, \rho_0 - r_0))} \fint_{B(x_0,2R)} v \, d\mu \\ &\lesssim \left(\frac{R}{\rho_0 - r_0} \right)^D \fint_{B(x_0,2R)} v \, d\mu, \end{aligned} \tag{7.4}$$

where we used the constant D from the doubling dimension in the last line. Set $\gamma := (R/(\rho_0 - r_0))^D$.

We repeat *Step 3* as it is. In *Step 4*, we define three functions

$$U(x,r) := \fint_{B(x,r)} u \, d\mu, \quad F(x,r) := \left(\fint_{B(x,r)} f^q \, d\mu \right)^{1/q}, G(x,r) := \left(\fint_{B(x,r)} g^q \, d\mu \right)^{1/q},$$

and for $\lambda > \lambda_0$, we denote the relevant level sets by

$$U_\lambda := B_{r_0} \cap \{u > \lambda\}, \quad F_\lambda := B_{r_0} \cap \{f > \lambda\}, \quad G_\lambda := B_{r_0} \cap \{g > \lambda\}.$$

We set

$$\lambda_0 := C\gamma m_u^\Omega(2B) + C \left(\gamma m_{f^q}^\Omega(2B) \right)^{1/q} + C(2R)^\beta \left(\gamma m_{h^s}^\Omega(2B) \right)^{1/s},$$

where C is a constant independent of u and the ball B, chosen such that, by an inclusion relation as in (7.4) we obtain

$$U(x, \rho_0 - r_0) + F(x, \rho_0 - r_0) + G(x, \rho_0 - r_0) \le \lambda_0 \tag{7.5}$$

for all $x \in B(x_0, \rho_0)$. Finally, we define as before

$$\Omega_\lambda := \left\{ x \in U_\lambda \cup F_\lambda \cup G_\lambda : x \text{ is a Lebesgue point for } u, f^q \text{ and } g^q \right\}.$$

In *Step 5*, we note, as before, that if $x \in \Omega_\lambda$ then

$$\lim_{r \to 0} U(x,r) + F(x,r) + G(x,r) > \lambda,$$

and thus for $x \in \Omega_\lambda$ we can define the stopping time radius, this time continuously, as

$$r_x := \sup \left\{ r < \rho_0 - r_0 : \ U(x,r) + F(x,r) + G(x,r) > \lambda \right\}.$$

Remark that (7.5) implies that $r_x < \rho_0 - r_0$. Of course $\Omega_\lambda \subset \cup_{x \in \Omega_\lambda} B(x, r_x/5)$. By the Vitali Covering Lemma (5r-Covering Lemma) there exists a countable collection of balls $\{B(x_i, r_i)\} = \{B_i\}$ with $r_i = r_{x_i}$ such that $\{\frac{1}{5}B_i\}$ are pairwise disjoint and $\mathcal{G}_\lambda \subset \cup_i B_i$.

We make three observations:

(i) For each i, either $\fint_{B_i} u\,d\mu \geq \frac{\lambda}{3}$, $(\fint_{B_i} f^q\,d\mu)^{1/q} \geq \frac{\lambda}{3}$, or $(\fint_{B_i} g^q\,d\mu)^{1/q} \geq \frac{\lambda}{3}$.

(ii) The radius of each B_i is less than $\rho_0 - r_0$ and $x_i \in B(x_0, r_0)$ so $B_i \subset B(x_0, \rho_0)$.

(iii) Each $r \in [r_i, \rho_0 - r_0)$ is 'above' or at the stopping time and

$$\fint_{B(x_i,r)} u\,d\mu + \left(\fint_{B(x_i,r)} f^q\,d\mu\right)^{1/q} + \left(\fint_{B(x_i,r)} g^q\,d\mu\right)^{1/q} \lesssim \lambda.$$

We obtain from (7.2) that

$$\begin{aligned} u^q(U_\lambda) &\leq u^q(U_\lambda \cup F_\lambda \cup G_\lambda) \leq \sum_i u^q(B_i) \\ &\lesssim \sum_i \mu(B_i)\big(m_u^{\Omega,loc}(B_i) + (m_{f^q}^{\Omega,loc}(B_i))^{1/q} + r_i^\beta (m_{h^s}^{\Omega,loc}(B_i))^{1/s}\big)^q. \end{aligned}$$

We handle the term involving f, to begin we split $m_{f^q}^{\Omega,loc}(B_i)$ as

(7.6)
$$\begin{aligned} m_{f^q}^{\Omega,loc}(B_i) &= \sup_{r\in[r_i, \frac{1}{2}\mathrm{dist}\,(x_i,\Omega^c))} \fint_{B(x_i,r)} f^q\,d\mu \\ &= \max\left(\sup_{r\in[r_i,\rho_0-r_0)} \fint_{B(x_i,r)} f^q\,d\mu, \sup_{r\in[\rho_0-r_0,\frac{1}{2}\mathrm{dist}\,(x_i,\Omega^c))} \fint_{B(x_i,r)} f^q\,d\mu \right). \end{aligned}$$

By observation (iii), we see that

$$\sup_{r\in[r_i,\rho_0-r_0)} \fint_{B(x_i,r)} f^q\,d\mu \lesssim \lambda.$$

On the other hand,

$$\begin{aligned} \sup_{r\in[\rho_0-r_0,\frac{1}{2}\mathrm{dist}\,(x_i,\Omega^c))} \fint_{B(x_i,r)} f^q\,d\mu &= \sup_{k\in[1,\frac{1}{2(\rho_0-r_0)}\mathrm{dist}\,(x_i,\Omega^c))} \fint_{B(x_i,k(\rho_0-r_0))} f^q\,d\mu \\ \lesssim \sup_{k\in[1,\frac{1}{2(\rho_0-r_0)}\mathrm{dist}\,(x_i,\Omega^c))} &\left(\frac{k(\rho_0-r_0)+r_0}{k(\rho_0-r_0)}\right)^D \fint_{B(x_0,k(\rho_0-r_0)+r_0)} f^q\,d\mu \\ \lesssim \left(\frac{R}{\rho_0-r_0}\right)^D & m_{f^q}^{\Omega}(B) \lesssim \lambda_0^q < \lambda^q, \end{aligned}$$

where the last line is justified as follows: By the upper bound on k in the supremum, we always have

$$R \leq k(\rho_0 - r_0) + r_0 \leq \frac{1}{2}\mathrm{dist}\,(x_i, \Omega^c) + r_0 \leq \frac{1}{2}\mathrm{dist}\,(x_0, \Omega^c) + 3R < \frac{3}{4}\mathrm{dist}\,(x_0, \Omega^c),$$

where we used $|x_i - x| < r_0 < \rho_0 \leq 2R$ and $B(x_0, 12R) \subset \Omega$. Hence every $B(x_0, k(\rho_0 - r_0) + r_0)$ is admissible in the definition of $m_{f^q}^{\Omega}(B)$ and we get the bound claimed before since $m_{f^q}^{\Omega}(B) \leq 2^D m_{f^q}^{\Omega}(2B)$. Altogether,

$$(m_{f^q}^{\Omega,loc}(B_i))^{1/q} \lesssim \lambda.$$

Terms with u and h are estimated similarly. The rest of *Step 5* follows as before and we obtain

$$\cup_i B_i \subset \{M(u\mathbb{1}_{B_{\rho_0}}) > \lambda/3\} \cup \{M(f^q\mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^q\} \cup \{M(g^q\mathbb{1}_{B_{\rho_0}}) > (\lambda/3)^q\}.$$

Step 6 involving maximal function arguments to estimate the measure of the set in the above display for $\lambda > \lambda_0$ as well as the overall contribution for $\lambda < \lambda_0$ is repeated without changes. In the end, we reach an inequality of the form (2.21). Indeed, set

$$\varphi(t) := \int_{B(x_0,t)} u_m^{p-q} u^q \, d\mu, \quad \alpha_p := \left(m_u^\Omega(2B) + (m_{f^q}^\Omega(2B))^{1/q} + (R)^\beta (m_{h^s}^\Omega(2B))^{1/s}\right)^p$$

and for $p \in (q, 2q)$ we may summarize our estimates as

$$\varphi(r_0) \lesssim \mu(B)\left(\frac{R}{\rho_0 - r_0}\right)^\eta \alpha_p + \epsilon_p \varphi(\rho_0) + \varepsilon_p^{-1}\int_{2B} f^p \, d\mu + \varepsilon_p^{-1}\int_{2B} g^p \, d\mu,$$

whenever $R \le r_0 < \rho_0 \le 2R$. Here $\epsilon_p = p - q$ and $\eta > 0$ is independent of u and B. The claim (7.3) then follows from a well known iteration argument (see e.g. Lemma 6.1 in [13]) or from modifying the argument in *Step 7*. □

Note that the proof for the maximal-function-like object $m_u^{\Omega,loc}$ is actually simpler than for the tailed a_u. Several choices of how to discretize the scale parameters can be omitted. This setup is also very close but not comparable to Gehring's original assumption

$$(Mu^q)^{1/q} \lesssim Mu$$

where $q > 1$ and M the Hardy–Littlewood maximal operator. Indeed, the left-hand side here does not have a maximal function and the right hand side is a maximal function restricted to large scales (a non-local maximal function).

7.3. **Domains.** We can define the tail functional $a_u^{\Omega,loc}$ restricted to an open set Ω, for example

$$a_u^{\Omega,loc}(B) := \sum_{\substack{k\ge 0 \\ 2^{k+4}B\subset\Omega}} \alpha_k \fint_{2^k B} u \, d\mu$$

and

$$a_u^{\Omega}(B) := \sum_{\substack{k\ge 0 \\ 2^{k+1}B\subset\Omega}} \alpha_k \fint_{2^k B} u \, d\mu,$$

where as before $(\alpha_k)_k$ is a non-increasing and summable sequence of positive numbers. Then we can localize the assumptions of Theorem 2.2 to Ω.

Corollary 7.7. *Let $\Omega \subset X$ be an open set in a metric space (X, d, μ) with doubling measure. Let $s, \beta > 0$ and $q > 1$ be such that $s < q$ and $\beta \ge D(1/s - 1/q)$ where D is any number satisfying (2.1). Suppose that $u, f, h \ge 0$ with $u^q, f^q, h^s \in L^1_{loc}(\Omega, d\mu)$ and $A \ge 0$ is a constant such that for every ball $B = B(x, R)$ with $16B \subset \Omega$*

$$\left(\fint_B u^q \, d\mu\right)^{1/q} \le A a_u^{\Omega,loc}(B) + (a_{f^q}^{\Omega,loc}(B))^{1/q} + R^\beta (a_{h^s}^{\Omega,loc}(B))^{1/s}. \tag{7.8}$$

Then there exists $p > q$ such that for all balls B with $32B \subset \Omega$,

$$\left(\fint_B u^p \, d\mu\right)^{1/p} \lesssim a_u^\Omega(4B) + (a_{f^q}^\Omega(4B))^{1/q} + R^\beta (a_{h^s}^\Omega(4B))^{1/s} + \left(\fint_{4B} f^p \, d\mu\right)^{1/p} + R^\beta \left(\fint_{4B} h^{ps/q} \, d\mu\right)^{q/sp}, \tag{7.9}$$

where both p and the implicit constant depend on A, β, s, q, D.

Proof. Theorem 2.2 shows how to deal with the tail. Corollary 7.1 shows how to adapt the proof to the setting relative to Ω. The proof of this Corollary can be reconstructed following the proof of Theorem 2.2 and carefully adapting the estimation in (2.15) in the spirit of estimating (7.6) to make sure that all relevant balls appearing in the estimates are contained in Ω. □

7.4. **Convolutions.** In the Euclidean setting where (X, d, μ) is $\mathbb{R}^n$ equipped with the usual distance and the Lebesgue measure, we can realize the functionals a_u as convolutions

$$a_u(B(x, r)) = (\varphi_r * u)(x)$$

where φ has suitable decay and integrability and $\varphi_r(x) = r^{-n}\varphi(x/r)$. More precisely, our assumptions correspond to φ being bounded, radial, decreasing and globally integrable. A convolution makes sense in certain groups, so this kind of special functional can also be considered, for instance, in nilpotent Lie groups as in [27].

8. Very weak A_∞ weights

For a weight (that is, a non-negative locally integrable function), the condition

$$\int_B M(\mathbb{1}_B w) \, d\mu \le C \int_B w \, d\mu$$

valid for some $C < \infty$ and all balls B of X can be taken as a definition of the A_∞ class, where M is the uncentered maximal operator, see [9, 28] for the Euclidean case with Lebesgue measure. In spaces of homogeneous type, this condition implies higher integrability with an exponent that can be computed from the constant C and the structural constants of X, see [16]. This was extended in [2] to weights in the weak A_∞ class defined by

$$\fint_B M(\mathbb{1}_B w) \, d\mu \le C \fint_{\sigma B} w \, d\mu. \tag{8.1}$$

where $\sigma > 1$ is given. The classes are shown to be independent of σ provided $\sigma > K$, K being the quasi-metric constant, and their elements still have a higher integrability. The methods passing through a dyadic analog yield an accurate estimate of the exponent in terms of the best C in the definition. We note that the dilation parameter σ is uniform: it is the same for all balls. Our methods allow us to remove the uniformity, that is we define the *very weak A_∞ class* as the set of weights such that for all balls,

$$\fint_B M(\mathbb{1}_B w) \, d\mu \le C \sup_{\sigma \ge 1} \fint_{\sigma B} w \, d\mu < \infty. \tag{8.2}$$

The quantity in the middle is the same functional as the one defined in Section 7.2 when $\Omega = X$.

We denote by A_∞^{vw} this class. As the right hand side of (8.2) requires boundedness of all averages on large balls, this rules out weights growing at ∞. For this reason, it is neither contained in, nor containing the class A_∞^{weak} introduced in [2]. Typically, such very weak A_∞ weights arise from fractional equations. See the next section.

Theorem 8.3. *For any very weak A_∞ weight w, there exists $p > 1$ and $C' < \infty$ such that for all balls B,*

$$\left(\fint_B M(\mathbb{1}_B w)^p \, d\mu \right)^{1/p} \leq C' \sup_{\sigma \geq 1} \fint_{\sigma B} w \, d\mu. \tag{8.4}$$

Remark 8.5. The improvement of integrability on a given ball B only depends on the finiteness of the right hand side for that same ball and nothing else, as the proof will show. Hence, one can also define the *very weak A_∞ class on B* by the condition (8.2) on that very ball. The theorem remains valid if one replaces (in the assumption and the conclusion) the supremum by a tail as before. That variant leads to the class C_p (see Section 8.1). The advantage is to allow some possible growth for which the tail is finite while the supremum is not. Finally our argument works with the supremum replaced by one average with a fixed dilation parameter. We leave these extensions to the interested reader. They will not be needed here.

Proof. To simplify we do the proof in the metric case. Again the trick to reduce the quasi-metric case to the metric case applies, see Section 3. The argument follows again that of Theorem 2.2 with $f, h = 0$ but with some changes.

We pick $N = 2$. We ignore *Step 1* and have the setup of *Step 2*. Having fixed the ball $B = B(x_0, R)$, the parameters ρ_0, r_0, and ℓ such that $2^\ell(\rho_0 - r_0) = R$, define

$$\widetilde{M}v(x) := \sup_{k \in \mathbb{Z}} \fint_{B(x, 2^k(\rho_0 - r_0))} |v| \, d\mu.$$

Then if M_c designates the centered maximal operator,

$$\widetilde{M}v \leq M_c v \leq Mv \leq \kappa' M_c v \leq \kappa \widetilde{M} v.$$

Indeed, $Mv \leq \kappa' M_c v$ is classical, while $M_c v \lesssim \widetilde{M}v$ follows from the doubling property and κ does not depend on $\rho_0 - r_0$ in particular. By the same token, in the right hand side of (8.2) we may restrict to the supremum over all $\sigma = 2^k$ for integers $k \geq 0$. This only causes a change in the constant C.

We modify *Step 3* as follows. With the truncation of the maximal function at level m,

$$\begin{aligned} \int_{B_{r_0}} (M(\mathbb{1}_{B_{r_0}} w))_m^p \, d\mu &\leq \kappa^p \int_{B_{r_0}} (\widetilde{M}(\mathbb{1}_{B_{r_0}} w))_{m/\kappa}^{p-1} \, \widetilde{M}(\mathbb{1}_{B_{r_0}} w) \, d\mu \\ &\leq \kappa^p \int_{B_{r_0}} (\widetilde{M}(\mathbb{1}_{B_{\rho_0}} w))_{m/\kappa}^{p-1} \, \widetilde{M}(\mathbb{1}_{B_{\rho_0}} w) \, d\mu \\ &= \kappa^p (p-1) \int_0^{m/\kappa} \lambda^{p-2} u(B_{r_0} \cap \{u > \lambda\}) \, d\lambda \end{aligned} \tag{8.6}$$

with $u := \widetilde{M}(\mathbb{1}_{B_{\rho_0}} w)$.

In *Step 4*, we pick $\lambda_0 := C_d^{\ell+2} \sup_{\sigma\geq 1} ⨍_{\sigma B} w\,d\mu$ (which is assumed finite otherwise there was nothing to prove), where we recall that C_d is the doubling constant. We observe that for $x \in B_{r_0} = B(x_0, r_0)$ and $k \geq 0$,

$$⨍_{B(x,2^k(\rho_0-r_0))} \mathbb{1}_{B_{\rho_0}} w\,d\mu \leq C_d^{\ell+2} ⨍_{B(x_0,2^{k+1}R)} w\,d\mu \leq \lambda_0. \tag{8.7}$$

The stopping time of *Step 5* is slightly different. Let $\lambda > \lambda_0$. Pick $x \in B_{r_0} \cap \{u > \lambda\}$. As $u(x) = \widetilde{M}(\mathbb{1}_{B_{\rho_0}} w)(x) > \lambda > \lambda_0$, the observation above and $B(x, 2^k(\rho_0 - r_0)) \subset B_{\rho_0}$ when $k < 0$ imply

$$u(x) = \sup_{k<0} ⨍_{B(x,2^k(\rho_0-r_0))} \mathbb{1}_{B_{\rho_0}} w\,d\mu = \sup_{k<0} ⨍_{B(x,2^k(\rho_0-r_0))} w\,d\mu.$$

Let $k_x < 0$ be the supremum of those $k < 0$ for which $⨍_{B(x,2^k(\rho_0-r_0))} w\,d\mu > \lambda$. We extract the covering $B_i = B(x_i, 2^{k_{x_i}}(\rho_0 - r_0))$ of $B_{r_0} \cap \{u > \lambda\}$, where all B_i are subballs of B_{ρ_0} with the $\frac{1}{5}B_i$ pairwise disjoint. We claim that if $B_i^* = 2B_i$, then for all $x \in B_i \cap B_{r_0}$, we have $u(x) \leq C_d^2 M(\mathbb{1}_{B_i^*} w)(x)$.

Indeed, fix $x \in B_i \cap B_{r_0}$ and pick $k \in \mathbb{Z}$. In the case where $k \geq 0$ we have by (8.7),

$$⨍_{B(x,2^k(\rho_0-r_0))} \mathbb{1}_{B_{\rho_0}} w\,d\mu \leq \lambda_0 < \lambda < ⨍_{B_i} w\,d\mu \leq M(\mathbb{1}_{B_i} w)(x).$$

In the case where $0 > k \geq k_{x_i}$, we have either by the stopping time or again by (8.7) if $k = -1$,

$$\begin{aligned} ⨍_{B(x,2^k(\rho_0-r_0))} \mathbb{1}_{B_{\rho_0}} w\,d\mu &\leq \frac{\mu(B(x_i, 2^{k+1}(\rho_0 - r_0)))}{\mu(B(x, 2^k(\rho_0 - r_0)))} ⨍_{B(x_i,2^{k+1}(\rho_0-r_0))} w\,d\mu \\ &\leq C_d^2 \lambda < C_d^2 ⨍_{B_i} w\,d\mu \leq C_d^2 M(\mathbb{1}_{B_i} w)(x). \end{aligned}$$

In the case where $k < k_{x_i}$, $B(x, 2^k(\rho_0 - r_0)) \subset B_i^*$ and $B(x, 2^k(\rho_0 - r_0)) \subset B_{\rho_0}$, hence

$$⨍_{B(x,2^k(\rho_0-r_0))} \mathbb{1}_{B_{\rho_0}} w\,d\mu = ⨍_{B(x,2^k(\rho_0-r_0))} w\,d\mu \leq M(\mathbb{1}_{B_i^*} w)(x).$$

Thus, the intermediate claim is proved.

Now, using this together with (8.2) and the opening remark, we obtain

$$\begin{aligned} u(B_{r_0} \cap \{u > \lambda\}) \leq \sum_i u(B_i \cap B_{r_0}) &\leq \sum_i \int_{B_i \cap B_{r_0}} u\,d\mu \\ &\leq \sum_i C_d^2 \int_{B_i \cap B_{r_0}} M(\mathbb{1}_{B_i^*} w)\,d\mu \\ &\leq \sum_i C_d^2\, \mu(B_i^*) ⨍_{B_i^*} M(\mathbb{1}_{B_i^*} w)\,d\mu \\ &\leq C C_d^2 \sum_i \mu(B_i^*) \sup_{k\geq 0} ⨍_{2^k B_i^*} w\,d\mu \end{aligned}$$

$$\leq CC_d^2 \sum_i \mu(B_i^*)\, \lambda$$
$$\lesssim \lambda\, \mu(\cup B_i).$$

The next to last inequality is by definition of $B_i^* = 2B_i$, hence all the averages do not exceed $\lambda_0 < \lambda$ by (8.7), and the last inequality uses doubling and the fact that $\frac{1}{5}B_i$ are disjoint. As $B_i \subset B_{\rho_0}$ and $\lambda < \fint_{B_i} w\, d\mu$ we have $\cup B_i \subset B_{\rho_0} \cap \{M(\mathbb{1}_{B_{\rho_0}} w) > \lambda\}$ and we have obtained

$$u(B_{r_0} \cap \{u > \lambda\}) \lesssim \lambda\, \mu(B_{\rho_0} \cap \{M(\mathbb{1}_{B_{\rho_0}} w) > \lambda\}).$$

Step 6 is now done as follows by cutting the rightmost integral in (8.6) at λ_0. Let $\varphi(r_0) := \int_{B_{r_0}} (M(\mathbb{1}_{B_{r_0}} w))_m^p\, d\mu$. Then

$$\begin{aligned}
\varphi(r_0) &\leq \kappa^p \lambda_0^{p-1}\, u(B_{r_0}) + \kappa^p(p-1) \int_{\lambda_0}^{m/\kappa} \lambda^{p-2} u(B_{r_0} \cap \{u > \lambda\})\, d\lambda \\
&\lesssim \mu(B) C_d^{p\ell} \left(\sup_{\sigma \geq 1} \fint_{\sigma B} w\, d\mu \right)^p + (p-1) \int_{\lambda_0}^{m/\kappa} \lambda^{p-1} \mu(B_{\rho_0} \cap \{M(\mathbb{1}_{B_{\rho_0}} w) > \lambda\})\, d\lambda \\
&\lesssim \mu(B) C_d^{p\ell} \left(\sup_{\sigma \geq 1} \fint_{\sigma B} w\, d\mu \right)^p + \frac{p-1}{p} \int_{B_{\rho_0}} M(\mathbb{1}_{B_{\rho_0}} w)_{m/\kappa}^p\, d\mu.
\end{aligned}$$

We recall that ℓ was defined by $2^\ell(\rho_0 - r_0) = R$. As $\kappa > 1$, we have obtained

$$\varphi(r_0) \lesssim \mu(B) C_d^{p\ell} \, (\sup_{\sigma \geq 1} \fint_{\sigma B} w\, d\mu)^p + \epsilon_p \varphi(\rho_0).$$

From there, we do as in *Step 7* an iteration provided $p - 1$ is small and finally let $m \to \infty$ to deduce (8.4). □

Having this theorem at hand, we can proceed as in [2] and show the equality of the class A_∞^{vw} with other classes. We say that a weight is a *very weak $\mathcal{A}_\infty$ weight*, if there exist an exponent $1 < p < \infty$ and a constant C such that for all balls B and Borel subsets E of B,

$$0 < \inf_{\sigma \geq 1} \frac{w(E)}{w(\sigma B)} \frac{\mu(\sigma B)}{\mu(B)} \leq C \left(\frac{\mu(E)}{\mu(B)} \right)^{1/p}. \tag{8.8}$$

We call $\mathcal{A}_\infty^{vw}$ this class. We say that a weight w is a *very weak reverse Hölder weight* if there exist an exponent $1 < q < \infty$ and a constant $C < \infty$ such that for all balls B,

$$\left(\fint_B w^q\, d\mu \right)^{1/q} \leq C \sup_{\sigma \geq 1} \fint_{\sigma B} w\, d\mu < \infty. \tag{8.9}$$

We call $\mathcal{RH}^{vw}$ this class.

Theorem 8.10. *Let w be a weight and B be a ball of X. The condition* (8.2)*,* (8.8) *for some $p \in (1, \infty)$ and* (8.9) *for some $q \in (1, \infty)$ are equivalent (with different constants). In particular, we have coincidence of A_∞^{vw}, $\mathcal{A}_\infty^{vw}$ and $\mathcal{RH}^{vw}$.*

Proof. Adapt the proof of Lemma 8.2 in [2] together with our Theorem 8.3 as the proper replacement for Theorem 5.6 therein. □

8.1. C_p **weights.** Let $X = \mathbb{R}^n$ equipped with Euclidean distance and Lebesgue measure and write $|E|$ for the Lebesgue measure of a set E. Fix a weight w. Upon replacing the supremum $\sup_{\sigma>1} w(\sigma B)/|\sigma B|$ by the tail functional

$$\begin{aligned} a_{C_p}(B) &:= \frac{1}{|B|}\int_{\mathbb{R}^n} M(\mathbb{1}_B)^p w\,dx \approx \fint_B w\,dx + \frac{1}{|B|}\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B}\left(\frac{|B|}{|2^k B|}\right)^p w\,dx \\ &\approx \sum_{k=1}^{\infty} 2^{-kn(p-1)}\fint_{2^k B} w\,dx \end{aligned}$$

with $1 < p < \infty$ in the definition of $\mathcal{A}_\infty^{vw}$ in (8.4), we recover the C_p condition of Muckenhoupt [22] and Sawyer [24]. Namely, we say that $w \in C_p$ if there are $\delta > 0$ and $C > 0$ so that

$$w(E) \le C\left(\frac{|E|}{|B|}\right)^{\delta}\int_{\mathbb{R}^n} M(\mathbb{1}_B)^p w\,dx < \infty \tag{8.11}$$

holds for all balls B and measurable $E \subset B$.

Following the proof of Lemma 8.2 in [2], we see that $w \in C_p$ if and only if there are $\delta' > 0$ and $C > 0$ such that for all balls B,

$$\left(\fint_B w^{1+\delta'}\,dx\right)^{1/(1+\delta')} \le C a_{C_p}(B) < \infty.$$

Modifying the proof of Theorem 8.3 (see Remark 8.5), one can append

$$\fint_B M(\mathbb{1}_B w)\,dx \le C a_{C_p}(B) < \infty$$

holding for some $C > 0$ and all balls B to the list of equivalent definitions of the C_p class. In conclusion, the class C_p gives examples of functions satisfying a reverse Hölder inequality with tail as in Theorem 2.2. Conversely, as we prove next a reverse Hölder inequality with a tail of the form $a_{C_p}(B)$ for fractional derivatives of solutions to certain fractional equations, we see that solutions produce examples of C_p weights.

9. An application to fractional equations

Throughout this section let $\alpha \in (0, 1/2)$. For $u \in L^2 = L^2(\mathbb{R}^n)$, we define the *fractional Laplacian* $(-\Delta)^\alpha u$ in the sense of tempered distributions through $\mathcal{F}(-\Delta)^\alpha u = (4\pi)^{2\alpha}|\xi|^{2\alpha}\widehat{u}$, where we use the normalization

$$\widehat{u}(\xi) := \mathcal{F}u(\xi) := \int_{\mathbb{R}^n} u(x)e^{-2\pi i\xi\cdot x}\,dx$$

for the Fourier transform. For simplicity, we shall always assume the dimension to be $n \ge 3$. The *Bessel potential spaces* $H^{2\alpha,2} = H^{2\alpha,2}(\mathbb{R}^n)$ consists of tempered distributions u with $u, (-\Delta)^\alpha u \in L^2$. For $u \in H^{2\alpha,2}$, we also have the singular integral representation for almost every $x \in \mathbb{R}^n$

$$(-\Delta)^\alpha u(x) = c\int_{\mathbb{R}^n}\frac{u(x) - u(y)}{|x-y|^{n+2\alpha}}\,dy,$$

where integral is understood in the principal value sense and $c = c(\alpha, n)$ can be computed explicitly. For further background on we refer e.g. to [20, 26].

Fractional Sobolev embedding theorems give us two important indices for every $p \in (1, \infty)$:

$$p^* := \frac{pn}{n - 2\alpha p} \qquad \text{and} \qquad p_* := \frac{pn}{n + 2\alpha p},$$

that satisfy $1/p^* + 1/p_* = 2/p$ and $(p_*)^* = (p^*)_* = p$. The value of α in the definition of p_* and p^* will usually be clear from the context. Otherwise it will be given explicitly as $p_{*,2\alpha}$ or $p^*_{2\alpha}$. We are interested in the weak solutions of the equation

$$(-\Delta)^\alpha \left(a(-\Delta)^\alpha u\right) = (-\Delta)^\alpha F + f, \tag{9.1}$$

where

- $a : \mathbb{R}^n \to \mathbb{C}$ satisfies $\lambda^{-1} \leq \operatorname{Re} a(x)$ and $|a(x)| \leq \lambda$ for some $\lambda > 1$ and all $x \in \mathbb{R}^n$,
- $\alpha \in (0, 1/2)$,
- $F \in L^2$ and $f \in L^{2_{*,2\alpha}}$.

Note that this fractional equation is different from those studied in [5] and [19].

Definition 9.2. A function $u \in H^{2\alpha,2}$ is a weak solution to (9.1) if for all $\varphi \in H^{2\alpha,2}$,

$$\int_{\mathbb{R}^n} (a(-\Delta)^\alpha u)\overline{(-\Delta)^\alpha \varphi}\, dx = \int_{\mathbb{R}^n} \left(F\overline{(-\Delta)^\alpha \varphi} + f\overline{\varphi}\right)\, dx.$$

Weak solutions to (9.1) satisfy a reverse Hölder inequality with tails. This will allow us to apply the non-local Gehring lemma to prove a self-improving property of weak solutions.

Lemma 9.3. *Let $u \in H^{2\alpha,2}$ be a weak solution to (9.1). Then there exists $\epsilon = \epsilon(n, \alpha) > 0$ and $\gamma = \gamma(n, \alpha) > 0$ such that for every ball $B(x, r) \subset \mathbb{R}^n$,*

$$\begin{aligned} \left(\fint_{B(x,r)} |(-\Delta)^\alpha u|^2\right)^{1/2} \lesssim &\left(\sum_{k=0}^\infty 2^{-k\gamma} \fint_{2^k B(x,r)} |(-\Delta)^\alpha u|^{2-\epsilon}\right)^{1/(2-\epsilon)} \\ &+ \left(\sum_{k=0}^\infty 2^{-k\gamma} \fint_{2^k B(x,r)} |F|^2\right)^{1/2} \\ &+ r^{2\alpha} \left(\sum_{k=0}^\infty 2^{-k\gamma} \fint_{2^k B(x,r)} |f|^{q'}\right)^{1/q'}, \end{aligned} \tag{9.4}$$

where $q' = 2_{*,2\alpha}$ and the implicit constants only depend on λ, n and α.

Proof. Throughout we allow the value of γ and ϵ to change from line to line, noting that we will only alter their values a finite amount of times and that estimates (9.4) and (9.6) become weaker for smaller γ and ϵ. We reduce the proof to the following claim.

Claim 9.5. Given any $\eta \in (0,1)$ we have

$$
\begin{aligned}
\left(\fint_{B(x,r)} |(-\Delta)^\alpha u|^2\right)^{1/2} &\leq \eta \left(\fint_{B(x,8r)} |(-\Delta)^\alpha u|^2\right)^{1/2} \\
&\quad + C\left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} |(-\Delta)^\alpha u|^{2-\epsilon}\right)^{1/(2-\epsilon)} \\
&\quad + C\left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} |F|^2\right)^{1/2} \\
&\quad + Cr^{2\alpha}\left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} |f|^{q'}\right)^{1/q'},
\end{aligned}
\tag{9.6}
$$

where $C = C(\eta, \lambda, n, \alpha)$, $\epsilon = \epsilon(n,\alpha) > 0$ and $\gamma = \gamma(n,\alpha) > 0$.

Taking the claim for granted momentarily, for any ball B set

$$
M(B) := \left(\fint_B |(-\Delta)^\alpha u|^2\right)^{1/2}
$$

and

$$
\begin{aligned}
A(B) := C&\left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B} |(-\Delta)^\alpha u|^{2-\epsilon}\right)^{1/(2-\epsilon)} \\
+ C&\left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B} |F|^2\right)^{1/2} \\
+ Cr^{2\alpha}&\left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B} |f|^{q'}\right)^{1/q'}.
\end{aligned}
$$

Then (9.6) is equivalent to

$$
M(B) \leq \eta M(8B) + A(B).
$$

Iterating this inequality, we obtain for $j \geq 1$

$$
\begin{aligned}
M(B) &\leq \eta^j M(8^j B) + \sum_{\ell=0}^{j-1} \eta^\ell A(8^\ell B) \\
&\leq \eta^j M(8^j B) + \sum_{\ell=0}^{\infty} \eta^{\ell/3} A(2^\ell B).
\end{aligned}
$$

Letting $j \to \infty$ and noting that $\eta^j M(8^j B) \to 0$ since $(-\Delta)^\alpha u \in L^2$, we obtain

$$
M(B) \leq \sum_{\ell=0}^{\infty} \eta^{\ell/3} A(2^\ell B).
$$

Letting $\eta = 2^{-3a}$ where $a - 2\alpha = 2\gamma$, we immediately recover (9.4). To see this, let us examine the following sum (which is the "hardest" to analyze)

$$\sum_{\ell=0}^{\infty} \eta^{\ell/3} C(2^\ell r)^{2\alpha} \left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^{k+\ell}B(x,r)} |f|^{q'} \right)^{1/q'}.$$

Using Hölder's inequality we have

$$\begin{aligned}
&\sum_{\ell=0}^{\infty} \eta^{\ell/3} C(2^\ell r)^{2\alpha} \left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^{k+\ell}B(x,r)} |f|^{q'} \right)^{1/q'} \\
&\quad = Cr^{2\alpha} \sum_{\ell=0}^{\infty} 2^{-2\ell\gamma} \left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^{k+\ell}B(x,r)} |f|^{q'} \right)^{1/q'} \\
&\quad \lesssim Cr^{2\alpha} \left(\sum_{\ell=0}^{\infty} 2^{-2\ell\gamma} \sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^{k+\ell}B(x,r)} |f|^{q'} \right)^{1/q'} \\
&\quad \lesssim Cr^{2\alpha} \left(\sum_{m=0}^{\infty} 2^{-m\gamma} \sum_{\ell=0}^{m} 2^{-\ell\gamma} \fint_{2^{m}B(x,r)} |f|^{q'} \right)^{1/q'} \\
&\quad \lesssim Cr^{2\alpha} \left(\sum_{m=0}^{\infty} 2^{-m\gamma} \fint_{2^{m}B(x,r)} |f|^{q'} \right)^{1/q'},
\end{aligned} \tag{9.7}$$

as desired. The bounds for the other terms are simpler. Thus, it suffices to prove Claim 9.5.

Proof of Claim 9.5 When proving the claim we may assume (by scaling) that $B = B(x, r) = B(0, 1)$ and that $u_{4B} = \fint_{4B} u = 0$ as $u - u_{4B}$ solves the same equation. Of course, strictly speaking, $u - u_{4B}$ is usually not contained in L^2 but setting $(-\Delta)^\alpha 1 := 0$, we immediately see that all all conclusions drawn for u in the previous section remain true upon adding a constant. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $1_B \leq \varphi \leq 1_{\frac{4}{3}B}$.

Before continuing we introduce some convenient notation for "error" terms: When the quantity exists we set

$$E_\alpha(u, \psi)(x) := \int_{\mathbb{R}^n} u(y) \frac{\psi(y) - \psi(x)}{|x - y|^{n+2\alpha}} \, dy.$$

We will justify absolute convergence of all such integrals on our way. Notice that for almost every $x \in \mathbb{R}^n$,

$$\begin{aligned}
c\varphi^2(x)(-\Delta)^\alpha u(x) &= \int_{\mathbb{R}^n} \varphi^2(x) \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} \, dy \\
&= \int_{\mathbb{R}^n} \frac{u(x)\varphi^2(x) - u(y)\varphi^2(y)}{|x - y|^{n+2\alpha}} \, dy + \int_{\mathbb{R}^n} \frac{u(y)\varphi^2(y) - u(y)\varphi^2(x)}{|x - y|^{n+2\alpha}} \, dy \\
&= (-\Delta)^\alpha (u\varphi^2)(x) + E_\alpha(u, \varphi^2)(x) \\
&= (-\Delta)^\alpha (u\varphi^2)(x) + \varphi(x) E_\alpha(u, \varphi)(x) + E_\alpha(u\varphi, \varphi)(x),
\end{aligned} \tag{9.8}$$

where we used the difference of squares formula in the last line.

Using the ellipticity of a and (9.8), we have

$$\begin{aligned}\int |(-\Delta)^\alpha u|^2\varphi^2 &\lesssim \left|\int (-\Delta)^\alpha u a\overline{(-\Delta)^\alpha u\varphi^2}\right| \\ &\lesssim \left|\int (-\Delta)^\alpha u a\overline{(-\Delta)^\alpha (u\varphi^2)}\right| + \left|\int (-\Delta)^\alpha u a\varphi\overline{E_\alpha(u,\varphi)}\right| \\ &\quad + \left|\int (-\Delta)^\alpha u a\overline{E_\alpha(u\varphi,\varphi)}\right| \\ &=: I + II + III.\end{aligned} \tag{9.9}$$

We intend on using the equation, (9.1), on term I, but we estimate II and III first as the estimates will be useful in handling I. Consider first term III. By the mean value theorem and the fact that $|\nabla\varphi| \lesssim 1$,

$$\begin{aligned}|E_\alpha(u\varphi,\varphi)(x)| &\le \int |u\varphi(y)|\frac{|\varphi(y)-\varphi(x)|}{|x-y|^{n+2\alpha}}\,dy \\ &\lesssim 1_{2B}(x)\int_{\mathbb{R}^n}\frac{|(u\varphi)(y)|}{|x-y|^{n+2\alpha-1}}\,dy \\ &\quad + 1_{(2B)^c}(x)\frac{1}{\operatorname{dist}(x,\frac{4}{3}B)^{n+2\alpha}}\int_{\mathbb{R}^n}|u\varphi|.\end{aligned} \tag{9.10}$$

By the fractional Sobolev-Poincaré inequality, see Remark 10.3 with $p = q = 1$ in the next section, we have

$$\int_{\mathbb{R}^n}|u\varphi| \le \int_{4B}|u| = \int_{4B}|u - u_{4B}| \lesssim \sum_{k=1}^{\infty} 2^{-k\gamma}\fint_{2^kB}|(-\Delta)^\alpha u|.$$

This yields the estimate

$$\begin{aligned}&\int_{\mathbb{R}^n}|(-\Delta)^\alpha u(x)|1_{(2B)^c}(x)\frac{1}{\operatorname{dist}(x,\frac{4}{3}B)^{n+2\alpha}}\int_{\mathbb{R}^n}|(u\varphi)(y)|\,dy\,dx \\ &\quad\le \left(\sum_{k=1} 2^{-k\gamma}\fint_{2^kB}|(-\Delta)^\alpha u|\right)^2,\end{aligned} \tag{9.11}$$

where we broke the integral in x over dyadic annuli. Also, by Hölder's inequality

$$\begin{aligned}&\int_{\mathbb{R}^n}|(-\Delta)^\alpha u(x)|1_{2B}(x)\int_{\mathbb{R}^n}\frac{|(u\varphi)(y)|}{|x-y|^{n+2\alpha-1}}\,dy\,dx \\ &\quad\lesssim \left(\int_{2B}|(-\Delta)^\alpha u|^{q'}\right)^{1/q'}\left(\int_{\mathbb{R}^n} I_{1-2\alpha}(|u|1_{2B})^q\right)^{1/q} \\ &\quad\lesssim \left(\sum_{k=0}^{\infty} 2^{-k\gamma}\fint_{2^kB}|(-\Delta)^\alpha u|^{2-\epsilon}\right)^{2/(2-\epsilon)}.\end{aligned} \tag{9.12}$$

Here we used in last line $q' < 2$ and that $I_{1-2\alpha} : L^{q_{*,1-2\alpha}} \to L^q$ with $1 < q_{*,1-2\alpha} < q = 2^*_{2\alpha}$ due to $\alpha < 1/2$; from this and Proposition 10.1 proved below we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n} I_{1-2\alpha}(|u|1_{2B})^q\right)^{1/q} &\lesssim \left(\int_{2B} |u|^{q_{*,1-2\alpha}}\right)^{\frac{1}{q_{*,1-2\alpha}}}\\ &\lesssim \left(\int_{4B} |u-u_{4B}|^{q_{*,1-2\alpha}}\right)^{\frac{1}{q_{*,1-2\alpha}}}\\ &\lesssim \left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^kB} |(-\Delta)^\alpha u|^{2-\epsilon}\right)^{1/(2-\epsilon)}.\end{aligned}$$

Combining (9.10), (9.11) and (9.12), we have obtained a desirable bound for III.

Next, we handle term II. Again, using the mean value theorem we obtain

$$\begin{aligned}II &= \left|\int (-\Delta)^\alpha u(x) a(x)\cdot \overline{\varphi(x)\int_{\mathbb{R}^n} u(y)\frac{\varphi(y)-\varphi(x)}{|x-y|^{n+2\alpha}}\,dy}\,dx\right|\\ &\lesssim \int |(-\Delta)^\alpha u(x)|\varphi(x)\int_{B(x,8/3)} \frac{|u(y)|}{|x-y|^{n+2\alpha-1}}\,dy\,dx\\ &\quad + \left|\int (-\Delta)^\alpha u(x)a(x)\varphi^2(x)\int_{B(x,8/3)^c}\frac{\overline{u(y)}}{|x-y|^{n+2\alpha}}\,dy\,dx\right|\\ &=: A_1 + A_2.\end{aligned} \tag{9.13}$$

To bound A_1 we may use that $\operatorname{supp}\varphi \subset \frac{4}{3}B$ and the L^p bounds for the Riesz potential $I_{1-2\alpha}(|u|1_{4B})$ and proceed exactly as in (9.12). For A_2 we have

$$\begin{aligned}A_2 &\le \left|\int (-\Delta)^\alpha u(x)a(x)\varphi^2(x)\int_{B(x,8/3)^c}\frac{\overline{u(y)-u(x)}}{|x-y|^{n+2\alpha}}\,dy\,dx\right|\\ &\quad + \left|\int (-\Delta)^\alpha u(x)a(x)\varphi^2(x)\overline{u(x)}\int_{B(x,8/3)^c}\frac{1}{|x-y|^{n+2\alpha}}\,dy\,dx\right|\\ &=: A_{2,1} + A_{2,2}.\end{aligned}$$

Using Young's inequality with δ's and Proposition 10.1 we have

$$\begin{aligned}A_{2,2} &\lesssim \delta\int |(-\Delta)^\alpha u|^2\varphi^2 + \frac{1}{\delta}\int_{2B}|u|^2\\ &\lesssim \delta\int |(-\Delta)^\alpha u|^2\varphi^2 + \frac{1}{\delta}\left(\sum_{k=0}^{\infty} 2^{-k\gamma}\fint_{2^kB}|(-\Delta)^\alpha u|^{2-\epsilon}\right)^{2/(2-\epsilon)},\end{aligned}$$

where we an hide the first term by choice of δ when returning to (9.9). Turning our attention to $A_{2,1}$ we recall the definition of $w(x)$ (with $r = \frac{8}{3}$) and apply Young's inequality with δ's and Proposition 10.4 to obtain

$$\begin{aligned} A_{2,1} &= \left| \int_{\mathbb{R}^n} (-\Delta)^\alpha u(x) a(x) \varphi^2(x) \overline{w(x)}\, dx \right| \\ &\lesssim \delta \int |(-\Delta)^\alpha u|^2 \varphi^2 + \frac{1}{\delta} \int_{2B} |w|^2 \\ &\lesssim \delta \int |(-\Delta)^\alpha u|^2 \varphi^2 + \frac{1}{\delta} \left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B} |(-\Delta)^\alpha u|^{2-\epsilon} \right)^{2/(2-\epsilon)} . \end{aligned}$$

Hiding the first term above we have a desirable bound for $A_{2,1}$.

We are left with handling term I. Using the equation (9.1) we have

$$\begin{aligned} I &\leq \left| \int F \overline{(-\Delta)^\alpha (u\varphi^2)} \right| + \left| \int f \overline{u\varphi^2} \right| \\ &=: I' + I''. \end{aligned}$$

For I'', using the "sharp" version of Proposition 10.1 and Young's inequality we have

$$\begin{aligned} I'' &\leq \left(\int_{2B} |f|^{q'} \right)^{1/q'} \left(\int_{4B} |u|^q \right)^{1/q} \\ &\leq \tilde{\eta}^{-1} \left(\int_{2B} |f|^{q'} \right)^{2/q'} + \tilde{\eta} \left(\int_{4B} |u|^q \right)^{2/q} \\ &\lesssim \tilde{\eta}^{-1} \left(\int_{2B} |f|^{q'} \right)^{2/q'} + \tilde{\eta} \int_{8B} |(-\Delta)^\alpha u|^2 \\ &\quad + \tilde{\eta} \left(\sum_{k=4}^{\infty} 2^{-k\gamma} \fint_{2^k B} |(-\Delta)^\alpha u| \right)^2 , \end{aligned}$$

which by a choice of $\tilde{\eta} = c\eta$ is a desirable bound. To bound I', we use the same techniques as we did for II and III. Using Young's inequality and (9.8), we have

$$\begin{aligned} I' &\leq \left| \int F \overline{((-\Delta)^\alpha u)\varphi^2} \right| + \left| \int F \overline{E_\alpha(u, \varphi^2)} \right| \\ &\leq \delta \int |(-\Delta)^\alpha u|^2 \varphi^2 + \frac{1}{\delta} \int_{2B} |F|^2 + \left| \int F \overline{E_\alpha(u, \varphi^2)} \right| . \end{aligned}$$

Hiding the first term above it only remains to bound

$$\begin{aligned} \left| \int F \overline{E_\alpha(u, \varphi^2)} \right| &\leq \left| \int F \overline{\varphi E_\alpha(u, \varphi)} \right| + \left| \int F \overline{E_\alpha(u\varphi, \varphi)} \right| \\ &=: \mathcal{A}_1 + \mathcal{A}_2. \end{aligned}$$

We handle $\mathcal{A}_2$ as we did III, recalling our bound for $|E_\alpha(u\varphi,\varphi)(x)|$ from (9.10). Using this bound we obtain (in similar fashion to III)

$$
\begin{aligned}
\mathcal{A}_2 &\lesssim \int_{\mathbb{R}^n} F(x) 1_{2B}(x) I_{1-2\alpha}(|u| 1_{2B})(x)\,dx \\
&\qquad + \int_{\mathbb{R}^n} F(x) 1_{(2B)^c}(x) \frac{1}{\operatorname{dist}(x,\frac{4}{3}B)^{n+2\alpha}} \int_{\mathbb{R}^n} |(u\varphi)(y)|\,dy\,dx \\
&\lesssim \int_{2B} |F|^2 + \int I_{1-2\alpha}(|u|1_{2B})^2 \\
&\qquad + \sum_{k=0}^{\infty} 2^{-k\gamma} \left(\fint_{2^kB} |F|^2 \right)^{1/2} + \left(\int |u\varphi| \right)^2 \\
&\lesssim \sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^kB} |F|^2 + \left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^kB} |(-\Delta)^\alpha u|^{2-\epsilon} \right)^{2/(2-\epsilon)},
\end{aligned}
$$

where we used that $I_{1-2\alpha} : L^{2_{*,1-2\alpha}} \to L^2$ with $2_{*,1-2\alpha} < 2$ $(< q)$. We now bound $\mathcal{A}_1$. Proceeding as we did for II, the mean value theorem gives the bound

$$
\begin{aligned}
\mathcal{A}_1 &\leq \int |F(x)| \varphi(x) \int_{B(x,8/3)} \frac{|u(y)|}{|x-y|^{n+2\alpha-1}}\,dy\,dx \\
&\qquad + \left| \int F(x) \varphi^2(x) \int_{B(x,8/3)^c} \frac{\overline{u(y)}}{|x-y|^{n+2\alpha}}\,dy\,dx \right| \\
&=: \mathcal{A}_1' + \mathcal{A}_1''.
\end{aligned}
$$

As before, since $B(x,8/3) \subset 4B$ for all $x \in \operatorname{supp}\varphi \subset \frac{4}{3}B$,

$$
\begin{aligned}
\mathcal{A}_1' &\leq \int_{2B} |F|^2 + \int I_{1-2\alpha}(|u|1_{4B})^2 \\
&\leq \int_{2B} |F|^2 + \left(\sum_{k=0}^{\infty} 2^{-k\gamma} |(-\Delta)^\alpha u|^{2-\epsilon} \right)^{2/(2-\epsilon)}.
\end{aligned}
$$

Finally, appealing to previous estimates we have

$$
\begin{aligned}
\mathcal{A}_1'' &\leq \left| \int F(x)\varphi^2(x) \int_{B(x,8/3)^c} \frac{\overline{u(y)-u(x)}}{|x-y|^{n+2\alpha}}\,dy\,dx \right| \\
&\qquad + \left| \int F(x)\varphi^2(x)\overline{u(x)} \int_{B(x,8/3)^c} \frac{1}{|x-y|^{n+2\alpha}}\,dy\,dx \right| \\
&\lesssim \left| \int_{\mathbb{R}^n} F(x)\varphi^2(x)\overline{w(x)}\,dx \right| + \left| \int_{2B} F(x) u(x)\,dx \right| \\
&\lesssim \int_{2B} |F|^2 + \left(\sum_{k=0}^{\infty} 2^{-k\gamma} |(-\Delta)^\alpha u|^{2-\epsilon} \right)^{2/(2-\epsilon)}.
\end{aligned}
$$

Combining our estimates for I, II and III we have proved the claim (by choice of $\tilde{\eta} = c\eta$) and hence the lemma. □

With Lemma 3.2 at hand, we can apply Theorem 2.2 to obtain improvement of the integrability of $(-\Delta)^\alpha u$. The first application concerns the case when the right hand side of the equations exhibits higher local integrability:

Theorem 9.14. *Let $u \in H^{2\alpha,2}$ and $p > 2$. Suppose u is a weak solution to (9.1) where $f \in L^{2_*} \cap L^{p_*}_{loc}$ and $F \in L^2 \cap L^p_{loc}$. Then there is $\epsilon_0 = \epsilon_0(\lambda, n, \alpha, p) > 0$ so that $|(-\Delta)^\alpha u| \in L^{2+\epsilon_0}_{loc}$.*

Proof. Let ϵ and γ as in Lemma 9.3 and put $v := |(-\Delta)^\alpha u|^{2-\epsilon}$ and similarly $\tilde{F} := |F|^{2-\varepsilon}$ and $\tilde{f} := |f|^{2-\varepsilon}$. In terms of $v, \tilde{F}, \tilde{f}$ the conclusion of that lemma reads

$$\left(\fint_{B(x,r)} v^{\frac{2}{2-\epsilon}}\right)^{\frac{2-\varepsilon}{2}} \lesssim \sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} v + \left(\sum_{k=1}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} \tilde{F}^{\frac{2}{2-\epsilon}}\right)^{\frac{2-\varepsilon}{2}} + r^{2\alpha(2-\varepsilon)} \left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} \tilde{f}^{\frac{2_*}{2-\epsilon}}\right)^{\frac{2-\varepsilon}{2_*}},$$

with implicit constants depending on λ, n and α. Now that the exponent of v on the right hand side is 1, the claim follows from Theorem 2.2 after checking the numerology. The parameters in that theorem are

$$(D, \beta, q, s) := \left(n, 2\alpha(2-\varepsilon), \tfrac{2}{2-\varepsilon}, \tfrac{2_*}{2-\varepsilon}\right),$$

and so the conditions $0 < s < q$, $q > 1$ and $\beta \geq D(1/s - 1/q)$ are satisfied. (Note that in fact $D(1/s - 1/q) = 2\alpha(2-\varepsilon) = \beta$ and that $s < 1$). Hence, Theorem 2.2 gives us local higher for v with exponent larger than q, provided $\tilde{F}$ and $\tilde{f}$ are globally integrable with exponents q and s and locally integrable to some higher exponents, respectively. By definition, this precisely means $F \in L^2 \cap L^p_{loc}$ and $f \in L^{2_*} \cap L^{p_*}_{loc}$ and for some $p > 2$, which is our assumption. Of course we can write the resulting estimate again in terms of the original functions: We get for all sufficiently small $\epsilon_0 = \epsilon_0(\lambda, n, \alpha, p) > 0$,

$$\begin{aligned}\left(\fint_{B(x,r)} |(-\Delta)^\alpha u|^{2+\epsilon_0}\right)^{\frac{1}{2+\epsilon_0}} \lesssim& \sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} |(-\Delta)^\alpha u|^2 \\ &+ \left(\sum_{k=1}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} |F|^2\right)^{1/2} \\ &+ r^{2\alpha}\left(\sum_{k=0}^{\infty} 2^{-k\gamma} \fint_{2^k B(x,r)} |f|^{2_*}\right)^{1/2_*} + \left(\fint_{2B(x,r)} |F|^p\right)^{\frac{1}{p}} \\ &+ r^{2\alpha}\left(\fint_{2B(x,r)} |f|^{p_*}\right)^{\frac{1}{p_*}}.\end{aligned}$$

□

A global version follows by replacing Theorem 2.2 by Theorem 5.1.

Theorem 9.15. *Suppose u is a weak solution to (9.1) where $f \in L^{2_*} \cap L^{p_*}$ and $F \in L^2 \cap L^p$ for some $p > 2$. Then there is $\epsilon_0 = \epsilon_0(\lambda, n, \alpha, p) > 0$ so that $|(-\Delta)^\alpha u| \in L^{2+\epsilon_0}$.*

10. Technical estimates

The Sobolev–Poincaré inequality is as follows:

Proposition 10.1. *Let $B = B(x, r)$ be a ball and $u \in H^{2\alpha,2}$. If $\alpha \in (0, 1/2)$ and $N > 1$, then*

$$\left(\fint_B |u - u_B|^q\right)^{1/q} \le Cr^{2\alpha}\left(\fint_{NB} |(-\Delta)^\alpha u|^p\right)^{1/p} + \frac{Cr^{2\alpha}}{(N-1)^{n+1-2\alpha}}\sum_{k=2}^{\infty} N^{-k(1-2\alpha)} \fint_{N^k B} |(-\Delta)^\alpha u|, \tag{10.2}$$

where $C = C(n, p, q, \alpha, N)$ for all $1 \le p \le 2$, $1 \le q \le p^*$ with exception of $p = 1, q = 1^*$, The constant C stays bounded as $N \to 1$.

Remark 10.3. In the next section we shall only use Proposition 10.1 with $N = 2$. In fact, we will only use this "strong" version of the inequality once and we will often use the inequality

$$\left(\fint_B |u - u_B|^q\right)^{1/q} \le Cr^{2\alpha}\left(\sum_{k=1}^{\infty} 2^{-k(1-2\alpha)} \fint_{2^k B} |(-\Delta)^\alpha u|^p\right)^{1/p},$$

which follows from (10.2) and Hölder's inequality.

Proof. We prove the claim for a Schwartz function u and $B = B(0, 1)$. The general claim follows by scaling and approximation via smooth truncation and convolution. Let $K(x) = |x|^{-n+2\alpha}$ be the kernel of the Riesz potential $I_{2\alpha}$. Using the formula $u = cI_{2\alpha}(-\Delta)^\alpha u$, see Chapter V in [26], we can write

$$|u(x) - u(y)| \lesssim \int |(-\Delta)^\alpha u(z)||K(x - z) - K(y - z)|\, dz$$

for all $x, y \in B$. Estimating ∇K and using the mean value theorem for $z \notin NB$, we obtain a uniform bound

$$\begin{aligned}
&\int_{(NB)^c} |(-\Delta)^\alpha u(z)||K(x - z) - K(y - z)|\, dz \\
&\quad\lesssim \left(\frac{N}{N-1}\right)^{n+1-2\alpha} \int_{(NB)^c} |(-\Delta)^\alpha u(z)||z|^{-n-1+2\alpha} \\
&\quad\lesssim \left(\frac{N}{N-1}\right)^{n+1-2\alpha} \sum_{k=2}^{\infty} N^{-(1-2\alpha)k} \fint_{N^k B} |(-\Delta)^\alpha u(z)|\, dz.
\end{aligned}$$

For the rest of the integral, we write

$$\int_{NB} |(-\Delta)^\alpha u(z)||K(x-z)-K(y-z)|\, dz \le I_{2\alpha}(1_{NB}|(-\Delta)^\alpha u|)(x) + I_{2\alpha}(1_{NB}|(-\Delta)^\alpha u|)(y).$$

Then

$$\iint_{B\times B} |u(x) - u(y)|^{p^*}\, dxdy \lesssim \left(\frac{N^{n+1-2\alpha}}{(N-1)^{n+1-2\alpha}} \sum_{k=2}^{\infty} N^{-(1-2\alpha)k} \fint_{N^k B} |(-\Delta)^\alpha u(z)|\, dz\right)^{p^*}$$

$$+ \left(\fint_{NB} |(-\Delta)^\alpha u(z)|^p \, dz \right)^{p^*/p},$$

where we have used the boundedness $I_{2\alpha} : L^p \to L^{p^*}$ if $1 < p \leq 2$. This proves the claim for the case $q = p^*$ and $1 < p \leq 2$. The intermediate cases for q are then obvious by Hölder's inequality. Finally, we need to consider $p = 1$ and $1 \leq q < 1^*$. Here, we use the boundedness $I_{2\alpha} : L^1 \to L^{1^*,\infty}$ and that the weak-type space $L^{1,\infty}(B)$ embeds into $L^q(B)$ since B has finite measure. □

We next prove an estimate for the truncated fractional Laplacian.

Proposition 10.4. *Let $u \in H^{2\alpha,2}(\mathbb{R}^n)$ and*

$$\omega(x) := \int_{|y-x|>r} \frac{u(y) - u(x)}{|x-y|^{n+2\alpha}} \, dy.$$

Given $p \in [2, 2^*)$, there are $\gamma > 0$ and $\epsilon > 0$ only depending on α, n and p such that

$$\left(\fint_{B(z,r)} |\omega(x)|^p \, dx \right)^{1/p} \lesssim \left(\sum_{k=1}^\infty 2^{-\gamma k} \fint_{2^k B(z,r)} |(-\Delta)^\alpha u|^{2-\epsilon} \, dx \right)^{1/(2-\epsilon)}$$

for all $z \in \mathbb{R}^n$ and $r > 0$.

Proof. Again, it suffices to treat the case $B(z, r) = B(0, 1)$ with u a Schwartz function. The general result follows by scaling and density as before. For $x \in \mathbb{R}^n$, we set $B_x := B(x, 1)$. Let $\widehat{v}(\xi) := |\xi|^{2\alpha} u$ so that $v = c(-\Delta)^\alpha u$. Then, writing $h = y - x$,

$$\omega(x) = \int_{\mathbb{R}^n} \left(\int_{|h|>1} \frac{e^{2\pi i \xi \cdot h} - 1}{|\xi|^{2\alpha} |h|^{n+2\alpha}} \, dh \right) e^{2\pi i \xi \cdot x} \widehat{v}(\xi) \, d\xi.$$

We analyze the multiplier

$$m(\xi) := \int_{|h|>1} \frac{e^{2\pi i \xi \cdot h} - 1}{|\xi|^{2\alpha} |h|^{n+2\alpha}} \, dh \tag{10.5}$$

and need to estimate $\omega = \check{m} * v$ in $L^p(B(0, 1))$. Let ψ be a smooth and radial function with

$$\operatorname{supp} \psi \subset \left\{ \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad \sum_{j \in \mathbb{Z}} \psi(2^{-j} \xi) = 1$$

whenever $\xi \neq 0$, and let $m_j(\xi) := m(\xi)\psi(2^{-j}\xi)$ be the piece of the multiplier localized near the frequency 2^j. We also put $\psi_j(\xi) := \psi(2^{-j}\xi)$.

Small frequencies. Now assume $|\xi| \leq 1$, that is, consider the pieces for $j \leq -1$. Using polar coordinates, we write

$$m(\xi) = \omega_n \int_{|\xi|}^\infty \int_{S^{n-1}} \frac{e^{2\pi i \rho \frac{\xi}{|\xi|} \cdot \theta} - 1}{\rho^{2\alpha}} \, d\theta \frac{d\rho}{\rho}.$$

The expression above is radial. Denote $\tilde{\theta} := \frac{\xi}{|\xi|} \cdot \theta$. Let $\tilde{m}(|\xi|) := m(\xi)$. Differentiating once, we see

$$\tilde{m}'(r) = \int_{S^{n-1}} \frac{e^{2\pi i r \tilde{\theta}} - 1}{r^{1+2\alpha}} \, d\theta = \int_{S^{n-1}} \sum_{j=1}^{\infty} \frac{(2\pi i \tilde{\theta})^j}{j!} r^{j-1-2\alpha} \, d\theta.$$

By symmetry,

$$\int_{S^{n-1}} \tilde{\theta} \, d\theta = 0.$$

Hence the term with $j = 1$ is zero, and the lowest power of r in the series is $1 - 2\alpha$. Consequently, for $r \in (0, 1)$ and $K \geq 1$ the K-th derivative is bounded by

$$|\tilde{m}^{(K)}(r)| \lesssim_K r^{1-K} r^{1-2\alpha}.$$

Hence a calculation yields for $\sigma \in \mathbb{N}_0^n$, a multi-index with $K = |\sigma|$ and $|\xi| < 1$, $|\partial_\xi^\sigma m(\xi)| \lesssim_K |\xi|^{1-K} |\xi|^{1-2\alpha}$

Take a frequency piece m_j with $j \leq -1$. Bounding the L^∞ norm of $\check{m}_j$ by the L^1 norm of its Fourier transform, we get

$$|x|^K |\check{m}_j(x)| \lesssim \sum_{|\sigma|=K} |x^\sigma \check{m}_j(x)| \leq \sum_{|\sigma|=K} \int |\partial_\xi^\sigma m_j(\xi)| \, d\xi \lesssim 2^{jn} \cdot 2^{j(1-K+1-2\alpha)}$$

by the support properties of m_j. Since $2\alpha < 1$, we can set $K = n + 1$ and sum over j to obtain

$$\sum_{j=-\infty}^{-1} |\check{m}_j(x)| \lesssim |x|^{-(n+1)} \sum_{j=-\infty}^{-1} 2^{j(1-2\alpha)} \lesssim |x|^{-(n+1)}.$$

This together with the integrability of $\sum_{j=-\infty}^{-1} \check{m}_j$ (by local integrability of m) shows

$$\left| \sum_{j=-\infty}^{-1} \check{m}_j(x) \right| \lesssim \frac{1}{(1 + |x|)^{n+1}}$$

and therefore, by splitting the integral into dyadic annuli, we obtain for almost $x \in B$ a desirable pointwise bound

$$\left| \left(\sum_{j=-\infty}^{-1} \check{m}_j \right) * v(x) \right| \lesssim \sum_{k=0^\infty} 2^{-k} ⨍_{2^k B} |v|. \tag{10.6}$$

Symbol estimate for large frequencies. To estimate the pieces m_j with $j \geq 0$, we write (10.5) with aid of a Bessel function (see Appendix B.4 of [14]) as

$$m(\xi) = \int_{|\xi|}^{\infty} \frac{2\pi J_{\frac{n-2}{2}}(2\pi\rho)}{\rho^{\frac{n-2}{2}}} \frac{d\rho}{\rho^{1+2\alpha}} - \frac{\omega_n}{2\alpha|\xi|^{2\alpha}}.$$

The part $m^0(\xi) = \omega_n / 2\alpha|\xi|^{2\alpha}$ is a constant multiple of the symbol of the Riesz potential $I_{2\alpha}$. Denoting $\psi_{-\infty} := \sum_{j=-\infty}^{-1} \psi_j$ (smooth and compactly supported), we can decompose

$$m^0 \sum_{j=0}^{\infty} \psi_j = m^0 - m^0 \psi_{-\infty}.$$

On the other hand, since we restrict to large frequencies, we have

$$\int \left|\partial^\beta(m^0\sum_{j=0}^\infty \psi_j)(\xi)\right| d\xi \lesssim_{|\beta|} \int_{|\xi|\geq\frac{1}{2}} |\xi|^{-|\beta|-2\alpha}\,d\xi \lesssim 1$$

for all multi-indices β with $|\beta| \geq n$, which, as in the previous step, implies a bound by $|x|^{-|\beta|}$ on the inverse Fourier transform. In total, we have

$$|k * v| := \left|\mathcal{F}^{-1}(vm^0\sum_{j=0}^\infty \psi_j)\right| \lesssim_K \min\left(\check{\psi}_{-\infty} * I_{2\alpha}(|v|) + I_{2\alpha}(|v|), \int \frac{|v(y)|}{|\cdot - y|^K}\,dy\right)$$

for any $K > n$. Now, we split $v = 1_{B(0,2)}v + 1_{B(0,2)^c}v$. For $p \in [2, 2^*)$, we see that convolution with $\check{\psi}_{-\infty}$ is bounded in L^p so that for $\epsilon := 2 - p_* > 0$,

(10.7)

$$\begin{aligned}\int_{B(0,1)} (k * |v|)(x)^p\,dx &\lesssim \int I_{2\alpha}(1_{B(0,2)}|v|)(x)^p\,dx + \left(\sum_{l=1}^\infty 2^{-l} \fint_{B(0,2^l)} |v|\,dy\right)^p \\ &\lesssim \left(\int_{B(0,2)} |v|^{2-\epsilon}\,dy\right)^{p/(2-\epsilon)} + \left(\sum_{l=1}^\infty 2^{-l} \fint_{B(0,2^l)} |v|\,dy\right)^p.\end{aligned}$$

Here we have chosen $K = n + 1$ in our estimate for $k * v$, used the boundedness of the Riesz potential from $L^{p_*} \to L^p$ and broke up the integral in y into dyadic annuli. This is a desirable estimate for $m^0 \sum_{j=0}^\infty \psi_j$.

It remains to estimate on the level of large frequencies the kernel of the multiplier

$$M(\xi) := (m - m^0)(\xi) = \int_{|\xi|}^\infty \frac{J_{\frac{n-2}{2}}(\rho)}{\rho^{\frac{n-2}{2}}} \frac{d\rho}{\rho^{1+2\alpha}}. \tag{10.8}$$

This is a radial function, and we denote by $\tilde{M}(r)$ its value at any ξ with $|\xi| = r$. Let $g_\nu(t) := t^{-\nu}J_\nu(t)$ so that the recursion $(g_\nu(t))' = tg_{\nu+1}(t)$ holds for $\nu > -1/2$ (item (1) of Appendix B.2 of [14]). Clearly

$$\tilde{M}'(r) = J_{\frac{n-2}{2}}(r)r^{-\frac{n-2}{2}-1-2\alpha} = g_{\frac{n-2}{2}}(r)r^{-1-2\alpha}$$

so that

$$\tilde{M}^{(l+1)}(r) = \sum_{l'=0}^l \binom{l}{l'}\left(\partial_r^{l-l'} g_{\frac{n-2}{2}}(r)\right)\left(\partial_r^{l'}\frac{1}{r^{1+2\alpha}}\right).$$

The Bessel functions satisfy $|J_\nu(r)| \lesssim r^{-1/2}$ for $r \geq 1/2$ provided $\nu > -1/2$ (Appendix B.7 of [14]). By this and the recursion formula for g, we see that repeated differentiation does not alter its decay rate so

$$|\tilde{M}^{(l+1)}(r)| \lesssim_l r^{-\frac{n-1}{2}-1-2\alpha} \tag{10.9}$$

for $r \geq 1/2$ and we can use the same technique as in the previous step to obtain for $j \geq 0$ and $M_j := M\psi_j$ the bounds

$$|\check{M}_j(x)| \lesssim_K \frac{2^{jn} \cdot 2^{-j(\frac{n-1}{2}+1+2\alpha)}}{|x|^K} \tag{10.10}$$

for all $K \geq 1$. This estimate does not allow for a pointwise bound for the associated kernel, but it will do for an averaged statement.

Pointwise bound for moderately large frequencies. We decompose v into pieces supported in annuli in space. Let $1_{B(0,1)} \leq \eta_1 \leq 1_{B(0,4)}$ be smooth and for each $k \geq 2$ let η_k be supported in $2^{k+1}B(0,1) \setminus 2^{k-1}B(0,1)$ and be such that

$$1 = \eta_1(y) + \sum_{k=2}^{\infty} \eta_k(y)$$

for all $y \in \mathbb{R}^n$.

Recall $M_j := M\psi_j$ for $j \geq 0$. We aim at estimating $\mathcal{F}^{-1}(\sum_{j=0}^{\infty} M_j v)$. Fix a scale $k \geq 2$ and denote $M_0^{k-1} := \sum_{j=0}^{k-1} M_j$. By the symbol estimate (10.10) and support of η_k, we have for $|x| \leq 1$,

$$\begin{aligned}
|\check{M}_0^{k-1} * (v\eta_k)(x)| &\lesssim \sum_{j=0}^{k-1} |\check{M}_j| * (|v|\eta_k)(x) \\
&\lesssim \sum_{j=0}^{k-1} 2^{-j(\frac{n-1}{2}+1+2\alpha)} \cdot 2^{jn} \int \frac{\eta_k(y)|v(y)|}{|x-y|^K} \, dy \\
&\lesssim 2^{-k} \fint_{B(0,2^{k+1})} |v| \, dy,
\end{aligned}$$

since $|x - y| \sim 2^k$ whenever $\eta_k \neq 0$ and choosing K large.

L^2 bound for very large frequencies. Finally, for $k \geq 2$, denote $M_k^{\infty} := \sum_{j=k}^{\infty} M_j$. By (10.9), where $\tilde{M}$ is the real function corresponding to the radial function M, and by now common arguments for the spatial derivatives, we see that for any multi-index $\beta \neq 0$,

$$\int_{\mathbb{R}^n} |\partial^\beta M_k^{\infty}|^2 \, d\xi \lesssim \int_{|\xi| \geq 2^{k-1}} |\xi|^{-(n+1+4\alpha)} \, d\xi \lesssim 2^{-k(1+4\alpha)}.$$

By Plancherel's formula, we infer

$$\int_{\mathbb{R}^n} |y^\beta \check{M}_k^{\infty}|^2 \, dy \lesssim 2^{-k(1+4\alpha)}.$$

Since $p < \infty$, there is $1 \leq q < 2$ so that $1 + \frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. Then by Young's inequality,

$$\begin{aligned}
\left(\int_{B(0,1)} |\check{M}_k^{\infty} * (\eta_k v)|^p \, dy \right)^{1/p} &\leq \left(\int \left| (\check{M}_k^{\infty} 1_{B(0,2^{k+2}) \setminus B(0,2^{k-2})}) * (\eta_k v) \right|^p dy \right)^{1/p} \\
&\leq \left\| \check{M}_k^{\infty} \right\|_{L^2(B(0,2^{k+2}) \setminus B(0,2^{k-2}))} \left\| \eta_k v \right\|_{L^q} \\
&\lesssim \left\| |x|^{n/q} \check{M}_k^{\infty} \right\|_{L^2} \left\| 2^{-nk/q} \eta_k v \right\|_{L^q} \\
&\lesssim 2^{-\frac{1}{2}(1+4\alpha)k} \left(\fint_{B(0,2^{k+2})} |v|^q \, dy \right)^{1/q}.
\end{aligned}$$

Conclusion for large frequencies. Summing over $k \geq 2$ the results from the previous two steps, we get the desirable bound for large frequencies

$$\begin{aligned}
&\left\| \sum_{j=0}^{\infty} \check{M}_j * ((1-\eta_1)v) \right\|_{L^p(B(0,1))} \\
&\qquad \lesssim \sum_{k=2}^{\infty} \left(\int_{B(0,1)} \left(|\check{M}_0^{k-1} * (\eta_k v)|^p + |\check{M}_k^{\infty} * (\eta_k v)|^p \right) dy \right)^{1/p} \\
&\qquad \lesssim \sum_{k=2}^{\infty} 2^{-\frac{1}{2}k} \left(\fint_{B(0,2^{k+2})} |v|^q \, dy \right)^{1/q}.
\end{aligned}$$

It still remains to estimate the piece with $k = 1$. In this case, we just note that the symbol from (10.8) is

$$M(\xi) = c|\xi|^{-2\alpha} \cdot |\xi|^{-\frac{n-2}{2}} \int_0^{\infty} \frac{1_{\{r>1\}}}{r^{n+2\alpha}} J_{\frac{n-2}{2}}(2\pi r|\xi|) r^{\frac{n}{2}} \, dr,$$

and the second factor can be regarded as the Fourier transform of the radial function $k_1 := 1_{B(0,1)^c}|x|^{-n-2\alpha}$ (cf. [14] Appendix B.5) whereas the first factor is the symbol of the Riesz potential $I_{2\alpha}$. Then, using again the smooth and compactly supported function $\psi_{-\infty} = \sum_{j=-\infty}^{-1} \psi_j$,

$$\begin{aligned}
\left| \sum_{j=0}^{\infty} \check{M}_j * (\eta_1 v)(x) \right| &\leq |k_1 * I_{2\alpha}(\eta_1 v)(x)| + |(k_1 * \psi_{-\infty}) * I_{2\alpha}(\eta_1 v)(x)| \\
&\leq \left(\|k_1\|_{L^2} + \|k_1\|_{L^2} \|\psi_{-\infty}\|_{L^1} \right) \|I_{2\alpha}(\eta_1 v)\|_{L^2} \\
&\lesssim \left(\int_{B(0,4)} |v|^{2_*} \, dx \right)^{1/2_*}.
\end{aligned}$$

This together with the bounds for the kernels of m with frequencies $|\xi| \lesssim 1$ in (10.6) and $m^0 = (m - M)$ with frequencies $|\xi| \gtrsim 1$ in (10.7) completes the proof. □

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