ON REGULARITY OF WEAK SOLUTIONS TO PARABOLIC SYSTEMS

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Abstract. We establish a new regularity property for weak solutions of parabolic systems with coefficients depending measurably on time as well as on all spatial variables. Namely, weak solutions are locally Hölder continuous L^p valued functions for some $p > 2$. Our methods are likely to adapt to other linear or non-linear systems.

1. Introduction

This work is concerned with local regularity of weak solutions to parabolic equations or systems in divergence form,

$$\partial_t u - \operatorname{div}_x A(t,x)\nabla_x u = f + \operatorname{div}_x F, \tag{1.1}$$

in absence of any regularity of the coefficients besides measurability. The system is considered in an open cylinder $(t,x) \in \Omega = I_0 \times Q_0 \subseteq \mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, ellipticity is imposed in the sense of a weak Gårding inequality and weak solutions belong to the usual Lions class $\mathrm{L}^2_{\mathrm{loc}}(I_0; \mathrm{W}^{1,2}_{\mathrm{loc}}(Q_0; \mathbb{C}^m))$. We note at this stage that we do not impose solutions to be in $\mathrm{L}^\infty_{\mathrm{loc}}(I_0; \mathrm{L}^2_{\mathrm{loc}}(Q_0))$.

From the classical theory of Nash and Moser we know that weak solutions to parabolic equations with real coefficients are Hölder continuous with respect to the parabolic distance. This is no longer true for equations with complex coefficients let alone systems, even in dimensions $n = 1, 2$ in contrast with elliptic equations. In fact, at this level of generality the only available results are continuity in time valued in spatial $\mathrm{L}^2_{\mathrm{loc}}$, see [19], and local L^p-integrability of $\nabla_x u$ for some $p > 2$, see [11]. In this paper we establish a new regularity property of weak solutions.

Theorem 1.1. *If $f = 0$ and $F = 0$, then u is locally bounded and α-Hölder continuous in time with values in $\mathrm{L}^p_{\mathrm{loc}}(Q_0; \mathbb{C}^m)$ for some $\alpha > 0$ and $p > 2$.*

Most of the regularity properties of solutions to parabolic systems have been established through the local variational methods emerging from the Lions theory [19]. It does not seem that those methods give access to our result. Instead we rely on a global variational approach based on this simple observation: We can extend the local solution u via multiplication with a smooth cut-off to a global function $v := u\chi$ and study the corresponding inhomogeneous problem

$$\partial_t v - \operatorname{div}_x A(t,x)\nabla_x v = \tilde{f} + \operatorname{div}_x \tilde{F}, \tag{1.2}$$

now on all of $\mathbb{R}^{n+1}$. Indeed, any local information of v carries over to u. However, the global setup with time describing the real line enables to bring powerful tools such as singular integral operators and the Fourier transform into play and it does not seem they have been fully exploited for local regularity up to now. Most notably, splitting $\partial_t = D_t^{1/2} H_t D_t^{1/2}$ according to the Fourier

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decomposition $-\mathrm{i}\tau = |\tau|^{1/2}(-\mathrm{i}\,\mathrm{sgn}(\tau))|\tau|^{1/2}$, there is a sesquilinear form

$$a(v,\phi) = \iint_{\mathbb{R}^{n+1}} A\nabla_x v \cdot \overline{\nabla_x \phi} + H_t D_t^{1/2} v \cdot \overline{D_t^{1/2} \phi} \, dx dt \tag{1.3}$$

corresponding to (1.2) which, in contrast to the Lions theory on finite time intervals, admits a hidden coercivity on a natural energy space. This observation due to [15] uses in a crucial way the algebraic properties of the Hilbert transform on the real line and has been key also to recent developments on parabolic boundary value problems, see [4] and the references therein, or maximal regularity [3, 8]. Compared to local weak solutions, whose t-derivative is understood in a weak sense only through the equation, we can therefore study on the global level the exact amount of differentiability that v should admit through a locally integrable function – the fractional derivative $D_t^{1/2} v(t,x)$. In fact, we have $|\nabla_x v| + |D_t^{1/2} v| + |v| \in \mathrm{L}^2(\mathbb{R}^{n+1})$. Since time is a one-dimensional variable, square-integrability of $D_t^{1/2} v$ is already the borderline case from the view-point of Sobolev embeddings, not enough for continuity in time though, which probably explains why $D_t^{1/2} v$ has not been exploited before. But its higher integrability would do. Theorem 1.1 follows, therefore, from the following result.

Theorem 1.2. *If $f = 0$ and $F = 0$, then there exists $p > 2$ such that $|\nabla_x v| + |D_t^{1/2} v| + |v| \in \mathrm{L}^p(\mathbb{R}^{n+1})$.*

Theorem 1.2 provides us with a self-improvement of integrability for the spatial gradient and the half-order time derivative simultaneously. Note that we shall not use the scale of parabolic Sobolev space $\mathrm{W}_p^{1,1/2}$ but Sobolev embeddings *a posteriori* furnishes $v \in \mathrm{W}_p^{1,1/2}(\mathbb{R}^{n+1}; \mathbb{C}^m)$ as $p \geq 2$.

We present two proofs of Theorem 1.2 relying on rather different methods, both using the global variational formulation explained above. We think they each have their own interest, with potential applicability to non-linear systems for the first one and to other types of parabolic equations as well as fractional elliptic equations for the second one.

The idea of the proof presented in Section 6 is to use, as in the analogous result for elliptic equations [20], self-improvement properties of reverse Hölder inequalities known as Gehring's lemma. First, we prove a new and delicate reverse Hölder inequality for the quantity $g := |\nabla_x v| + |D_t^{1/2} v| + |H_t D_t^{1/2} v|$, by extending ideas from [4]. The presence of the Hilbert transform comes from our argument. The non-locality of the fractional derivative reflects in local L^2 averages of g being controlled only by a weighted infinite sum of L^1 averages. Hence, we need a substantial extension of the classical Gehring lemma, which we shall prove in Section 5 and could be of independent interest. An unrelated Gehring type lemma "with tail" recently obtained in the context of fractional elliptic equations [18] has been inspiring to us. We mention that we shall explore further such extensions in a forthcoming work [2].

For the second proof presented in Section 7, we consider the operator $\mathcal{L}$ associated with the sesquilinear form (1.3) in virtue of the Lax-Milgram lemma and use the analytic perturbation argument of Šneĭberg [22]. More precisely, exploiting in a crucial way the hidden coercivity, $\mathcal{L}$ plus a large constant turns out to be invertible from the natural L^2 energy space to its dual and extends boundedly to the corresponding L^p-based spaces. The higher integrability of g then follows from the fact that invertibility of a bounded operator between complex interpolation scales extrapolates.

All this is for homogeneous equations so far, that is $f = 0$ and $F = 0$, on $I_0 \times Q_0$. However, as the extension to $\mathbb{R}^{n+1}$ forces us to work with inhomogeneous equations anyway, there is no real obstacle to start with an inhomogeneous equation right away. For the general situation, our main result reads as follows.

Theorem 1.3. *Let $q = 2 + \frac{4}{n}$, with dual exponent $q' = \frac{2n+4}{n+4}$. Assume that $u \in \mathrm{L}^2_{\mathrm{loc}}(I_0; \mathrm{W}^{1,2}_{\mathrm{loc}}(Q_0; \mathbb{C}^m))$ is a weak solution to (1.1) on $\Omega = I_0 \times Q_0$ with right-hand side $f \in \mathrm{L}^{q'}_{\mathrm{loc}}(\Omega; \mathbb{C}^m)$ and $F \in \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbb{C}^{mn})$. Let χ be a smooth cut-off with support in Ω.*

(i) *It holds*

$$u \in \mathrm{L}^q_{\mathrm{loc}}(\Omega; \mathbb{C}^m) \cap \mathrm{C}(I_0; \mathrm{L}^2_{\mathrm{loc}}(Q_0; \mathbb{C}^m)).$$

Moreover, $t \mapsto \|(u\chi)(t, \cdot)\|^2$ is absolutely continuous on $\mathbb{R}$ and $D_t^{1/2}(u\chi) \in \mathrm{L}^2(\mathbb{R}^{n+1}; \mathbb{C}^m)$.

(ii) *There exists $p > 2$ depending only on ellipticity and dimensions such that if $F \in \mathrm{L}^p_{\mathrm{loc}}(\Omega; \mathbb{C}^{mn})$ and $f \in \mathrm{L}^{p_*}_{\mathrm{loc}}(\Omega; \mathbb{C}^m)$ with $p_* = \frac{p(n+2)}{n+p+2}$, then $|\nabla_x(u\chi)| + |D_t^{1/2}(u\chi)| + |u\chi| \in \mathrm{L}^p(\mathbb{R}^{n+1})$.*

(iii) *With the assumption of (ii), u is locally bounded and Hölder continuous in time of exponent $\alpha = \frac{1}{2} - \frac{1}{p}$ with values in $\mathrm{L}^p_{\mathrm{loc}}(Q_0; \mathbb{C}^m)$.*
(iv) *With the assumption of (ii), the spatial gradient $\nabla_x u$ satisfies reverse Hölder inequalities with higher exponent p.*

Theorems 1.2 and 1.1 are a consequence of (ii) and (iii) and so we only need to concentrate on the final theorem. As mentioned before, the higher integrability of $\nabla_x u$ was proved in [11] when $f = 0, F = 0$ by means of the classical Gehring lemma and was generalized to non zero right-hand side in [7], but with stronger requirement on f compared to (iv) and $F = 0$. Such results have impact on partial regularity of nonlinear systems [7, 10, 11]. We present precise versions of (iii) and (iv) including local estimates in Sections 7 and 8.

To finish this introduction, let us draw the reader's attention to the following observation concerning (i). The usual starting point of Lions variational approach is $\mathrm{L}^2_{\mathrm{loc}}$-valued continuity in time of solutions $u \in \mathrm{L}^2_{\mathrm{loc}}(I_0; \mathrm{W}^{1,2}_{\mathrm{loc}}(Q_0; \mathbb{C}^m))$ for which $\partial_t u \in \mathrm{L}^2_{\mathrm{loc}}(I_0; \mathrm{W}^{-1,2}_{\mathrm{loc}}(Q_0; \mathbb{C}^m))$; here f in (1.1) has integrability exponent $q' < 2$, so that $\partial_t u$ belongs to a larger space and the argument does not apply. One way to overcome this difficulty would be to begin with $u \in \mathrm{L}^\infty_{\mathrm{loc}}(I_0; \mathrm{L}^2_{\mathrm{loc}}(Q_0; \mathbb{C}^m))$ as well, which by a standard Gagliardo-Nirenberg inequality implies $u \in \mathrm{L}^q_{\mathrm{loc}}(\Omega; \mathbb{C}^m)$ and allows one to run the Lions argument. This is the approach taken for instance in [21]. However, and this is an observation we have not found in the literature, both the local L^2 boundedness and local L^q integrability follow from the hypotheses, again thanks to the global variational approach and the use of half-order derivatives. We shall present a proof of this result in Section 3. The estimates that come with this fact are essentially used in our proofs. For example, we obtain in Section 4 the important Caccioppoli inequalities under our assumption.

2. Notation

Most of our notation is standard. One exception is that for X a Banach space we let X^* be the (anti-)dual space of conjugate linear functionals on X. With regard to parabolic systems, we use the following notions.

Ellipticity. In what follows we assume that

$$A(t,x) = (A^{\alpha,\beta}_{i,j}(t,x))^{\alpha,\beta=1,\ldots,m}_{i,j=1,\ldots,n} \in \mathrm{L}^\infty(\mathbb{R}^{n+1}; \mathcal{L}(\mathbb{C}^{mn})) \tag{2.1}$$

and that there exist $\lambda > 0$ and $\kappa \geq 0$ such that the (weak) Gårding inequality

$$\mathrm{Re} \int_{\mathbb{R}^n} A(t,x)\nabla_x u(x) \cdot \overline{\nabla_x u(x)} \, dx \geq \lambda \int_{\mathbb{R}^n} |\nabla_x u(x)|^2 \, dx - \kappa \int_{\mathbb{R}^n} |u(x)|^2 \, dx \tag{2.2}$$

holds for all $u \in \mathrm{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$, uniformly in $t \in \mathbb{R}$. Our notation is $A(t,x)\nabla_x u(x) \cdot \overline{\nabla_x u(x)} := A^{\alpha,\beta}_{i,j}(t,x)\partial_j u^\beta(x)\overline{\partial_i u^\alpha(x)}$ with summation convention on repeated indices. We shall refer to λ, κ and an upper bound for the L^∞-norm of A as *ellipticity* and to n and the number $m \geq 1$ of equations as *dimensions*.

Let us remark that or the local results we are after, it is no serious restriction to define A on all of $\mathbb{R}^{n+1}$. Indeed, if, for some open interval $I_0 \subset \mathbb{R}$ and ball $Q_0 \subset \mathbb{R}^n$, $A \in \mathrm{L}^\infty(I_0 \times Q_0; \mathcal{L}(\mathbb{C}^{mn}))$ satisfies (2.2) for all $u \in \mathrm{W}^{1,2}_0(Q_0; \mathbb{C}^m)$ uniformly in $t \in I_0$, then given $\varepsilon \in (0,1)$, there are coefficients $\tilde{A}$ with $\tilde{A} = A$ on $(1-\varepsilon)^2 I_0 \times (1-\varepsilon)Q_0$ that satisfy (2.2) for all $u \in \mathrm{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ uniformly in $t \in \mathbb{R}$. Ellipticity for $\tilde{A}$ is possibly different and may depend on ε, see Lemma A.1 in the appendix. Of course, if $A \in \mathrm{L}^\infty(I_0 \times Q_0; \mathcal{L}(\mathbb{C}^{mn}))$ is *strongly elliptic* as in [11], then we could simply extend by the identity matrix outside of $I_0 \times Q_0$.

Weak solutions. Let I_0 be an open interval, Q_0 be an open ball of $\mathbb{R}^n$ and $\Omega := I_0 \times Q_0$. We denote by $\ell(I_0)$ the length of I_0 and by $r(Q_0)$ the radius of Q_0. Given $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathbb{C}^m)$ and $F \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathbb{C}^{mn})$, we say that u is a *weak solution* to $\partial_t u - \mathrm{div}_x A(t,x)\nabla_x u = f + \mathrm{div}_x F$ in Ω if $u \in \mathrm{L}^2_{\mathrm{loc}}(I_0, \mathrm{W}^{1,2}_{\mathrm{loc}}(Q_0; \mathbb{C}^m))$

and for all $\phi \in \mathrm{C}_0^\infty(\Omega;\mathbb{C}^m)$,

$$\begin{aligned}\iint_\Omega A(t,x)\nabla_x u(t,x)\cdot\overline{\nabla_x\phi(t,x)}\,dxdt - \iint_\Omega u(t,x)\overline{\partial_t\phi(t,x)}\,dxdt \\ = \iint_\Omega f(t,x)\overline{\phi(t,x)}\,dxdt - \iint_\Omega F(t,x)\cdot\overline{\nabla_x\phi(t,x)}\,dxdt.\end{aligned} \tag{2.3}$$

Here, our notation is $(F\cdot\overline{\nabla_x\phi})^\alpha = F^{\alpha,i}\overline{\partial_i\phi}$. Having posed the setup, we are going to ignore the target spaces $\mathbb{C}^m$ or $\mathbb{C}^{mn}$ in our notation whenever the context will be clear. Similarly, we do not write the Lebesgue measures dx and dt when the context is clear and we abbreviate $\nabla := \nabla_x$ and $\operatorname{div} := \operatorname{div}_x$ for the gradient and divergence in the spatial variables x, respectively.

Fractional time derivatives. In the following $D_t^{1/2}$ and H_t denote the *half-order time derivative* and *Hilbert transform in time* defined on $\mathcal{S}'(\mathbb{R})/\mathbb{C}$, the tempered distributions modulo constants, through the Fourier symbols $|\tau|^{1/2}$ and $-\mathrm{i}\operatorname{sgn}(\tau)$, respectively, see Section 3 in [4] for summarizing properties. In particular, the time derivative factorizes as $\partial_t = D_t^{1/2}H_tD_t^{1/2}$. We shall use the space $\mathrm{H}^{1/2}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^n))$ of functions in $\mathrm{L}^2(\mathbb{R}^{n+1})$ such that $D_t^{1/2}f\in\mathrm{L}^2(\mathbb{R}^{n+1})$. Here, we identify $\mathrm{L}^2(\mathbb{R}^{n+1})$ with $\mathrm{L}^2(\mathbb{R};\mathrm{L}^2(\mathbb{R}^n))$ and having said this, we extend $D_t^{1/2}$ and H_t to $\mathbb{R}^{n+1}$ by acting only on the time variable.

More generally, for $1<p<\infty$ we introduce the spaces $\mathrm{H}^{1/2,p}(\mathbb{R};\mathrm{L}^p(\mathbb{R}^n))$ of functions in $\mathrm{L}^p(\mathbb{R}^{n+1})$ such that $D_t^{1/2}f\in\mathrm{L}^p(\mathbb{R}^{n+1})$ with norm $(\|f\|_p^p+\|D_t^{1/2}f\|_p^p)^{1/p}$. For the sake of completeness only, we remark that up to equivalent norms these are the (vector-valued) Bessel potential spaces usually denoted by the same symbol [5]. We also note that $\mathrm{C}_0^\infty(\mathbb{R}^{n+1})$ is dense in these spaces using smooth convolution and truncation. Lebesgue space norms are denoted with the usual symbol $\|\cdot\|_p$.

3. Proof of Theorem 1.3(i)

We begin with a result on $\mathbb{R}^{n+1}$. Consider the Hilbert space $V := \mathrm{L}^2(\mathbb{R};\mathrm{W}^{1,2}(\mathbb{R}^n))$ with norm $\|u\|_V := (\|u\|_2^2+\|\nabla u\|_2^2)^{1/2}$. We recall that $q = 2+\frac{4}{n}$ and let q' be its dual exponent, $\frac{1}{q}+\frac{1}{q'}=1$.

Proposition 3.1. *Let $f\in\mathrm{L}^{q'}(\mathbb{R}^{n+1})$ and $F\in\mathrm{L}^2(\mathbb{R}^{n+1})$. Consider $v\in V$ a weak solution to $\partial_t v-\operatorname{div}A(t,x)\nabla v = f+\operatorname{div}F$ in $\mathbb{R}^{n+1}$. Then*

(i) $v\in\mathrm{H}^{1/2}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^n))$,
(ii) $v\in\mathrm{L}^q(\mathbb{R}^{n+1})$,
(iii) $v\in\mathrm{C}_0(\mathbb{R};\mathrm{L}^2(\mathbb{R}^n))$ *and $t\mapsto\|v(t,\cdot)\|_2^2$ is absolutely continuous on $\mathbb{R}$,*

with

$$\sup_{t\in\mathbb{R}}\|v(t,\cdot)\|_2+\|v\|_q+\|D_t^{1/2}v\|_2\lesssim\|v\|_V+\|f\|_{q'}+\|F\|_2.$$

The implicit constant depends only on dimensions and ellipticity.

We need a few lemmas to prepare the proof of Proposition 3.1.

Lemma 3.2. *Let $1<p<n+2$ and $\frac{1}{p^*}:=\frac{1}{p}-\frac{1}{n+2}$. Then for all $\phi\in\mathrm{C}_0^\infty(\mathbb{R}^{n+1})$,*

$$\|\phi\|_{p^*}\lesssim\|\nabla\phi\|_p+\|D_t^{1/2}\phi\|_p.$$

Remark 3.3. Note that in the particular case $p=2$ we have $p^*=q$.

Proof. Let $\mathcal{F}$ be the Fourier transform on $\mathbb{R}^{n+1}$ and let (τ,ξ) be the Fourier variable corresponding to (t,x). The Sobolev inequality in parabolic scaling from [14] gives $\|\phi\|_{p^*}\lesssim\|\mathcal{F}^{-1}((\mathrm{i}\tau+|\xi|^2)^{1/2}\mathcal{F}\phi)\|_p$. So, in order to conclude, it suffices to remark that the operators defined on the Fourier side by multiplication with $(\mathrm{i}\tau+|\xi|^2)^{1/2}/(|\tau|^{1/2}+|\xi|)$ and $\xi/|\xi|$ are bounded on $\mathrm{L}^p(\mathbb{R}^{n+1})$ by the Marcinkiewicz multiplier theorem, see Corollary 5.2.5 in [13]. □

Consider now the Hilbert space $E := V\cap\mathrm{H}^{1/2}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^n))$ with norm $\|u\|_E := (\|u\|_2^2+\|\nabla u\|_2^2+\|D_t^{1/2}u\|_2^2)^{1/2}$. The following result is basically that of [15] but we repeat the short argument for the reader's convenience.

Lemma 3.4. *The operator $\mathcal{L} := \partial_t - \operatorname{div} A(t,x)\nabla + \kappa + 1$ can be defined as a bounded operator from E to its dual E^* via*

$$\langle \mathcal{L}u, v\rangle := \iint_{\mathbb{R}^{n+1}} A\nabla u \cdot \overline{\nabla v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} + (\kappa+1) u \cdot \overline{v}\, dxdt, \qquad u, v \in E.$$

This operator is invertible and its norm as well as the norm of the inverse depend only on ellipticity and dimensions.

Proof. The $E \to E^*$ boundedness of $\mathcal{L}$ is clear by definition. Next, for the invertibility, the form

$$a_\delta(u,v) := \iint_{\mathbb{R}^{n+1}} A\nabla u \cdot \overline{\nabla(1+\delta H_t)v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2}(1+\delta H_t)v} + (\kappa+1)u \cdot \overline{(1+\delta H_t)v}\, dxdt,$$

for $u, v \in E$, is bounded and satisfies an accretivity bound for $\delta > 0$ sufficiently small. Indeed, from the ellipticity condition and the fact that the Hilbert transform is L^2-isometric, skew-adjoint and commutes with $D_t^{1/2}$ and ∇,

$$\operatorname{Re} a_\delta(u,u) \geq (\lambda - \|A\|_\infty \delta)\|\nabla u\|_2^2 + \delta \|D_t^{1/2} u\|_2^2 + \|u\|_2^2.$$

As

$$\langle \mathcal{L}u, (1+\delta H_t)v\rangle = a_\delta(u,v), \qquad u, v \in E,$$

and since $(1+\delta^2)^{-1/2}(1+\delta H_t)$ is isometric on E as is seen using its symbol $(1+\delta^2)^{-1/2}(1 - \mathrm{i}\delta \operatorname{sgn} \tau)$, it follows from the Lax-Milgram lemma that $\mathcal{L}$ is invertible from E onto E^*. □

In the following, we shall denote $V_p := V \cap \mathrm{L}^p(\mathbb{R}^{n+1})$, $1 < p < \infty$, which becomes a Banach space for $\|v\|_{V_p} = \max(\|v\|_V, \|v\|_p)$. Of course, we have $V_2 = V$.

Lemma 3.5. *Let $1 < p < \infty$ and consider a function $v \in V_p$ such that $\partial_t v \in V_p^*$. Then $v \in \mathrm{C}_0(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^n))$ and $t \mapsto \|v(t,\cdot)\|_2^2$ is absolutely continuous on $\mathbb{R}$ with*

$$\sup_{t\in\mathbb{R}} \|v(t,\cdot)\|_2^2 \leq 2\|v\|_{V_p}\|\partial_t v\|_{V_p^*}.$$

Proof. Since $\mathrm{C}_0^\infty(\mathbb{R}^{n+1})$ is dense in both V and $\mathrm{L}^p(\mathbb{R}^{n+1})$, the dual V_p^* of V_p can be identified with $V^* + \mathrm{L}^{p'}(\mathbb{R}^{n+1}) = \mathrm{L}^2(\mathbb{R}; \mathrm{W}^{-1,2}(\mathbb{R}^n)) + \mathrm{L}^{p'}(\mathbb{R}^{n+1})$ via the sesquilinear duality given by

$$(f, v) = \int_{\mathbb{R}} \langle f_1(t,\cdot), v(t,\cdot)\rangle\, dt + \iint_{\mathbb{R}^{n+1}} f_2(t,x)\overline{v(t,x)}\, dxdt,$$

where $v \in V_p$, $f_1 \in \mathrm{L}^2(\mathbb{R}; \mathrm{W}^{-1,2}(\mathbb{R}^n))$, $f_2 \in \mathrm{L}^{p'}(\mathbb{R}^{n+1})$ and $f = f_1 + f_2$. Here, $\langle\cdot,\cdot\rangle$ is the sesquilinear duality between $\mathrm{W}^{-1,2}(\mathbb{R}^n)$ and $\mathrm{W}^{1,2}(\mathbb{R}^n)$ extending the $\mathrm{L}^2(\mathbb{R}^n)$ inner product. The norm on V_p^* is given by the infimum over $\|f_1\|_{V^*} + \|f_2\|_{p'}$ for all such f_1, f_2, see Theorem 2.7.1 in [5]. Moreover, $\int_{\mathbb{R}} |\langle f_1(t,\cdot), v(t,\cdot)\rangle|\, dt \leq \|f_1\|_{V^*}\|v\|_V$ and $\iint_{\mathbb{R}^{n+1}} |f_2(t,x)\overline{v(t,x)}|\, dxdt \leq \|f_2\|_{p'}\|v\|_p$. Hence, the function

$$h : t \mapsto \langle f_1(t,\cdot), v(t,\cdot)\rangle + \int_{\mathbb{R}^n} f_2(t,x)\overline{v(t,x)}\, dx \tag{3.1}$$

is integrable on $\mathbb{R}$ with

$$\int_{\mathbb{R}} |h(t)|\, dt \leq \|f_1\|_{V^*}\|v\|_V + \|f_2\|_{p'}\|v\|_p. \tag{3.2}$$

It does not depend on the choice of the decomposition of f. We omit the straightforward argument.

Assume now that $v \in V_p$ and $\partial_t v \in V_p^*$. Using smooth convolution and truncation, we can approximate v by the smooth function with compact support $v_{\varepsilon,R} := \chi_R(v * \varphi_\varepsilon)$ in V_p. Here $\varepsilon > 0$ tends to 0 and $R > 0$ tends to ∞. Next, take an arbitrary real-valued $\eta \in \mathrm{C}_0^\infty(\mathbb{R})$. Integration by parts shows that

$$\iint_{\mathbb{R}^{n+1}} |v_{\varepsilon,R}(t,x)|^2 \partial_t \eta(t)\, dxdt = -2\operatorname{Re} \iint_{\mathbb{R}^{n+1}} \partial_t v_{\varepsilon,R}(t,x)\overline{v_{\varepsilon,R}(t,x)\eta(t)}\, dxdt = -2\operatorname{Re}(\partial_t v_{\varepsilon,R}, v_{\varepsilon,R}\eta).$$

Write $\partial_t v_{\varepsilon,R}(t,x) = \partial_t \chi_R(t,x)\,(v*\varphi_\varepsilon)(t,x) + \chi_R(t,x)\,(\partial_t v * \varphi_\varepsilon)(t,x)$. Clearly, $\partial_t \chi_R\,(v*\varphi_\varepsilon) \to 0$ in $\mathrm{L}^2(\mathbb{R}^{n+1})$, while $v_{\varepsilon,R}\eta \to v\eta$ in $\mathrm{L}^2(\mathbb{R}^{n+1})$. Also $\chi_R\,(\partial_t v * \varphi_\varepsilon)$ is bounded in V_p^* with weak* convergence to $\partial_t v$, while $v_{\varepsilon,R}\eta \to v\eta$ in V_p. Hence at the limit, we obtain

$$\iint_{\mathbb{R}^{n+1}} |v(t,x)|^2 \partial_t \eta(t)\,dx dt = -2\operatorname{Re}(\partial_t v, v\eta).$$

Setting $f := \partial_t v$, this precisely means that the distributional derivative of $t \mapsto \|v(t,\,\cdot\,)\|_2^2$ is the function $2\operatorname{Re} h$ with h defined in (3.1). As h is integrable, this shows that $t \mapsto \|v(t,\,\cdot\,)\|_2^2$ is absolutely continuous on $\mathbb{R}$, has limits $\ell_{\pm\infty}$ at $\pm\infty$ and satisfies

$$\|v(t,\,\cdot\,)\|_2^2 = 2\operatorname{Re}\int_{-\infty}^t h(s)\,ds + \ell_{-\infty} = -2\operatorname{Re}\int_t^{+\infty} h(s)\,ds + \ell_{+\infty}, \qquad t \in \mathbb{R}.$$

Now, $t \mapsto \|v(t,\,\cdot\,)\|_2^2$ is integrable on $\mathbb{R}$ since $v \in \mathrm{L}^2(\mathbb{R}^{n+1})$ and so the limits must be 0. Finally, the sup-norm estimate follows from (3.2) and minimization over f_1, f_2. □

We can now give the

Proof of Proposition 3.1. It follows from Lemma 3.2 and density of $\mathrm{C}_0^\infty(\mathbb{R}^{n+1})$ in E (standard mollification and truncation), that E embeds into $\mathrm{L}^q(\mathbb{R}^{n+1})$. Hence, $\mathrm{L}^{q'}(\mathbb{R}^{n+1})$ embeds into E^*. Thus, $f + (\kappa+1)v + \operatorname{div} F \in E^*$ and it follows from Lemma 3.4 that there exists $\tilde{v} \in E$ such that $\mathcal{L}\tilde{v} = f + (\kappa+1)v + \operatorname{div} F$. By definition of the respective embeddings, this means that for all $\phi \in E$,

$$\iint_{\mathbb{R}^{n+1}} A\nabla\tilde{v}\cdot\overline{\nabla\phi} + H_t D_t^{1/2}\tilde{v}\cdot\overline{D_t^{1/2}\phi} + (\kappa+1)\tilde{v}\cdot\overline{\phi}\,dx dt = \iint_{\mathbb{R}^{n+1}} (f + (\kappa+1)v)\overline{\phi} - F\cdot\overline{\nabla\phi}\,dx dt.$$

Restricting to $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^{n+1})$, we can write $\partial_t\phi = D_t^{1/2} H_t D_t^{1/2}\phi$ and see in particular that $u := v - \tilde{v} \in V$ is a weak solution to $\partial_t u - \operatorname{div} A(t,x)\nabla u + (\kappa+1)u = 0$ in $\mathbb{R}^{n+1}$. We may now apply the standard Caccioppoli inequality, see Remark 4.2 below for convenience,

$$\iint |\nabla(u\chi)|^2\,dx dt \lesssim \iint |u|^2(|\nabla\chi|^2 + |\partial_t\chi|)\,dx dt \tag{3.3}$$

for any $\chi \in \mathrm{C}_0^\infty(\mathbb{R}^{n+1})$. Choosing suitable test functions χ that converge to 1 reveals $\nabla u = 0$ as $u \in \mathrm{L}^2(\mathbb{R}^{n+1})$. Hence, u depends only on t. Again, as $u \in \mathrm{L}^2(\mathbb{R}^{n+1})$, u must be 0. It follows that $v = \tilde{v} \in E$, hence (i) is proved and $\|D_t^{1/2}v\|_2 \lesssim \|\mathcal{L}v\|_{E^*} \le (\kappa+1)\|v\|_2 + \|f\|_{q'} + \|F\|_2$ by Lemma 3.4. Applying Lemma 3.2 again and density yields (ii). Finally (iii) follows from Lemma 3.5 as $v \in V_q$ and $\partial_t v \in V_q^*$. □

We close with the

Proof of Theorem 1.3(i). Let $\chi \in \mathrm{C}_0^\infty(\Omega)$. Set $v := u\chi$. Clearly, $v \in V$ is a weak solution to $\partial_t v - \operatorname{div} A(t,x)\nabla v = \tilde{f} + \operatorname{div}\tilde{F}$ in $\mathbb{R}^{n+1}$ with

$$\begin{cases} \tilde{f} &= \chi f + \partial_t\chi u - A\nabla u\cdot\nabla\chi - F\cdot\nabla\chi, \\ \tilde{F} &= -A(u\nabla\chi) + F\chi. \end{cases} \tag{3.4}$$

Here, we suggestively use the notation for the scalar case $m = 1$ also when $m > 1$ as we shall only be interested in norm estimates. Using the assumption on f, F, the properties of u and $q' < 2$, we see that $\tilde{f} \in \mathrm{L}^{q'}(\mathbb{R}^{n+1})$, $\tilde{F} \in \mathrm{L}^2(\mathbb{R}^{n+1})$. The conclusion follows from Proposition 3.1. In particular, we get

$$\|u\chi\|_q \lesssim \|\tilde{f}\|_{q'} + \|\tilde{F}\|_2 + \|u\chi\|_2 + \|\nabla(u\chi)\|_2. \tag{3.5}$$ □

4. Elementary estimates

In this section, we will assume $u \in \mathrm{L}^2_{\mathrm{loc}}(I_0; \mathrm{W}^{1,2}_{\mathrm{loc}}(Q_0))$ is a weak solution of (1.1) in Ω with $f \in \mathrm{L}^{q'}_{\mathrm{loc}}(\Omega)$ and $F \in \mathrm{L}^2_{\mathrm{loc}}(\Omega)$. The estimates provided here are quite standard but our unconventional assumption on f requires to provide proofs, though. Throughout, we let $I \times Q \subset \Omega$ be a parabolic cylinder with $\ell \sim r^2$, where $\ell := \ell(I)$, $r := r(Q)$, and we let $\gamma > 1$ be such that $\overline{\gamma^2 I \times \gamma Q} \subset \Omega$.

Let us begin with the Caccioppoli estimate.

Proposition 4.1. *For u, f, F as above, one has*

$$\left(\iint_{I\times Q} |\nabla u|^2\right)^{1/2} \lesssim \frac{1}{r}\left(\iint_{\gamma^2 I\times\gamma Q} |u|^2\right)^{1/2} + \left(\iint_{\gamma^2 I\times\gamma Q} |F|^2\right)^{1/2} + r\left(\iint_{\gamma^2 I\times\gamma Q} |f|^{q'}\right)^{1/q'}, \tag{4.1}$$

where the implicit constant depends only on ellipticity, dimensions, γ and the constants controlling the ratio r^2/ℓ.

Proof. By scaling we may assume that $r = 1$ as our hypotheses are invariant under dilations. We pick $\chi \in \mathrm{C}_0^\infty(\mathbb{R}^{n+1})$, real-valued, with $\chi = 1$ on $I \times Q$ and support contained in $\gamma^2 I \times \gamma Q$. Write the equation satisfied by $v := u\chi$ as $\partial_t v = \tilde{f} + \operatorname{div}(A\nabla v + \tilde{F})$ with $\tilde{f}, \tilde{F}$ given by (3.4). We have $v \in V_q$ thanks to Proposition 3.1 and from (3.4) and the assumptions on f,F we can infer $\partial_t v \in V_q^*$. Thus, we know from the proof of Lemma 3.5 that for almost every $t \in \mathbb{R}$,

$$\frac{d}{dt}\|v(t,\,\cdot)\|_2^2 = -2\operatorname{Re}\int (A\nabla v + \tilde{F})\cdot\overline{\nabla v}\,dx + 2\operatorname{Re}\int \tilde{f}\,\overline{v}\,dx. \tag{4.2}$$

Integration with respect to t, together with Fubini's theorem and limits 0 at $\pm\infty$, yields

$$0 = -2\operatorname{Re}\iint (A\nabla v + \tilde{F})\cdot\overline{\nabla v}\,dxdt + 2\operatorname{Re}\iint \tilde{f}\,\overline{v}\,dxdt. \tag{4.3}$$

Isolating the term $A\nabla v \cdot \overline{\nabla v}$ and using Young's inequality leads to

$$2\operatorname{Re}\iint A\nabla v\cdot\overline{\nabla v} \le \iint \lambda|\nabla v|^2 + \lambda^{-1}|\tilde{F}|^2 + 2|\tilde{f}v|.$$

We now apply Gårding's inequality (2.2) and hide one term with $\lambda|\nabla v|^2$ to obtain

$$\lambda\iint |\nabla v|^2 \le \iint \lambda^{-1}|\tilde{F}|^2 + 2|\tilde{f}v| + \kappa|v|^2. \tag{4.4}$$

Let now $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^{n+1})$ be another real-valued function, supported in $\gamma^2 I \times \gamma Q$ with $\phi = 1$ on the support of χ. Unraveling the terms using the definitions of v, $\tilde{f}$ and $\tilde{F}$ gives us

$$\begin{aligned}\lambda\iint |\nabla(u\chi)|^2 &\lesssim \iint |u|^2(|\nabla\chi|^2 + |\partial_t\chi| + \kappa|\chi|^2) + \iint |u\chi A\nabla u\cdot\nabla\chi| \\ &\quad + \iint |F|^2(|\chi|^2 + |\nabla\chi|^2) + \|f\phi\|_{q'}\|u\chi^2\|_q,\end{aligned}$$

where we wrote $\chi f\overline{v} = f\phi\,\overline{u\chi^2}$ and applied Hölder's inequality to obtain the final term on the right. Now, we write $\chi A\nabla u\cdot\nabla\chi = A\nabla(u\chi)\cdot\nabla\chi - A(u\nabla\chi)\cdot\nabla\chi$ on the right and use Young's inequality to hide again the contribution of $\nabla(u\chi)$ appearing on the right. The result is

$$\iint |\nabla(u\chi)|^2 \lesssim \iint |u|^2(|\nabla\chi|^2 + |\partial_t\chi| + \kappa|\chi|^2) + \iint |F|^2(|\chi|^2 + |\nabla\chi|^2) + \|f\phi\|_{q'}\|u\chi^2\|_q. \tag{4.5}$$

Next, we invoke $u \in \mathrm{L}^q_{\mathrm{loc}}(\Omega)$ with the bound (3.5) as proved in Theorem 1.3(i),

$$\|u\chi^2\|_q \lesssim \|f'\|_{q'} + \|F'\|_2 + \|u\chi^2\|_2 + \|\nabla(u\chi^2)\|_2.$$

Here, f', F' are defined as in (3.4) but with χ replaced by χ^2. Again from the explicit form of f', F' we see that in the product $\|f\phi\|_{q'}\|u\chi^2\|_q$ the only terms that are not welcome on the right-hand side of (4.5) are $\|f\phi\|_{q'}\|\nabla(u\chi^2)\|_2$ and $\|f\phi\|_{q'}\|A\nabla u\cdot\nabla(\chi^2)\|_2$. (For some of the other terms we keep in mind that $q' \le 2$ holds by definition.) For the first of these remaining terms we write $\nabla(u\chi^2) = \chi\nabla(u\chi) + u\chi\nabla\chi$ and for the second one $A\nabla u\cdot\nabla(\chi^2) = 2A\nabla(u\chi)\cdot\nabla\chi - 2A(u\nabla\chi)\cdot\nabla\chi$. Hence, in both cases we can hide $\iint|\nabla(u\chi)|^2$ using Young's inequality from (4.5) and we conclude for (4.1) from the properties of χ and ϕ. □

Remark 4.2. Suppose we had the equation $\partial_t u - \operatorname{div} A(t,x)\nabla u + (\kappa+1)u = 0$ in Ω to start with. Then $v \in V_q$, $\partial_t v \in V_q^*$ (allowing us to use (4.2)) and in the argument above we would get the extra negative term $-(\kappa+1)\iint v\cdot\overline{v}$ on the right-hand side of (4.3). Hence, we can ignore the integral $\iint \kappa|v|^2$ in the upper bound (4.4) and all subsequent estimates. In particular, we obtain (4.5) with $f = 0$, $F = 0$ and without $|u|^2\kappa|\chi|^2$, which is exactly (3.3).

In combination with (3.5), we can prove further local estimates. In particular, we obtain a reverse Hölder inequality for u.

Proposition 4.3. *For u, f, F as above, one has*

$$\left(\fint\!\!\fint_{I\times Q} |\nabla u|^2\right)^{1/2} \lesssim \frac{1}{r}\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |u| + \left(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |F|^2\right)^{1/2} + r\left(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |f|^{q'}\right)^{1/q'} \tag{4.6}$$

and

$$\left(\fint\!\!\fint_{I\times Q} |u|^q\right)^{1/q} \lesssim \fint\!\!\fint_{\gamma^2 I\times\gamma Q} |u| + r\left(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |F|^2\right)^{1/2} + r^2\left(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |f|^{q'}\right)^{1/q'}, \tag{4.7}$$

where the implicit constants depend only on ellipticity, dimensions, γ and the constants controlling the ratio r^2/ℓ.

Proof. The equation (4.7) with $\big(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |u|^2\big)^{1/2}$ on the right-hand side follows from (3.5) and Proposition 4.1 – at least when $r = 1$, which suffices since we can rescale as before. The improvement to $\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |u|$ follows from a classical self-improvement feature of reverse Hölder inequalities, see Theorem 2 in [16]. A simple proof that applies in our situation can be found in Theorem B.1 of [6]. Thus, incorporating this in (4.1) gives us (4.6). □

5. A Gehring type lemma with tail

We provide here the main real analysis lemma to obtain our estimates. For a ball $Q \subset \mathbb{R}^n$ and an interval $I \subset \mathbb{R}$ with $\sqrt{\ell(I)} = r(Q) := r$ we write $B := I \times Q$. If (t, x) is the center of B, we also use the notation $B = B((t, x), r)$ and $r = r(B)$ for such a parabolic cylinder (that is, a cylinder which is a ball in the parabolic (quasi-)metric $d((t, x),(s, y)) := \max(\sqrt{2|t-s|}, |x-y|))$. For $u \geq 0$ locally integrable and $\gamma > 1$ we let

$$a_u(B) := \sum_{j=0}^{\infty} 2^{-j-1} \fint\!\!\fint_{4^j I\times\gamma Q} u \,dxdt := \sum_{j=0}^{\infty} 2^{-j-1} \fint_{4^j I\times\gamma Q} u \,d\mu,$$

where for this section μ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$ and we use the single integral notation for simplicity. The functional a_u is an approximate identity indexed over radii of parabolic cylinders when $u \in \mathrm{L}^p(\mathbb{R}^{n+1})$ for some $p \in (1,\infty)$ in the sense that $a_u(B((t, x), r)) \to u(t, x)$ as $r \to 0$ for almost every (t, x). Indeed, introduce the maximal operators M_x and M_t on space and time variables separately. For each $j \geq 0$ we have

$$\left|\fint_{4^j I\times\gamma Q} u \,d\mu\right| \leq M_t M_x u(t, x)$$

and as $M_t M_x$ is bounded on $\mathrm{L}^p(\mathbb{R}^{n+1})$, this average converges to $u(t, x)$ for almost every point. So, the claim follows from the dominated convergence theorem for series and $\sum_{j=0}^{\infty} 2^{-j-1} = 1$. In addition, we have $a_u(B((t, x), r)) \to 0$ when $r \to \infty$ by Hölder's inequality. This last point also holds when $u \in \mathrm{L}^1(\mathbb{R}^{n+1})$.

Lemma 5.1. *Let g, f, h be be non-negative functions with $g^2, f^2, h^s \in \mathrm{L}^1(\mathbb{R}^{n+1})$, $1 < s < n+2$, and suppose that for some $A \geq 1$,*

$$\left(\fint_B g^2 \,d\mu\right)^{1/2} \leq A a_g(B) + (a_{f^2}(B))^{1/2} + r(B)(a_{h^s}(B))^{1/s}$$

holds for all parabolic cylinders B. Let $p > 2$ and suppose there are $\alpha, \beta \geq 0$ and $q_\alpha, q_\beta > 1$ (depending on p) such that

$$2\alpha + \beta = s \qquad \text{and} \qquad \frac{1}{q_\alpha} - \alpha = \frac{s}{p} = \frac{1}{q_\beta} - \frac{\beta}{n}. \tag{5.1}$$

If $|p-2|$ is sufficiently small (depending on A and dimension), then

$$\|g\|_{\mathrm{L}^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{\mathrm{L}^p(\mathbb{R}^{n+1})} + \|h^s\|^{1/s}_{\mathrm{L}^{q_\alpha}(\mathbb{R};\mathrm{L}^{q_\beta}(\mathbb{R}^n))}.$$

The implicit constant depends on A, α, β, q_α, q_β and dimension.

Proof. Let $m > 0$. Denote $g_m := \min(g, m)$. By the Cavalieri principle we have

$$\int_{\mathbb{R}^{n+1}} g_m^{p-2} g^2 \, d\mu = (p-2) \int_0^m \lambda^{p-2-1} g^2(\{g > \lambda\}) \, d\lambda,$$

where $g^2(A) := \int_A g^2 \, d\mu$. We define three functions

$$\begin{aligned} G(t,x,r) &:= a_g(B((t,x),r)), \\ F(t,x,r) &:= a_{f^2}(B((t,x),r))^{1/2}, \\ H(t,x,r) &:= r \cdot a_{h^s}(B((t,x),r))^{1/s} \end{aligned}$$

and for $\lambda > 0$, we denote $E_\lambda := \{g > \lambda\}$. We have

$$\lim_{r \to 0} G(t,x,r) = g(t,x)$$

for almost every (t,x) by the discussion before the statement of the lemma and we define $\tilde{E}_\lambda$ as the subset of E_λ where this holds. We also note

$$\lim_{r \to \infty} \left(G(t,x,r) + F(t,x,r) + H(t,x,r) \right) = 0$$

for all (t,x), using the global assumptions on g, f, h and $s < n+2$.

By definition, if $(t,x) \in \tilde{E}_\lambda$, then

$$\lim_{r \to 0} G(t,x,r) > \lambda,$$

and thus for $(t,x) \in \tilde{E}_\lambda$ we can define the stopping time radius

$$r_{t,x} := \sup\{r > 0 : G(t,x,r) + F(t,x,r) + H(t,x,r) > \lambda\}.$$

We readily see that $\sup_{(t,x) \in \tilde{E}_\lambda} r_{t,x} < \infty$. Indeed, since G, F, H are continuous functions of $r > 0$ for fixed (t,x), we have at $r = r_{t,x}$ equality $G(t,x,r) + F(t,x,r) + H(t,x,r) = \lambda$ and thus either $G(t,x,r) \geq \lambda/3$ or $F(t,x,r) \geq \lambda/3$ or $H(t,x,r) \geq \lambda/3$. In the last case for example, we obtain

$$r^{n+2-s} (\lambda/3)^s \lesssim \int_{\mathbb{R}^{n+1}} h^s \, d\mu < \infty$$

and the other cases give us an upper bound on r in a similar manner. By the Vitali covering lemma, there exists an absolute constant K and a countable collection of balls $\{B((t_i,x_i),r_i)\} = \{B_i\}$ with $r_i = r_{t_i,x_i}$ such that the $\frac{1}{K} B_i$ are pairwise disjoint and $\tilde{E}_\lambda \subset \cup_i B_i$. (A value of K can be computed explicitly by following the usual proofs in this particular quasi-metric.)

Now, using the hypothesis for each B_i and pairwise disjointness of the balls $\frac{1}{K} B_i$, we find

$$\begin{aligned} g^2(\tilde{E}_\lambda) &\leq \sum_i g^2(B_i) \leq \sum_i \mu(B_i) \Big(A a_g(B_i) + (a_{f^2}(B_i))^{1/2} + r_i (a_{h^s}(B_i))^{1/s} \Big)^2 \\ &\leq A^2 \sum_i \mu(B_i) \lambda^2 \leq A^2 K^{n+2} \sum_i \mu(\tfrac{1}{K} B_i) \lambda^2 \leq A^2 K^{n+2} \lambda^2 \mu\Big(\bigcup_i B_i \Big). \end{aligned}$$

Let M_x^β be the fractional maximal function with respect to the x-variable:

$$M_x^\beta v(t,x) := \sup_{Q \ni x} r(Q)^\beta \fint_Q |v(t,y)| \, dy.$$

Similarly, define M_t^α with respect to the t-variable. Since $2\alpha + \beta = s$, the parabolic scaling $r(B) = r(Q) = \sqrt{\ell(I)}$ yields $r(B)^s = r(Q)^\beta \times \ell(I)^\alpha$. Thus, it follows from the definition of r_i that

$$\bigcup_i B_i \subset \{M_t M_x g \geq \lambda/3\} \cup \{M_t M_x (f^2) \geq (\lambda/3)^2\} \cup \{M_t^\alpha M_x^\beta (h^s) \geq (\lambda/3)^s\} =: S_\lambda.$$

We thus have established

$$g^2(E_\lambda) = g^2(\tilde{E}_\lambda) \leq A^2 K^{n+2} \lambda^2 \mu(S_\lambda).$$

Going back to the start of the proof, so far we have found

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} g_m^{p-2} g^2 \, d\mu &= (p-2) \int_0^m \lambda^{p-3} g^2(\{g > \lambda\}) \, d\lambda \\ &\leq A^2 K^{n+2} (p-2) \int_0^m \lambda^{p-1} \mu(S_\lambda) \, d\lambda \\ &\leq A^2 K^{n+2} (I + II + III),\end{aligned} \tag{5.2}$$

where the integrals I, II, III correspond to the decomposition of S_λ above. By the Cavalieri principle, we obtain for $p > 2$,

$$II \leq \frac{p-2}{p} \int_{\mathbb{R}^{n+1}} M_t M_x (f^2)^{p/2} \, d\mu \lesssim \frac{p}{p-2} \int_{\mathbb{R}^{n+1}} f^p \, d\mu$$

by iterating the two maximal function $\mathrm{L}^{p/2}$ bounds, so that the implicit constant depends only on the dimension n. Note that $p > 2$ and that p is determined by the other parameters in (5.1). Similarly,

$$III \leq \frac{p-2}{p} \int_{\mathbb{R}^{n+1}} M_t^\alpha M_x^\beta (h^s)^{p/s} \, d\mu.$$

By hypothesis, we have exponents $q_\alpha, q_\beta > 1$ such that

$$\frac{1}{q_\alpha} - \alpha = \frac{s}{p} = \frac{1}{q_\beta} - \frac{\beta}{n}.$$

With a slight abuse in our notation, ignoring the other variable, these are precisely the conditions guaranteeing that $M_t^\alpha : \mathrm{L}^{q_\alpha}(\mathbb{R}) \to \mathrm{L}^{p/s}(\mathbb{R})$ and $M_x^\beta : \mathrm{L}^{q_\beta}(\mathbb{R}^n) \to \mathrm{L}^{p/s}(\mathbb{R}^n)$ are bounded, see Theorem 3.1.4 in [1]. Now, using this and Minkowski's inequality along with $sq_\alpha/p = 1 - \alpha q_\alpha \leq 1$ in the second step, we see that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} M_t^\alpha M_x^\beta (h^s)^{p/s} \, d\mu &\lesssim \int_{\mathbb{R}^n} \bigg(\int_{\mathbb{R}} M_x^\beta (h^s)^{q_\alpha} \, dt \bigg)^{p/(sq_\alpha)} \, dx \\ &\leq \bigg(\int_{\mathbb{R}} \bigg(\int_{\mathbb{R}^n} M_x^\beta (h^s)^{p/s} \, dx \bigg)^{sq_\alpha/p} \, dt \bigg)^{p/(sq_\alpha)} \lesssim \|h^s\|_{\mathrm{L}^{q_\alpha}(\mathbb{R}; \mathrm{L}^{q_\beta}(\mathbb{R}^n))}^{p/s}\end{aligned}$$

with implicit constant depending on $\alpha, \beta, q_\alpha, q_\beta$ and dimension. The remaining term is

$$I = (p-2) \int_0^m \lambda^{p-1} \mu(\{M_t M_x g \geq \lambda/3\}) \, d\lambda.$$

To handle I, we first notice that

$$\{M_t M_x g \geq \lambda/3\} \subset \{M_t M_x (g \mathbf{1}_{\{g > \lambda/6\}}) \geq \lambda/6\}.$$

From this estimate and the weak type $(\frac{3}{2}, \frac{3}{2})$-bound for the iterated maximal function (which follows from the strong type $(\frac{3}{2}, \frac{3}{2})$), we obtain

$$\mu(\{M_t M_x g \geq \lambda/3\}) \leq \frac{C}{\lambda^{3/2}} \int_{\{g > \lambda/6\}} g^{3/2} \, d\mu$$

for a dimensional constant C. Using this bound in I yields

$$\begin{aligned}I &\leq C(p-2) \int_0^m \lambda^{p-5/2} \int_{\{g > \lambda/6\}} g^{3/2} \, d\mu \, d\lambda \\ &= C(p-2) \int_{\mathbb{R}^{n+1}} g^{3/2} \int_0^{\min(m, 6g)} \lambda^{p-5/2} \, d\lambda \, d\mu \\ &= C 6^{p-3/2} \frac{p-2}{p-3/2} \int_{\mathbb{R}^{n+1}} g_{m/6}^{p-3/2} g^{3/2} \, d\mu \\ &\leq C 6^{p-3/2} \frac{p-2}{p-3/2} \int_{\mathbb{R}^{n+1}} g_m^{p-2} \, g^2 \, d\mu.\end{aligned}$$

Choosing $p-2>0$ small enough, depending on A and dimension, we see from (5.2) that

$$\int_{\mathbb{R}^{n+1}} g_m^{p-2} g^2 \, d\mu \leq \frac{1}{2} \int_{\mathbb{R}^{n+1}} g_m^{p-2} g^2 \, d\mu + C\Big(\|f\|_{\mathrm{L}^p(\mathbb{R}^{n+1})}^p + \|h^s\|_{\mathrm{L}^{q_\alpha}(\mathbb{R};\mathrm{L}^{q_\beta}(\mathbb{R}^n))}^{p/s} \Big)$$

for some constant C depending on $\alpha, \beta, q_\alpha, q_\beta$ and dimension. This finishes the proof after hiding the first term and taking the limit $m \to \infty$. □

Remark 5.2. The same estimate also holds with the mixed norm in different order as we are free to interchange the fractional maximal functions. If we want $q_\alpha = q_\beta$, then (5.1) reveals that α, β are uniquely determined by s, p and n. Hence, for each p there is at most one such pair.

In the application to our parabolic PDE, we consider special values for the auxiliary parameters in Lemma 5.1.

Corollary 5.3. *Suppose the setup of Lemma 5.1 with $s = q' = (2+\frac{4}{n})' = \frac{2n+4}{n+4}$. Then for $p>2$ with $p-2$ small enough depending on A and dimension,*

$$\|g\|_{\mathrm{L}^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{\mathrm{L}^p(\mathbb{R}^{n+1})} + \|h\|_{\mathrm{L}^{p_*}(\mathbb{R}^{n+1})},$$

where $p_* = \frac{p(n+2)}{n+p+2}$. The implicit constant depends on A, p and n.

Proof. We have $s = q' = (2+\frac{4}{n})' = \frac{2n+4}{n+4}$. We want $q_\alpha = q_\beta$ in Lemma 5.1, and so we can solve in (5.1) for

$$q_\alpha = q_\beta = \frac{p(n+2)}{s(p+n+2)}$$

corresponding to

$$\alpha = \frac{1}{q_\alpha} - \frac{s}{p}, \qquad \beta = \frac{n}{q_\beta} - \frac{ns}{p}.$$

Indeed, we have $\alpha, \beta > 0$ due to $q_\alpha = q_\beta < \frac{p}{s}$ and $q_\alpha = q_\beta > 1$ follows from

$$s = \frac{2(n+2)}{n+4} < \frac{p(n+2)}{n+p+2} = q_\alpha s = q_\beta s,$$

since we have $p > 2$. □

6. The first proof of Theorem 1.3(ii)

Our key lemma is the following non-local reverse Hölder inequality. A special case was proved in [4] for the purpose of obtaining non-tangential maximal estimates for the half-order time derivative of certain weak solutions.

Lemma 6.1. *Let $\tilde{f} \in \mathrm{L}^{q'}(\mathbb{R}^{n+1})$ and $\tilde{F} \in \mathrm{L}^2(\mathbb{R}^{n+1})$. Consider $v \in V$ a weak solution to $\partial_t v - \operatorname{div} A(t,x)\nabla v = \tilde{f} + \operatorname{div} \tilde{F}$ in $\mathbb{R}^{n+1}$. Let $\gamma > 1$ and let $I \times Q$ be a parabolic cylinder with $\ell(I) \sim r(Q)^2$. Then for $g := |\nabla v| + |D_t^{1/2} v| + |H_t D_t^{1/2} v|$,*

$$\Big(\fint\!\!\fint_{I\times Q} g^2 \Big)^{1/2} \lesssim \sum_{k\in\mathbb{Z}} \frac{1}{1+|k|^{3/2}} \fint\!\!\fint_{I_k \times \gamma Q} g + \sum_{k\in\mathbb{Z}} \frac{1}{1+|k|^{3/2}} \Big(\fint\!\!\fint_{I_k\times\gamma Q} |\tilde{F}|^2 \Big)^{1/2} + r(Q) \sum_{k\in\mathbb{Z}} \frac{1}{1+|k|^{3/2}} \Big(\fint\!\!\fint_{I_k\times\gamma Q} |\tilde{f}|^{q'} \Big)^{1/q'}. \tag{6.1}$$

Here, $I_k := k\ell(I) + I$ are the disjoint translates of I covering the real line up to a countable set. The implicit constant depends only on ellipticity, dimensions, γ and the constants controlling the ratio $r(Q)^2/\ell(I)$.

The proof follows the argument presented in Section 8 of [4] (for $f = 0$ and $F = 0$) with a few differences that we point out along the way. Duplicated arguments with [4] are omitted. In this reference, the order of variables was (x, t) and an additional spatial dimension was carried through the argument, both for the purpose of treating boundary value problems. The latter plays no role here and can be ignored. Next, u in [4] has become v here and the extra property $D_t^{1/2} v \in \mathrm{L}^2(\mathbb{R}^{n+1})$ provided by Proposition 3.1 means that v is a reinforced weak solution in the terminology there.

Proof. We remark that $g \in \mathrm{L}^2(\mathbb{R}^{n+1})$ due to Proposition 3.1 applied to $\partial_t v - \operatorname{div} A(t,x)\nabla v = \tilde{f} + \operatorname{div} \tilde{F}$ and the fact that H_t is isometric on $\mathrm{L}^2(\mathbb{R}^{n+1})$.

It suffices to prove the claim for $\gamma = 16$ since *a posteriori* a covering argument, which we leave to the reader, gives us the inequality with any $\gamma > 1$.

For simplicity, we are also going to assume $r(Q) \sim 1$ and that $I \times Q$ is centered at $(0, 0)$ as scaling and translating give us back the general estimate. Having normalized to scale 1, averages are integrals (up to numerical constants) and we reserve the use of averages for when this is necessary.

For the time being it will be enough to work with $\gamma = 8$, so that the parabolic enlargement is $64I \times 8Q$. We fix a smooth cut-off $\eta : \mathbb{R}^{n+1} \to [0, 1]$ with support in $4I \times 2Q$ that is 1 on an enlargement $\frac{9}{4}I \times \frac{3}{2}Q$. For a reason which will become clear later on, we choose η to have the product form

$$\eta(s, y) = \eta_I(s)\eta_Q(y),$$

where η_I is symmetric about 0 (the midpoint of I). For the sake of notational simplicity, we give a name to the translation sums

$$\sum(h) := \sum_{k\in\mathbb{Z}} \frac{1}{1 + |k|^{3/2}} \iint_{I_k \times 8Q} |h|.$$

Step 1: Caccioppoli and Poincaré. The estimate (4.6) with $v - c$ and $c := \fint\!\!\fint_{I\times 2Q} v$ yields

$$\Big(\iint_{4I\times 2Q} |\nabla v|^2\Big)^{1/2} \lesssim \iint_{64I\times 8Q} |v - c| + \Big(\iint_{64I\times 8Q} |\tilde{F}|^2\Big)^{1/2} + \Big(\iint_{64I\times 8Q} |\tilde{f}|^{q'}\Big)^{1/q'}.$$

Now, we can follow line by line the proof of Lemma 8.4 in [4], which relies only on Poincaré inequalities and does not use the equation, to obtain with g as in the statement,

$$\iint_{64I\times 8Q} |v - c| \lesssim \sum(g). \tag{6.2}$$

Thus,

$$\begin{aligned}\Big(\iint_{4I\times 2Q} |\nabla v|^2\Big)^{1/2} &\lesssim \sum(g) + \Big(\iint_{64I\times 8Q} |\tilde{F}|^2\Big)^{1/2} + \Big(\iint_{64I\times 8Q} |\tilde{f}|^{q'}\Big)^{1/q'} \\ &\lesssim \sum(g) + \sum_{|k|\le 64} \frac{1}{1 + |k|^{3/2}} \bigg(\Big(\iint_{I_k\times 8Q} |\tilde{F}|^2\Big)^{1/2} + \Big(\iint_{I_k\times 8Q} |\tilde{f}|^{q'}\Big)^{1/q'}\bigg).\end{aligned} \tag{6.3}$$

It remains to estimate the $\mathrm{L}^2(I \times Q)$ integrals of $H_t D_t^{1/2} v$ and $D_t^{1/2} v$. As the fractional derivatives annihilate constants, we may replace v by $v - c$ and write $v - c = \eta(v - c) + (1 - \eta)(v - c)$.

Step 2: Local terms. For the local term $w := \eta(v - c)$,

$$\iint_{I\times Q} |H_t D_t^{1/2} w|^2 + |D_t^{1/2} w|^2 \le \iint_{\mathbb{R}^{n+1}} |H_t D_t^{1/2} w|^2 + |D_t^{1/2} w|^2 = 2 \iint_{\mathbb{R}^{n+1}} |D_t^{1/2} w|^2,$$

using that H_t is isometric on $\mathrm{L}^2(\mathbb{R}^{n+1})$. Since w solves an equation of the form $\partial_t w - \operatorname{div} A\nabla w = f' + \operatorname{div} F'$, Proposition 3.1 implies

$$\|D_t^{1/2} w\|_2 \le \|w\|_V + \|f'\|_{q'} + \|F'\|_2,$$

where

$$f' := \eta\tilde{f} + \partial_t\eta(v-c) - A\nabla v\cdot\nabla\eta - \tilde{F}\cdot\nabla\eta,$$
$$F' := -A((v-c)\nabla\eta) + \tilde{F}\eta,$$

as we can read off from (3.4). Using the formulæ for f', F' and $q' < 2$ along with Hölder's inequality, we easily find

$$\|D_t^{1/2}w\|_2 \lesssim \Big(\iint_{4I\times 2Q}|v-c|^2\Big)^{1/2} + \Big(\iint_{4I\times 2Q}|\nabla v|^2\Big)^{1/2} + \Big(\iint_{4I\times 2Q}|\tilde{F}|^2\Big)^{1/2} + \Big(\iint_{4I\times 2Q}|\tilde{f}|^{q'}\Big)^{1/q'}.$$

For the first and second terms, we use (4.6) and (4.7), respectively, and then (6.2). This yields

$$\Big(\iint_{I\times Q}|H_tD_t^{1/2}w|^2 + |D_t^{1/2}w|^2\Big)^{1/2} \lesssim \sum(g) + \Big(\iint_{64I\times 8Q}|\tilde{F}|^2\Big)^{1/2} + \Big(\iint_{64I\times 8Q}|\tilde{f}|^{q'}\Big)^{1/q'}$$

and decomposing $64I$ into translates of I as before gives an estimate of the required type.

Step 3: Error terms. In fact, this is the most delicate step in [4] since the non-locality of the operators $D_t^{1/2}$ and $H_tD_t^{1/2}$ cannot be circumvented anymore (as in the previous step). As $\eta_Q = 1$ on Q, we have

$$D_t^{1/2}((1-\eta)(v-c)) = D_t^{1/2}((1-\eta_I)(v-c))$$

on $I\times Q$. The same observation applies to $H_tD_t^{1/2}$ in lieu of $D_t^{1/2}$. We split

$$v - c = v - \fint_{2Q} v + \fint_{2Q} v - \fint\!\!\fint_{I\times 2Q} v.$$

For the terms involving $w_1 := (1-\eta_I)(v - \fint_{2Q} v)$ we follow the proof of Lemma 8.6 in [4] *verbatim* (using Poincaré inequalities), and obtain

$$\Big(\iint_{I\times Q}|H_tD_t^{1/2}w_1|^2 + |D_t^{1/2}w_1|^2\Big)^{1/2} \lesssim \sum_{j\in\mathbb{Z}}\frac{1}{1+|j|^{3/2}}\Big(\iint_{4I_j\times 2Q}|\nabla v|^2\Big)^{1/2}.$$

Inserting (6.3) for each $4I_j\times 2Q$ instead of $4I\times 2Q$ and using the convolution inequality

$$\sum_{j\in\mathbb{Z}}\frac{1}{1+|j|^{3/2}}\frac{1}{1+|k-j|^{3/2}} \lesssim \frac{1}{1+|k|^{3/2}}, \qquad k\in\mathbb{Z},$$

we obtain the desired bound by

$$\sum(g) + \sum_{k\in\mathbb{Z}}\frac{1}{1+|k|^{3/2}}\Big(\iint_{I_k\times 8Q}|\tilde{F}|^2\Big)^{1/2} + \sum_{k\in\mathbb{Z}}\frac{1}{1+|k|^{3/2}}\Big(\iint_{I_k\times 8Q}|\tilde{f}|^{q'}\Big)^{1/q'}.$$

The remaining average of $|H_tD_t^{1/2}w_2|^2 + |D_t^{1/2}w_2|^2$, where $w_2 := (1-\eta_I)(\fint_{2Q} v - \fint\!\!\fint_{I\times 2Q} v)$, is basically treated independently of knowing that v is a solution. In fact, it only involves the function $h(t) := \fint_{2Q} v - \fint\!\!\fint_{I\times 2Q} v$ of one real variable. First, since η_I is symmetric about the midpoint of I and identically 1 on $\frac{9}{4}I$, Lemma 8.7 in [4] is applicable and provides the bound

$$|H_tD_t^{1/2}(1-\eta_I)h| \lesssim \sum_{k\in\mathbb{Z}}\frac{1}{1+|k|^{3/2}}\fint_{I_k}|H_tD_t^{1/2}h| + |D_t^{1/2}h| \tag{6.4}$$

almost everywhere on I. Now, we take the $\mathrm{L}^2(I\times Q)$ average and use that $H_tD_t^{1/2}$ commutes with averages in the spatial variable to give

$$\Big(\iint_{I\times Q}|H_tD_t^{1/2}w_2|^2\Big)^{1/2} \lesssim \sum_{k\in\mathbb{Z}}\frac{1}{1+|k|^{3/2}}\iint_{I_k\times 2Q}|H_tD_t^{1/2}v| + |D_t^{1/2}v|.$$

We remark that at this stage the required bound for the $\mathrm{L}^2(I\times Q)$ average of $|H_tD_t^{1/2}v|^2$ has been completed, taking $\gamma = 8$ on the right-hand side. It only remains to consider the bound for the

$\mathrm{L}^2(I \times Q)$ average of $D_t^{1/2} w_2 = D_t^{1/2}((1-\eta_I)h)$. To this end, following the proof of Lemma 8.9 in [4] line by line yields that $(\int_I |D_t^{1/2}(1-\eta_I)h)|^2)^{1/2}$ is controlled by

$$\Big(\int_{4I} |H_t D_t^{1/2} h|^2\Big)^{1/2} + \Big(\int_{4I} |H_t D_t^{1/2}(1-\eta_{4I})h|^2\Big)^{1/2} + \sum (|H_t D_t^{1/2} h| + |D_t^{1/2} h|).$$

Here, η_{4I} has the same properties as η_I but for the interval $4I$ instead of I. But the first two terms have already been controlled. Indeed, for the first one we can use $H_t D_t^{1/2} h = \fint_{2Q} H_t D_t^{1/2} v$ and the completed reverse Hölder estimate for $H_t D_t^{1/2} v$ on the parabolic cylinder $4I \times 2Q$, which is results in finally using $\gamma = 16$ on the right-hand side. For the second one we can again rely on (6.4), the change from η_I to η_{4I} being only a technicality as is explained in the proof of Lemma 8.9 in [4]. $\square$

Corollary 6.2. *Let $\tilde f \in \mathrm{L}^{q'}(\mathbb{R}^{n+1})$, $\tilde F \in \mathrm{L}^2(\mathbb{R}^{n+1})$ and let $v \in V$ be a weak solution to $\partial_t v - \operatorname{div} A(t,x)\nabla v = \tilde f + \operatorname{div} \tilde F$ in $\mathbb{R}^{n+1}$. Let g be as in Lemma 6.1. Then, there exists $p > 2$ such that*

$$\|g\|_{\mathrm{L}^p(\mathbb{R}^{n+1})} \lesssim \|\tilde F\|_{\mathrm{L}^p(\mathbb{R}^{n+1})} + \|\tilde f\|_{\mathrm{L}^{p_*}(\mathbb{R}^{n+1})},$$

where $p_* = \frac{p(n+2)}{n+p+2}$. The implicit constant as well as p depends only on ellipticity and dimensions.

Proof. Rearranging unions of translates of an interval I into unions of its dilates, and vice versa, reveals that for any positive function h on the real line we have

$$\sum_{k\in\mathbb{Z}} \frac{1}{1+|k|^{3/2}} \fint_{I_k} h \sim \sum_{j=0}^{\infty} 2^{-j} \fint_{4^j I} h,$$

with absolute implicit constants. Lemma 6.1 together with this observation and Hölder's inequality yields

$$\Big(\fint\!\!\fint_{I\times Q} g^2\Big)^{1/2} \lesssim \sum_{j=0}^{\infty} 2^{-j} \fint\!\!\fint_{4^j I\times Q} g + \Big(\sum_{j=0}^{\infty} 2^{-j} \fint\!\!\fint_{4^j I\times Q} |\tilde F|^2\Big)^{1/2} + r\Big(\sum_{j=0}^{\infty} 2^{-j} \fint\!\!\fint_{4^j I\times Q} |\tilde f|^{q'}\Big)^{1/q'}.$$

Thus, we have the setup of Lemma 5.1 with $s = q'$ and we conclude by Corollary 5.3. $\square$

Having this at hand, we can immediately give the

Proof of Theorem 1.3(ii). The function $v := u\chi \in V$ is a weak solution to $\partial_t v - \operatorname{div} A(t,x)\nabla v = \tilde f + \operatorname{div} \tilde F$ on $\mathbb{R}^{n+1}$ with the relations (3.4) and $\tilde f \in \mathrm{L}^{q'}(\mathbb{R}^{n+1})$, $\tilde F \in \mathrm{L}^2(\mathbb{R}^{n+1})$. Notice that $\tilde f \in \mathrm{L}^{p_*}(\mathbb{R}^{n+1})$, $\tilde F \in \mathrm{L}^p(\mathbb{R}^{n+1})$ from the hypothesis in Theorem 1.3(ii). Hence, Corollary 6.2 applies. $\square$

7. The second proof of Theorem 1.3(ii)

Throughout, $p^* := \frac{p(n+2)}{n+2-p}$ and $p_* := \frac{p(n+2)}{n+2+p}$ denote the upper and lower Sobolev exponents of p with respect to the parabolic scaling. Remark that $q = 2^*$ and $q' = 2_*$. For $1 < p < \infty$, set $E_p := \mathrm{L}^p(\mathbb{R}; \mathrm{W}^{1,p}(\mathbb{R}^n)) \cap \mathrm{H}^{1/2,p}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^n))$ with norm $\|u\|_{E_p} := (\|u\|_p^p + \|\nabla u\|_p^p + \|D_t^{1/2} u\|_p^p)^{1/p}$, so that in particular $E = E_2$. These are Banach spaces with $\mathrm{C}_0^\infty(\mathbb{R}^{n+1})$ as a common dense subspace as is seen by approximation via smooth convolution and truncation.

We have the following extension of Lemma 3.4.

Lemma 7.1. *The operator $\mathcal{L} = \partial_t - \operatorname{div} A(t,x)\nabla + \kappa + 1 : E_2 \to E_2^*$ extends by density to a bounded operator from E_p to $(E_{p'})^*$ for $1 < p < \infty$. This extension is invertible for $|p-2|$ small enough and its inverse agrees with the one calculated when $p = 2$ on $(E_2)^* \cap (E_{p'})^*$. The norm of the inverse and the smallness of $|p-2|$ depend only on ellipticity and dimensions.*

Proof. By definition, $\mathcal{L} : E_2 \to E_2^*$ acts via

$$\langle \mathcal{L}u, v\rangle = \iint_{\mathbb{R}^{n+1}} A\nabla u \cdot \overline{\nabla v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} + (\kappa+1) u \cdot \overline{v} \, dx dt.$$

Thus, $E_p \to (E_{p'})^*$ boundedness of $\mathcal{L}$ follows from Hölder's inequality and the norm depends only on ellipticity and dimension. That $(E_p)_{1<p<\infty}$ and the dual scale $(E_p^*)_{1<p<\infty}$ are complex interpolation scales is a special case of Lemma 6.1 in [4]. As $\mathcal{L}$ is invertible when $p = 2$ by Lemma 3.4, the invertibility for $|p-2|$ small enough follows from Šneĭberg's result, see [22] or Theorem 1.3.25 in [9] for a qualitative version revealing that the smallness of $|p-2|$ and the bound for the inverse depend only on ellipticity and dimensions. Finally, the compatibility of the inverses is an abstract feature of complex interpolation, see Theorem 8.1 in [17]. ☐

A simple but important consequence is

Lemma 7.2. *Let $\tilde f \in \mathrm{L}^{p_*}(\mathbb{R}^{n+1})$ and $\tilde F \in \mathrm{L}^{p}(\mathbb{R}^{n+1})$. Then $\mathcal{L}^{-1}(\tilde f + \operatorname{div} \tilde F) \in E_p$ when $|p-2|$ is sufficiently small (depending only on ellipticity and dimensions) and in this case*

$$\|\mathcal{L}^{-1}(\tilde f + \operatorname{div} \tilde F)\|_{E_p} \lesssim \|\tilde f\|_{\mathrm{L}^{p_*}(\mathbb{R}^{n+1})} + \|\tilde F\|_{\mathrm{L}^{p}(\mathbb{R}^{n+1})},$$

with an implicit constant depending only on ellipticity and dimensions.

Proof. From Lemma 3.2, $E_{p'}$ embeds into $\mathrm{L}^{p'^*}(\mathbb{R}^{n+1})$ when $1 < p' < n+2$. As the dual exponent of p'^* is p_*, we obtain that $\mathrm{L}^{p_*}(\mathbb{R}^{n+1})$ embeds into $(E_{p'})^*$. Thus $\tilde f + \operatorname{div} \tilde F \in (E_{p'})^*$ and the conclusion follows from Lemma 7.1. ☐

With this at hand, we are ready to give another

Proof of Theorem 1.3(ii). Let $\chi \in \mathrm{C}_0^\infty(\Omega)$. As before, $v := u\chi \in V$ is a weak solution to $\partial_t v - \operatorname{div} A(t,x)\nabla v = \tilde f + \operatorname{div} \tilde F$ on $\mathbb{R}^{n+1}$ with $\tilde f, \tilde F$ as in (3.4). By Proposition 3.1 we know that $v \in E$ and $\mathcal{L}v = (\kappa+1)v + \tilde f + \operatorname{div} \tilde F$.

Let now $p > 2$ be such that we have Lemma 7.2 at our disposal. We may also suppose $p \le q$, which is equivalent to $p_* \le q_* = 2$. If we assume $f \in \mathrm{L}^{p_*}_{\mathrm{loc}}(\Omega)$ and $F \in \mathrm{L}^{p}_{\mathrm{loc}}(\Omega)$, then $\tilde f + (\kappa+1)v \in \mathrm{L}^{p_*}(\mathbb{R}^{n+1})$ and $\tilde F \in \mathrm{L}^{p}(\mathbb{R}^{n+1})$, using also Theorem 1.3(i) to control $u\chi$ in the formula for $\tilde F$. Hence, by compatibility of the inverses (Lemma 7.1) and Lemma 7.2, we obtain $v \in E_p$ with

$$\|v\|_{E_p} \lesssim \|v\|_{p_*} + \|\tilde f\|_{p_*} + \|\tilde F\|_{p}. \tag{7.1}$$

The left-hand side controls $\|u\chi\|_p + \|\nabla(u\chi)\|_p + \|D_t^{1/2}(u\chi)\|_p$ and we are done. ☐

8. Proof of Theorem 1.3(iii) and (iv)

We next prove part (iii) of Theorem 1.3 in the following more precise result.

Proposition 8.1. *Consider the setup of Theorem 1.3. There exists $p > 2$ depending only on ellipticity and dimensions such that, with $\alpha = 1/2 - 1/p$,*

$$\begin{aligned} &\Big(\fint\!\!\fint_{I\times Q} |\nabla u|^p\Big)^{1/p} + \sup_{t\in I}\Big(\fint_Q |u(t,\cdot)|^p\Big)^{1/p} + \sup_{t,s\in I}\Big(\fint_Q \frac{|u(t,\cdot)-u(s,\cdot)|^p}{|t-s|^{\alpha p}}\Big)^{1/p} \\ &\quad\lesssim \frac{1}{r(Q)}\Big(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |u|^2\Big)^{1/2} + \Big(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |F|^p\Big)^{1/p} + r(Q)\Big(\fint\!\!\fint_{\gamma^2 I\times\gamma Q} |f|^{p_*}\Big)^{1/p_*}. \end{aligned} \tag{8.1}$$

Here, $\gamma > 1$ and $I \times Q$ is a parabolic cylinder with $\ell(I) \sim r(Q)^2$ and $\overline{\gamma^2 I \times \gamma Q} \subseteq \Omega$. The implicit constant depends on ellipticity, dimensions, γ and the constants controlling the ratio $r(Q)^2/\ell(I)$.

Proof. We assume again $r(Q) = 1$ as rescaling will give us the right powers of $r(Q)$.

Let $\chi \in \mathrm{C}_0^\infty(\Omega)$, $\chi = 1$ on $I \times Q$, with support in $\gamma^2 I \times \gamma Q$. Then $v := u\chi \in V$ is a weak solution to $\partial_t v - \operatorname{div} A(t,x)\nabla v = \tilde f + \operatorname{div} \tilde F$ on $\mathbb{R}^{n+1}$ with the relations (3.4) and $\tilde f \in \mathrm{L}^{q'}(\mathbb{R}^{n+1})$, $\tilde F \in \mathrm{L}^{2}(\mathbb{R}^{n+1})$. By Corollary 6.2 we have if $p - 2 > 0$ is small enough,

$$\|\nabla v\|_p + \|D_t^{1/2} v\|_p \lesssim \|\tilde F\|_p + \|\tilde f\|_{p_*}. \tag{8.2}$$

Alternatively, we could have used (7.1) here at the expense of a term $\|v\|_{p_*} \lesssim \|v\|_p$ on the right, which turns out to be harmless. Indeed, we have from (4.7) if $p \le q$, as we may assume,

$$\|v\|_p \lesssim \|v\|_2 + \|\tilde F\|_2 + \|\tilde f\|_{q'} \lesssim \|u\chi\|_2 + \|\tilde F\|_p + \|\tilde f\|_{p_*}. \tag{8.3}$$

We have used $q' \le p_*$ and $2 < p$ in the second step.

We have shown that $v, D_t^{1/2} v$ are controlled in $\mathrm{L}^p(\mathbb{R}^{n+1})$. Since $p > 2$, a Hölder norm estimate on v will follow from classical embeddings. Indeed, if $p \in (1,\infty)$ and $h : \mathbb{R} \to \mathbb{C}$ satisfies $\|h\|_2 + \|h\|_p < \infty$

and $\|D_t^{1/2}h\|_2 + \|D_t^{1/2}h\|_p < \infty$, then for any interval $J \subset \mathbb{R}$, and $\alpha = 1/2 - 1/p$, we have the fractional Poincaré inequality

$$\Big(\fint_J \Big|h - \fint_J h\Big|^p\Big)^{1/p} \lesssim \ell(J)^\alpha \|D_t^{1/2}h\|_p,$$

see for instance Lemma 8.3 in [4]. Moreover, if $p > 2$, then the Campanato characterization of Hölder regularity yields after redefining h on a Lebesgue null set,

$$\sup_{t\in J}|h(t)| + \sup_{t,s\in J}\frac{|h(t)-h(s)|}{|t-s|^\alpha} \lesssim \|h\|_p + \|D_t^{1/2}h\|_p,$$

where $J \subset \mathbb{R}$ is any interval and the implicit constant depends also on $\ell(J)$, see Theorem 2.9 in [12]. Both of these inequalities rely mainly on Hölder's inequality and Lebesgue differentiation. Their proofs pass *verbatim* to $\mathrm{L}^p(\mathbb{R}^n)$-valued functions, replacing absolute values by $\mathrm{L}^p(\mathbb{R}^n)$-norms.

Applying the vector-valued version to $h = v$ on the particular interval I, we find that the left hand side of (8.1) is bounded by $\|v\|_p + \|\nabla v\|_p + \|D_t^{1/2}v\|_p$. (Recall the normalization $r = 1 \sim \ell$.) In view of (8.2) and (8.3), we see that it remains to control $\|\tilde{F}\|_p + \|\tilde{f}\|_{p_*}$ from above by the right-hand side of (8.1).

We begin with $\tilde{F}$. Let $1 < \delta < \gamma$ be such that the support of χ is contained in $\delta^2 I \times \delta Q$. By (3.4),

$$\begin{aligned}\|\tilde{F}\|_p &\lesssim \Big(\iint_{\gamma^2 I\times\gamma Q}|F|^p\Big)^{1/p} + \Big(\iint_{\delta^2 I\times\delta Q}|u|^p\Big)^{1/p}\\ &\lesssim \Big(\iint_{\gamma^2 I\times\gamma Q}|F|^p\Big)^{1/p} + \Big(\iint_{\gamma^2 I\times\gamma Q}|u|^2\Big)^{1/2} + \Big(\iint_{\gamma^2 I\times\gamma Q}|f|^{p_*}\Big)^{1/p_*},\end{aligned}$$

where we used (4.7) in the second step along with $p \le q$, which we have already assumed and which implies $q' \le p_*$. Next, for $\tilde{f}$, we note that $p_* \le 2$ is equivalent to $p \le q$. Then by (3.4),

$$\|\tilde{f}\|_{p_*} \lesssim \Big(\iint_{\delta^2 I\times\delta Q}|f|^{p_*} + |u|^{p_*} + |\nabla u|^{p_*} + |F|^{p_*}\Big)^{1/p_*}$$

and we are done as $p_* \le 2$ and since the term $\iint_{\delta^2 I\times\delta Q}|\nabla u|^{p_*}$ can be treated using Proposition 4.3. □

We finally prove part (iv) of Theorem 1.3 in the following more precise result, which extends the result of [11] to our more general assumption.

Proposition 8.2. *Consider the setup of Theorem 1.3. There exists $p > 2$, depending only on ellipticity and dimensions, such that*

$$\Big(\fint\!\!\fint_{I\times Q}|\nabla u|^p\Big)^{1/p} \lesssim \fint\!\!\fint_{\gamma^2 I\times\gamma Q}|\nabla u| + \Big(\fint\!\!\fint_{\gamma^2 I\times\gamma Q}|F|^p\Big)^{1/p} + r(Q)\Big(\fint\!\!\fint_{\gamma^2 I\times\gamma Q}|f|^{p_*}\Big)^{1/p_*}. \tag{8.4}$$

Here, $\gamma > 1$ and $I \times Q$ is a parabolic cylinder with $\ell(I) \sim r(Q)^2$ and $\overline{\gamma^2 I \times \gamma Q} \subseteq \Omega$. The implicit constant depends on ellipticity, dimensions, γ and the constants controlling the ratio $r(Q)^2/\ell(I)$.

Proof. As in the proof of Proposition 4.3 it suffices to prove (8.4) with $\big(\fint\!\!\fint_{\gamma^2 I\times\gamma Q}|\nabla u|^2\big)^{1/2}$ on the right-hand side, see again [6, 16]. It is enough to assume as usual $r(Q) = 1$.

We use the weighted mean trick introduced by Struwe in [23]. We choose χ in the proof of Proposition 8.1 of the form $\chi(t,x) = \eta(t)\varphi(x)$ and define the weighted mean $\tilde{u}(t) := a\int u(t,x)\varphi(x)\,dx$, with $a := (\int\varphi)^{-1}$. Set $w(t,x) := (u(t,x) - \tilde{u}(t))\eta(t)\varphi(x)$. We remark that $\nabla u = \nabla w$ on $I \times Q$. It is thus enough to estimate $\big(\iint_{I\times Q}|\nabla w|^p\big)^{1/p}$. We proceed as follows.

It follows from the equation for u that $\tilde{u}$ is absolutely continuous on I with

$$\partial_t\tilde{u}(t) = a\int(-(A\nabla u + F)\cdot\nabla\varphi + f\varphi)\,dx \tag{8.5}$$

almost everywhere. Using (3.4) for u and $\tilde u$, we deduce $\partial_t w - \operatorname{div} A(t,x)\nabla w = \tilde f + \operatorname{div} \tilde F$ on $\mathbb{R}^{n+1}$, where, omitting the variables except for the integration,

$$\tilde f := \Big(f - a\int f\varphi\,dx\Big)\varphi\eta + \Big(a\varphi\int F\cdot\nabla\varphi\,dx - F\cdot\nabla\varphi\Big)\eta$$
$$+ (u-\tilde u)\varphi\partial_t\eta + \Big(a\varphi\int A\nabla u\cdot\nabla\varphi\,dx - A\nabla u\cdot\nabla\varphi\Big)\eta$$

and

$$\tilde F := -\eta A(u-\tilde u)\nabla\varphi + \eta\varphi F.$$

Inspecting the argument to prove (8.1) for the function w, we obtain

$$\Big(\iint_{I\times Q}|\nabla w|^p\Big)^{1/p} \lesssim \Big(\iint_{\gamma^2 I\times\gamma Q}|f|^{p_*}\Big)^{1/p_*} + \Big(\iint_{\delta^2 I\times\delta Q}|u-\tilde u|^p\Big)^{1/p} + \Big(\iint_{\gamma^2 I\times\gamma Q}|F|^p\Big)^{1/p}.$$

It remains to handle the middle term and to this end we need to introduce some further notation. Recall that the functions η and φ are supported in $\delta^2 I$ and δQ with $1<\delta<\gamma$ and equal to 1 on I and Q, respectively. For clarity, let us then denote $\tilde u_\delta := \tilde u$ this weighted mean and $w_\delta := w$ accordingly. Likewise, let us write $\tilde u_\gamma$, w_γ, replacing $1,\delta$ by δ,γ. Thus, we have to estimate $I_{\delta,p} := \big(\iint_{\delta^2 I\times\delta Q}|u-\tilde u_\delta|^p\big)^{1/p}$. Writing $u-\tilde u_\delta = w_\gamma + \tilde u_\gamma - \tilde u_\delta$ on $\delta^2 I\times\delta Q$, we get

$$I_{\delta,p} \le \Big(\iint_{\delta^2 I\times\delta Q}|w_\gamma|^p\Big)^{1/p} + \Big(\int_{\delta^2 I}|\tilde u_\gamma - \tilde u_\delta|^p\Big)^{1/p}. \tag{8.6}$$

Now, w_γ solves a parabolic equation of the same structure as w, except for a different choice of the cut-offs η and φ. Thus, invoking (4.7) along with $p\le q$, which we may assume, and the explicit formulæ for $\tilde f$ and $\tilde F$ above, we see that the first integral is controlled by $I_{\gamma,2}$ and integrals of $|\nabla u|^2$, $|F|^2$ and $|f|^{q'}$ on $\gamma^2 I\times\gamma Q$. Since the support of φ is contained in γQ, a variant of the L^2 Poincaré's inequality in the x-variable yields

$$I_{\gamma,2} = \Big(\iint_{\gamma^2 I\times\gamma Q}|u-\tilde u_\gamma|^2\Big)^{1/2} \lesssim \Big(\iint_{\gamma^2 I\times\gamma Q}|\nabla u|^2\Big)^{1/2},$$

see for instance Lemma 8.3.1 in [1] and keep in mind $r(Q)=1$. For the second integral on the right of (8.6), we apply the classical Sobolev embedding $\mathrm{W}^{1,\sigma}(\delta^2 I)\subset \mathrm{L}^\tau(\delta^2 I)$, $1\le\sigma\le\tau\le\infty$, to $\tilde u_\gamma - \tilde u_\delta$ with $\sigma = p_*$ and $\tau = p$. As $\ell(I)\sim 1$, we obtain with an implicit constant depending on δ,

$$\Big(\int_{\delta^2 I}|\tilde u_\gamma - \tilde u_\delta|^p\Big)^{1/p} \lesssim \Big(\int_{\delta^2 I}|\tilde u_\gamma - \tilde u_\delta|^{p_*}\Big)^{1/p_*} + \Big(\int_{\delta^2 I}|\partial_t\tilde u_\gamma - \partial_t\tilde u_\delta|^{p_*}\Big)^{1/p_*}.$$

Thanks to $p_*\le q_* = 2$, the first integral on the right can be controlled by $I_{\gamma,2}$ and we can conclude again using the variant of the L^2 Poincaré inequality in the x-variable. For the second one, we use the explicit calculation of $\partial_t\tilde u = \partial_t\tilde u_\delta$ in (8.5) and the analog for $\partial_t\tilde u_\gamma$ to obtain integrals on $\gamma^2 I\times\gamma Q$ of $|\nabla u|^{p_*}$, $|F|^{p_*}$ and $|f|^{p_*}$. Hence we obtain the desired right-hand side. □

Remark 8.3. Once Theorem 1.3(i) is established, it is also possible to prove directly (8.4) under our assumptions by adapting appropriately the original argument in [11] and invoking the usual Gehring lemma.

Appendix A. Extension of weakly elliptic coefficients

We provide here a simple lemma justifying the use of the global Gårding inequality in the context of local weak solutions.

Lemma A.1. *Let $Q_0\subset\mathbb{R}^n$ be an open set. Let $A\in\mathrm{L}^\infty(Q_0;\mathcal{L}(\mathbb{C}^{nm}))$ and suppose that there exist $\lambda>0$ and $\kappa\ge 0$ such that*

$$\operatorname{Re}\int_{Q_0} A\nabla u\cdot\overline{\nabla u} \ge \lambda\int_{Q_0}|\nabla u|^2 - \kappa\int_{Q_0}|u|^2, \qquad u\in\mathrm{W}^{1,2}_0(Q_0;\mathbb{C}^m).$$

Let Q be compact subset of Q_0. If $\sigma > 0$ is sufficiently large, depending only on λ, $\|A\|_\infty$, n, m and the distance $\operatorname{dist}(Q, \mathbb{R}^n \setminus Q_0)$, then $\tilde{A} := \mathbf{1}_Q A + \sigma \mathbf{1}_{\mathbb{R}^n \setminus Q}$ satisfies for some constant K depending on the same parameters,

$$\operatorname{Re} \int_{\mathbb{R}^n} \tilde{A} \nabla u \cdot \overline{\nabla u} \geq \frac{\lambda}{4} \int_{\mathbb{R}^n} |\nabla u|^2 - K \int_{\mathbb{R}^n} |u|^2, \qquad u \in \mathrm{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m).$$

Proof. Let $\varphi : \mathbb{R}^n \to [0,1]$ a smooth cut-off that is 1 on Q, has support in Q_0 and satisfies $\|\nabla \varphi\|_\infty \leq \frac{c}{\operatorname{dist}(Q, \mathbb{R}^n \setminus Q_0)}$ for some dimensional constant c. Let $u \in \mathrm{W}^{1,2}(\mathbb{R}^n; \mathbb{C}^m)$ and split $u = u_1 + u_2$, where $u_1 := \varphi u \in \mathrm{W}^{1,2}_0(Q_0; \mathbb{C}^m)$ and $u_2 := (1 - \chi) u$. Accordingly, we split

$$\int \tilde{A} \nabla u \cdot \overline{\nabla u} = \int \tilde{A} \nabla u_1 \cdot \overline{\nabla u_1} + \int \tilde{A} \nabla u_2 \cdot \overline{\nabla u_2} + \int (\tilde{A} \nabla u_1 \cdot \overline{\nabla u_2} + \tilde{A} \nabla u_2 \cdot \overline{\nabla u_1}) =: I + II + III.$$

Firstly, by assumption on A and since $\sigma \geq 0$, we have $\operatorname{Re} I \geq \lambda \|\nabla u_1\|_2^2 - \kappa \|u_1\|_2^2$. Secondly, since u_2 vanishes on Q, we get $\operatorname{Re} II \geq (\sigma - \|A\|_\infty) \|\nabla u_2\|_2^2$ from the definition of $\tilde{A}$. Thirdly, again by definition of $\tilde{A}$, we have

$$\operatorname{Re} III \geq -2 \|A\|_\infty \|\nabla u_1\|_2 \|\nabla u_2\|_2 + 2\sigma \operatorname{Re} \int \nabla u_1 \cdot \overline{\nabla u_2}.$$

Expanding

$$\nabla u_1 \cdot \overline{\nabla u_2} = (\varphi \nabla u + u \nabla \varphi) \cdot \overline{((1 - \varphi) \nabla u - u \nabla \varphi)}$$

and making the key observation that $\operatorname{Re}(\varphi \nabla u \cdot \overline{(1 - \varphi) \nabla u}) = \varphi (1 - \varphi) |\nabla u|^2$ is non-negative almost everywhere by the choice of φ, we see that for some constant $C > 0$ depending only on n, m and $\operatorname{dist}(Q, \mathbb{R}^n \setminus Q_0)$,

$$\operatorname{Re} \int \nabla u_1 \cdot \overline{\nabla u_2} \geq -C \|u\|_2 \|\nabla u\|_2 - C \|u\|_2^2.$$

Summing up, we discover

$$\begin{aligned} \int \tilde{A} \nabla u \cdot \overline{\nabla u} \geq \lambda \|\nabla u_1\|_2^2 + (\sigma - \|A\|_\infty) \|\nabla u_2\|_2^2 - 2 \|A\|_\infty \|\nabla u_1\|_2 \|\nabla u_2\|_2 \\ - 2\sigma C \|u\|_2 \|\nabla u\|_2 - (2\sigma C + \kappa) \|u\|_2^2. \end{aligned}$$

Note that $\|\nabla u\|_2^2 \leq 2 \|\nabla u_1\|_2^2 + 2 \|\nabla u_2\|_2^2$ as a consequence of $u = u_1 + u_2$. Hence, we can fix σ large enough depending on λ and $\|A\|_\infty$ and apply Young's inequality to deduce

$$\int \tilde{A} \nabla u \cdot \overline{\nabla u} \geq \frac{\lambda}{2} (\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2) - K \|u\|_2^2,$$

where K depends on all the other (by now fixed) parameters. The same estimate on the gradients as before yields the claim. ∎

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