# FINITELY RANDOMIZED DYADIC SYSTEMS AND BMO ON METRIC MEASURE SPACES 

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#### Abstract

We study the connection between BMO and dyadic BMO in metric measure spaces using finitely randomized dyadic systems, and give a Garnett-Jones type proof for a theorem of Uchiyama on a construction of certain BMO functions. We obtain a relation between the BMO norm of a suitable expectation over dyadic systems and the dyadic BMO norms of the original functions in different systems. The expectation is taken over only finitely randomized dyadic systems to overcome certain measurability questions. Applying our result, we derive Uchiyama's theorem from its dyadic counterpart, which we also prove.


Keywords: randomized dyadic system, bounded mean oscillation, dyadic BMO, dyadic cube, metric space

## 1. Introduction

Random dyadic systems have played an important role in several recent advances in harmonic analysis both in Euclidean and more abstract setting, see e.g. [16], [17], [21], [7], [12], [15]. In metric spaces, these systems were first constructed by Hytönen and Martikainen in [9]. The construction was elaborated and simplified in [8], [2] and [10]. A random dyadic system consists of a probability space $(\Omega, \mathbb{P})$ and dyadic systems $\mathcal{D}(\omega), \omega \in \Omega$, each having the same properties as the classical dyadic system of M. Christ [4]. Moreover, the probability of a point to end up near the boundary of a random dyadic cube is small. For an elementary construction of a dyadic system, see also [20].

In [3], Chen, Li and Ward studied the connection between BMO and dyadic BMO in metric measure spaces using random dyadic systems from [8]. They showed that if there is a family $\left\{f^{\omega}\right\}_{\omega \in \Omega}$ of functions with uniformly bounded dyadic BMO norms in $\omega$ such that the mapping $\omega \mapsto f^{\omega}$ is measurable, then the expectation $f=\mathbb{E}\left[f^{\omega}\right]$ over $\omega \in \Omega$ belongs to BMO and its BMO norm is controlled by the dyadic BMO norms of the original functions $f^{\omega}$; see [3, Theorem 3.1].

In this paper, we study the connection between BMO and dyadic BMO using finitely randomized dyadic systems. These systems are constructed as in [10] except that the randomization is applied only in finite number of generations. An advantage of this approach is that the underlying probability space is finite and so, for example, measurability with respect to the probability parameter $\omega$ is automatically satisfied. For finitely randomized systems, we get an essentially similar result as Chen, Li and Ward except that there occurs an extra
term, which vanishes as the number of generations used in the randomization goes to infinity.

As an application of our result, we give a proof for a theorem of Uchiyama on a construction of certain BMO functions, see [23] and [6] for the Euclidean setting. First we prove a dyadic version of the theorem and then apply finitely randomized dyadic systems to obtain the general version. In this situation, the method of Chen, Li and Ward cannot be easily applied since the dependence of the appearing functions on the probability parameter is so implicit that measurability is not obvious. We hope that our method can be useful also in other questions. For a different proof of Uchiyama's theorem, which does not utilize dyadic structures, see [11].

The connection between BMO and dyadic BMO was widely studied first in [6], in which Garnett and Jones gave new proofs for four theorems concerning BMO functions, including Uchiyama's theorem. Their idea was to first prove the easier dyadic version of each theorem and then obtain the general version by averaging over the dyadic results over translations in $\mathbb{R}^{n}$. A crucial part of their proofs is that the translation average of a suitable family of dyadic BMO functions belongs to BMO. Similar results were shown to hold for other BMO related spaces in [18] and [19]. For related results concerning the connection between BMO and dyadic BMO via finitely many translations, see [14], [5] and [13]. In [22], Treil gave a different way to get BMO from dyadic BMO by showing that the BMO norm is comparable to the expectation of dyadic BMO norms over suitable randomized dyadic systems. This approach showed that the translation structure of $\mathbb{R}^{n}$ is not necessarily essential for the connection between BMO and dyadic BMO.

The paper is organized as follows. In Section 2, we recall basic properties of dyadic systems in doubling metric measure spaces. In Section 3, we prove the dyadic version of Uchiyama's theorem. In Section 4, we prove a version of Chen-Li-Ward theorem for finitely randomized dyadic systems and apply it to deduce the general Uchiyama's theorem from the dyadic one.

## 2. Notation and preliminaries

Throughout the paper, $(X, d, \mu)$ is a metric measure space, where the measure $\mu$ is Borel regular and satisfies $0<\mu(U)<\infty$ whenever $U$ is non-empty, open and bounded. Moreover, we assume that $\mu$ is doubling, which means that there exists a constant $C_{\mu}$ such that

$$
\begin{equation*}
\mu(B(x, 2 r)) \leq C_{\mu} \mu(B(x, r)) \tag{2.1}
\end{equation*}
$$

for every ball $B(x, r)=\{y \in X: d(x, y)<r\} \subset X$.
As usual, the characteristic function of a set $A \subset X$ is denoted by $\chi_{A}$. The integral average of a locally integrable function $f$ over a bounded measurable set $A \subset X$ is denoted by each of the following:

$$
f_{A}=f_{A} f d \mu=\frac{1}{\mu(A)} \int_{A} f d \mu
$$

In general, $C$ is a positive constant whose value is not necessarily the same at each occurrence.

Under the assumptions above, there exist a set of points $\left\{z_{\alpha}^{k}\right\}_{k, \alpha}$ and a family of sets $\left\{Q_{\alpha}^{k}\right\}_{k, \alpha}, k \in \mathbb{Z}, \alpha \in \mathscr{A}_{k}$, in $X$ satisfying the following properties:

$$
\begin{align*}
& \text { if } l \leq k \text {, then either } Q_{\alpha}^{k} \subset Q_{\beta}^{l} \text { or } Q_{\alpha}^{k} \cap Q_{\beta}^{l}=\emptyset \text {, }  \tag{2.2}\\
& \text { for every } Q_{\alpha}^{k} \text { and } l \leq k \text {, there exists a unique } Q_{\beta}^{l} \supset Q_{\alpha}^{k} \text {, }  \tag{2.3}\\
& \text { for every } k \in \mathbb{Z} \text { and } \alpha \neq \beta, Q_{\alpha}^{k} \cap Q_{\beta}^{k}=\emptyset  \tag{2.4}\\
& \text { for every } k \in \mathbb{Z}, X=\bigcup_{\alpha \in \mathscr{A}_{k}} Q_{\alpha}^{k} \text {, }  \tag{2.5}\\
& \text { for every } Q_{\alpha}^{k}, B\left(z_{\alpha}^{k}, c_{0} \delta^{k}\right) \subset Q_{\alpha}^{k} \subset B\left(z_{\alpha}^{k}, C_{0} \delta^{k}\right) \text {, }  \tag{2.6}\\
& \text { if } l \leq k \text { and } Q_{\alpha}^{k} \subset Q_{\beta}^{l} \text {, then } B\left(z_{\alpha}^{k}, C_{0} \delta^{k}\right) \subset B\left(z_{\beta}^{l}, C_{0} \delta^{l}\right) \text {, }  \tag{2.7}\\
& \text { for every } Q_{\alpha}^{k}, \#\left\{Q_{\beta}^{k+1} \subset Q_{\alpha}^{k}: \beta \in \mathscr{A}_{k+1}\right\} \leq N_{0} \text {, } \tag{2.8}
\end{align*}
$$

where the constants $c_{0}>0, C_{0}>0,0<\delta<1$ and $N_{0} \in \mathbb{N}$ depend only on the doubling constant $C_{\mu}$, and $\mathscr{A}_{k}$ is a countable index set for each $k$. The set $Q_{\alpha}^{k}$ is called a dyadic cube and $z_{\alpha}^{k}$ the center of the cube. The family of dyadic cubes of generation $k$ and the family of all dyadic cubes are

$$
\mathcal{D}_{k}=\left\{Q_{\alpha}^{k}: \alpha \in \mathscr{A}_{k}\right\} \quad \text { and } \quad \mathcal{D}=\bigcup_{k \in \mathbb{Z}} \mathcal{D}_{k},
$$

respectively. For $x \in X$, we denote by $Q^{k}(x)$ the unique dyadic cube of generation $k$ containing $x$. If $Q_{\beta}^{k+1} \subset Q_{\alpha}^{k}$, then $Q_{\beta}^{k+1}$ is called a child of $Q_{\alpha}^{k}$ and $Q_{\alpha}^{k}$ the parent of $Q_{\beta}^{k+1}$. Every dyadic cube has exactly one parent by (2.3) and at most $N_{0}$ children by (2.8).

By (2.1), (2.6) and (2.7) we have the following dyadic doubling property. For every dyadic cube $Q$ and its parent $Q^{*} \supset Q$,

$$
\begin{equation*}
\mu\left(Q^{*}\right) \leq C^{*} \mu(Q) \tag{2.9}
\end{equation*}
$$

where $C^{*}$ depends only on the doubling constant $C_{\mu}$. Also, a dyadic version of Lebesgue's theorem holds.

Theorem 2.1 (Dyadic Lebesgue's theorem). Let $f$ be a locally integrable function in $(X, d, \mu)$. Then

$$
\lim _{j \rightarrow \infty} f_{Q^{j}(x)} f d \mu=f(x)
$$

for almost every $x \in X$.
The ancestor space (called quadrant in [1]) of a dyadic cube $Q$ is

$$
X(Q)=\bigcup_{Q^{\prime} \supset Q} Q^{\prime}
$$

where the union is taken over all dyadic cubes $Q^{\prime}$ containing $Q$. The construction of dyadic cubes can be done such that $X(Q)=X$ for every dyadic cube $Q$. Otherwise, $X$ is a disjoint union of finitely many different ancestor spaces.
A locally integrable function $f$ has bounded mean oscillation in $(X, d, \mu)$, denoted by $f \in \mathrm{BMO}$, if

$$
\begin{equation*}
\|f\|_{\mathrm{BMO}}:=\sup _{B} f_{B}\left|f-f_{B}\right| d \mu<\infty, \tag{2.10}
\end{equation*}
$$

where the supremum is taken over all balls $B \subset X$. Similarly, $f$ has bounded dyadic mean oscillation, denoted by $f \in \mathrm{BMO}_{\mathrm{d}}$, if

$$
\begin{equation*}
\|f\|_{\mathrm{BMO}_{\mathrm{d}}}:=\sup _{Q} f_{Q}\left|f-f_{Q}\right| d \mu<\infty \tag{2.11}
\end{equation*}
$$

where the supremum is taken over all dyadic cubes $Q \subset X$. When the suprema above are taken only over balls and dyadic cubes contained in a measurable subset $A \subset X$, we denote $f \in \mathrm{BMO}(A)$ and $f \in \mathrm{BMO}_{\mathrm{d}}(A)$, respectively, and the corresponding local BMO norms by $\|f\|_{\mathrm{BMO}(A)}$ and $\|f\|_{\mathrm{BMO}_{\mathrm{d}}(A)}$. It is easy to see that $\mathrm{BMO} \subset \mathrm{BMO}_{\mathrm{d}}$ by (2.6) and (2.1), but the converse is not true.

BMO norms can be estimated applying the following lemma. For the proof, see e.g. [11].

Lemma 2.2. Let $f \in$ BMO. Then

$$
\frac{1}{2}\|f\|_{\mathrm{BMO}} \leq \sup \left|\int_{X} f g d \mu\right| \leq\|f\|_{\mathrm{BMO}}
$$

where the supremum is taken over all functions $g \in L^{\infty}(X)$ for which there exists a ball $B \subset X$ such that

$$
\begin{equation*}
\operatorname{supp} g \subset B, \quad\|g\|_{L^{\infty}(X)} \leq \frac{1}{\mu(B)} \quad \text { and } \quad \int_{X} g d \mu=0 \tag{2.12}
\end{equation*}
$$

We have an analogous version for the dyadic BMO.
Lemma 2.3. Let $f \in \mathrm{BMO}_{\mathrm{d}}$. Then

$$
\frac{1}{2}\|f\|_{\mathrm{BMO}_{\mathrm{d}}} \leq \sup \left|\int_{X} f g d \mu\right| \leq\|f\|_{\mathrm{BMO}_{\mathrm{d}}}
$$

where the supremum is taken over all functions $g \in L^{\infty}(X)$ for which there exists a dyadic cube $Q \subset X$ such that

$$
\begin{equation*}
\operatorname{supp} g \subset Q, \quad\|g\|_{L^{\infty}(X)} \leq \frac{1}{\mu(Q)} \quad \text { and } \quad \int_{X} g d \mu=0 . \tag{2.13}
\end{equation*}
$$

Proof. For any $g$ satisfying (2.13), we have

$$
\left|\int_{X} f g d \mu\right|=\left|\int_{X}\left(f-f_{Q}\right) g d \mu\right| \leq f_{Q}\left|f-f_{Q}\right| d \mu \leq\|f\|_{\mathrm{BMO}_{\mathrm{d}}}
$$

giving the upper bound.
To obtain the lower bound, let $\varepsilon>0$ and let $Q$ be a dyadic cube such that

$$
\begin{equation*}
f_{Q}\left|f-f_{Q}\right| d \mu \geq\|f\|_{\mathrm{BMO}_{\mathrm{d}}}-\varepsilon . \tag{2.14}
\end{equation*}
$$

Denoting $h:=\operatorname{sgn}\left(f-f_{Q}\right)$, we have

$$
\begin{equation*}
\int_{Q}\left|f-f_{Q}\right| d \mu=\int_{Q}\left(f-f_{Q}\right) h d \mu=\int_{Q}\left(f-f_{Q}\right)\left(h-h_{Q}\right) d \mu . \tag{2.15}
\end{equation*}
$$

Define

$$
g:=\frac{\left(h-h_{Q}\right) \chi_{Q}}{2 \mu(Q)} .
$$

Then $g$ satisfies (2.13) with $Q$. Moreover,

$$
\begin{aligned}
\int_{X} f g d \mu & =\frac{1}{2} f_{Q} f\left(h-h_{Q}\right) d \mu=\frac{1}{2} f_{Q}\left(f-f_{Q}\right)\left(h-h_{Q}\right) d \mu \\
& =\frac{1}{2} f_{Q}\left|f-f_{Q}\right| d \mu \geq \frac{1}{2}\left(\|f\|_{\mathrm{BMO}_{\mathrm{d}}}-\varepsilon\right)
\end{aligned}
$$

by (2.15) and (2.14). The claim follows by estimating by the supremum on the left and letting $\varepsilon \rightarrow 0$.

## 3. Dyadic Uchiyama's theorem

In this section, we prove a metric space version of dyadic Uchiyama's theorem. The proof mainly follows the steps of the proof of its Euclidean counterpart in [6]. We start by proving the following lemma.

Lemma 3.1. Let $N \in \mathbb{N}$ and $\lambda>K_{N}:=\log _{C_{\mu}}(2 N)$. Let $Q_{0} \subset X$ be a dyadic cube and let $E_{1}, \ldots, E_{N}$ be measurable subsets of $Q_{0}$ such that

$$
\begin{equation*}
\min _{1 \leq i \leq N} \frac{\mu\left(Q \cap E_{i}\right)}{\mu(Q)} \leq C_{\mu}^{-2 \lambda} \tag{3.1}
\end{equation*}
$$

for all dyadic $Q \subset Q_{0}$, where $C_{\mu}$ is the doubling constant in (2.1). Then there exist families $\mathcal{G}_{n}$ of dyadic cubes and functions $\varphi_{i}^{n}, i=1, \ldots, N, n=0,1, \ldots$, such that

$$
\begin{equation*}
\varphi_{i}^{n}=\varphi_{i}^{n-1}+\sum_{Q \in \mathcal{G}_{n}} a_{i, Q} \chi_{Q}, \quad i=1, \ldots, N, \tag{3.2}
\end{equation*}
$$

for every $n=1,2, \ldots$,

$$
\begin{equation*}
\sum_{i=1}^{N} \varphi_{i}^{n}=\lambda, \quad 0 \leq \varphi_{i}^{n} \leq \lambda, \quad i=1, \ldots, N \tag{3.3}
\end{equation*}
$$

for every $n=0,1, \ldots$, the coefficients $a_{i, Q}$ satisfy

$$
\begin{equation*}
\left|a_{i, Q}\right| \leq A_{N}:=N\left(K_{N}+\log _{C_{\mu}} C^{*}\right), \quad i=1, \ldots, N, \tag{3.4}
\end{equation*}
$$

for every $n=1,2, \ldots$ and $Q \in \mathcal{G}_{n}$, and

$$
\begin{equation*}
\left(\varphi_{i}^{n}\right)_{Q} \leq \max \left\{0,-K_{N}+\log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)}\right)\right\}, \quad i=1, \ldots, N \tag{3.5}
\end{equation*}
$$

for every $n=0,1, \ldots$ and $Q \in \mathcal{G}_{n}$, where $C^{*}$ is the constant in (2.9).
Proof. We proceed by induction. Let $i(Q)$ be a minimizing index in (3.1) for $Q \subset Q_{0}$, which in particular implies

$$
\begin{equation*}
2 \lambda \leq \log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i(Q)}\right)}\right) \tag{3.6}
\end{equation*}
$$

for any dyadic $Q \subset Q_{0}$. Set $\mathcal{G}_{0}=\left\{Q_{0}\right\}$ and $\varphi_{i}^{0} \equiv a_{i, Q_{0}}$, where

$$
a_{i, Q_{0}}= \begin{cases}0, & i \neq i\left(Q_{0}\right) \\ \lambda, & i=i\left(Q_{0}\right)\end{cases}
$$

Then (3.3) holds trivially for $n=0$. If $i \neq i\left(Q_{0}\right)$, then $\left(\varphi_{i}^{0}\right)_{Q_{0}}=0$. If $i=i\left(Q_{0}\right)$, then by (3.6)

$$
\left(\varphi_{i}^{0}\right)_{Q_{0}}=\lambda \leq-\lambda+\log _{C_{\mu}}\left(\frac{\mu\left(Q_{0}\right)}{\mu\left(Q_{0} \cap E_{i}\right)}\right) \leq-K_{N}+\log _{C_{\mu}}\left(\frac{\mu\left(Q_{0}\right)}{\mu\left(Q_{0} \cap E_{i}\right)}\right)
$$

and (3.5) holds for $n=0$.
For $n \geq 1$, define $\mathcal{G}_{n}$ as the set of maximal dyadic cubes $Q \subset Q^{\prime} \in \mathcal{G}_{n-1}$ such that for some $i, 1 \leq i \leq N$,

$$
\begin{equation*}
\left(\varphi_{i}^{n-1}\right)_{Q}>\log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)}\right) \tag{3.7}
\end{equation*}
$$

Notice that by (3.5) the condition (3.7) cannot hold for any $Q^{\prime} \in \mathcal{G}_{n-1}$. Define $\varphi_{i}^{n}$ by (3.2) with

$$
a_{i, Q}= \begin{cases}-\min \left\{K_{N}+\log _{C_{\mu}} C^{*},\left(\varphi_{i}^{n-1}\right)_{Q}\right\}, & i \neq i(Q)  \tag{3.8}\\ -\sum_{j \neq i(Q)} a_{j, Q}, & i=i(Q)\end{cases}
$$

Then

$$
\begin{aligned}
\varphi_{i}^{n} & =\varphi_{i}^{n-1}+\sum_{Q \in \mathcal{G}_{n}, i(Q)=i} a_{i, Q} \chi_{Q}+\sum_{Q \in \mathcal{G}_{n}, i(Q) \neq i} a_{i, Q} \chi_{Q} \\
& \geq \varphi_{i}^{n-1}-\sum_{Q \in \mathcal{G}_{n}, i(Q) \neq i}\left(\varphi_{i}^{n-1}\right)_{Q} \chi_{Q} \\
& \geq \varphi_{i}^{n-1}-\varphi_{i}^{n-1} \chi_{Q_{0}} \geq 0
\end{aligned}
$$

since $a_{i(Q), Q} \geq 0, \mathcal{G}_{n}$ is disjoint by maximality and $\varphi_{i}^{n-1} \geq 0$ is constant in each $Q \in \mathcal{G}_{n}$. In addition, by (3.2) and (3.8)

$$
\sum_{i=1}^{N} \varphi_{i}^{n}=\sum_{i=1}^{N} \varphi_{i}^{n-1}+\sum_{Q \in \mathcal{G}_{n}}\left(\sum_{i=1}^{N} a_{i, Q}\right) \chi_{Q}=\lambda+0=\lambda,
$$

and (3.3) has been proven.

For $Q \in \mathcal{G}_{n}$ and $i \neq i(Q)$,

$$
\left|a_{i, Q}\right| \leq K_{N}+\log _{C_{\mu}} C^{*} \leq A_{N}
$$

and

$$
\left|a_{i(Q), Q}\right| \leq \sum_{j \neq i(Q)}\left|a_{j, Q}\right| \leq(N-1)\left(K_{N}+\log _{C_{\mu}} C^{*}\right) \leq A_{N}
$$

by (3.8) verifying (3.4). To establish (3.5), fix $Q \in \mathcal{G}_{n}$. For $i=i(Q)$, we estimate

$$
\left(\varphi_{i}^{n}\right)_{Q} \leq \lambda \leq-\lambda+\log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)}\right) \leq-K_{N}+\log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)}\right)
$$

by (3.3) and (3.6). For $i \neq i(Q)$, let $Q^{*}$ be the parent of $Q$. Then $\left(\varphi_{i}^{n-1}\right)_{Q^{*}}=$ $\left(\varphi_{i}^{n-1}\right)_{Q}$ since the accuracy of $Q$ appears not until the level $n$. In addition, by (2.9)

$$
\frac{\mu\left(Q^{*}\right)}{\mu\left(Q^{*} \cap E_{i}\right)} \leq C^{*} \frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)},
$$

and consequently

$$
\log _{C_{\mu}}\left(\frac{\mu\left(Q^{*}\right)}{\mu\left(Q^{*} \cap E_{i}\right)}\right) \leq \log _{C_{\mu}} C^{*}+\log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)}\right) .
$$

Further, (3.7) fails for $Q^{*}$ since $Q$ is maximal, and

$$
\begin{align*}
\left(\varphi_{i}^{n-1}\right)_{Q} & =\left(\varphi_{i}^{n-1}\right)_{Q^{*}} \leq \log _{C_{\mu}}\left(\frac{\mu\left(Q^{*}\right)}{\mu\left(Q^{*} \cap E_{i}\right)}\right) \\
& \leq \log _{C_{\mu}} C^{*}+\log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)}\right) \tag{3.9}
\end{align*}
$$

If $a_{i, Q}=-\left(\varphi_{i}^{n-1}\right)_{Q}$, then $\varphi_{i}^{n}=\varphi_{i}^{n-1}-\left(\varphi_{i}^{n-1}\right)_{Q}$ in $Q$ by (3.2), which implies $\left(\varphi_{i}^{n}\right)_{Q}=0$ and (3.5) holds. If $a_{i, Q}=-\left(K_{N}+\log _{C_{\mu}} C^{*}\right)$, then

$$
\varphi_{i}^{n}=\varphi_{i}^{n-1}-\left(K_{N}+\log _{C_{\mu}} C^{*}\right)
$$

in $Q$ implying

$$
\left(\varphi_{i}^{n}\right)_{Q}=\left(\varphi_{i}^{n-1}\right)_{Q}-\left(K_{N}+\log _{C_{\mu}} C^{*}\right) \leq \log _{C_{\mu}}\left(\frac{\mu(Q)}{\mu\left(Q \cap E_{i}\right)}\right)-K_{N}
$$

by (3.9). Thus, (3.5) holds in this case, as well, and the induction is complete.

Theorem 3.2 (Dyadic Uchiyama's theorem). Let $\lambda>0$, let $Q_{0} \subset X$ be a dyadic cube and let $E_{1}, \ldots, E_{N}$ be measurable subsets of $Q_{0}$ such that

$$
\begin{equation*}
\min _{1 \leq i \leq N} \frac{\mu\left(Q \cap E_{i}\right)}{\mu(Q)} \leq C_{\mu}^{-2 \lambda} \tag{3.10}
\end{equation*}
$$

for all dyadic $Q \subset Q_{0}$. Then there exist functions $f_{1}, \ldots, f_{N}$ satisfying

$$
\begin{equation*}
\sum_{i=1}^{N} f_{i}=1, \quad 0 \leq f_{i} \leq 1, \quad i=1, \ldots, N \tag{3.11}
\end{equation*}
$$

$$
\begin{equation*}
f_{i}=0 \quad \text { almost everywhere in } E_{i}, \quad i=1, \ldots, N \tag{3.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|f_{i}\right\|_{\mathrm{BMO}_{\mathrm{d}}\left(Q_{0}\right)} \leq \frac{C_{N}}{\lambda}, \quad i=1, \ldots, N \tag{3.13}
\end{equation*}
$$

where the constant $C_{N}$ depends only on $N$ and the doubling constant $C_{\mu}$.
Proof. Denote $K_{N}:=\log _{C_{\mu}}(2 N)$. If $\lambda \leq K_{N}$, we can set

$$
f_{i}=\frac{1-\chi_{E_{i}}}{\sum_{i=1}^{N}\left(1-\chi_{E_{i}}\right)} \in[0,1], \quad i=1, \ldots, N
$$

and $C_{N}=K_{N} \geq \lambda$. Then

$$
\sup _{Q \subset Q_{0}} f_{Q}\left|f_{i}-\left(f_{i}\right)_{Q}\right| d \mu \leq 1 \leq \frac{C_{N}}{\lambda}
$$

and (3.11), (3.12) and (3.13) hold. These functions $f_{i}$ are well-defined almost everywhere since

$$
\mu\left(\bigcap_{i=1}^{N} E_{i}\right)=0
$$

which follows by applying dyadic Lebesgue's theorem 2.1 to the inequality

$$
f_{Q} \chi_{\bigcap_{i=1}^{N} E_{i}} d \mu=\frac{\mu\left(Q \cap\left(\bigcap_{i=1}^{N} E_{i}\right)\right)}{\mu(Q)} \leq C_{\mu}^{-2 \lambda}<1, \quad Q \subset Q_{0}
$$

implied by (3.10); For almost every $x \in \bigcap_{i=1}^{N} E_{i}$, the left-hand side goes to $\chi_{\bigcap_{i=1}^{N} E_{i}}(x)$ as $Q$ shrinks to $x$, and thus $\chi_{\bigcap_{i=1}^{N} E_{i}}=0$ almost everywhere.

Then assume $\lambda>K_{N}$. Lemma 3.1 gives us families $\mathcal{G}_{n}$ of dyadic cubes and functions $\varphi_{i}^{n}, i=1, \ldots, N, n=0,1, \ldots$, satisfying (3.2)-(3.5). Let $n \geq 1$ and for $Q^{\prime} \in \mathcal{G}_{n-1}$, estimate $\sum_{Q \in \mathcal{G}_{n}\left(Q^{\prime}\right)} \mu(Q)$ when it is different from 0 , where $\mathcal{G}_{n}\left(Q^{\prime}\right):=\left\{Q \in \mathcal{G}_{n}: Q \subset Q^{\prime}\right\}$. Denote

$$
\mathcal{G}_{n}^{i}:=\left\{Q \in \mathcal{G}_{n}:(3.7) \text { holds for } i \text { and } Q\right\}
$$

For $Q \in \mathcal{G}_{n}^{i}$, we then have $\left(\varphi_{i}^{n-1}\right)_{Q^{\prime}}=\left(\varphi_{i}^{n-1}\right)_{Q}>0$. (3.7) also implies

$$
\mu(Q)<C_{\mu}^{\left(\varphi_{i}^{n-1}\right)_{Q}} \mu\left(Q \cap E_{i}\right)
$$

and thus

$$
\begin{align*}
\sum_{Q \in \mathcal{G}_{n}\left(Q^{\prime}\right)} \mu(Q) & \leq \sum_{i=1}^{N} \sum_{Q \in \mathcal{G}_{n}^{i}\left(Q^{\prime}\right)} \mu(Q) \\
& \leq \sum_{i=1}^{N} \sum_{Q \in \mathcal{G}_{n}^{i}\left(Q^{\prime}\right)} C_{\mu}^{\left(\varphi_{i}^{n-1}\right)_{Q}} \mu\left(Q \cap E_{i}\right)  \tag{3.14}\\
& \leq \sum_{i=1}^{N} C_{\mu}^{\left(\varphi_{i}^{n-1}\right)_{Q^{\prime}}} \mu\left(Q^{\prime} \cap E_{i}\right) \\
& \leq N C_{\mu}^{-K_{N}} \mu\left(Q^{\prime}\right)=\frac{1}{2} \mu\left(Q^{\prime}\right) .
\end{align*}
$$

In the third inequality, we have used disjointness of $\mathcal{G}_{n}^{i}$, and the fourth inequality follows from (3.5) for $\left(\varphi_{i}^{n-1}\right)_{Q^{\prime}}>0$.

For $Q^{\prime} \in \mathcal{G}_{m}, m \geq 0$, (3.14) implies by induction

$$
\begin{equation*}
\sum_{Q \in \mathcal{\mathcal { G } _ { n }}\left(Q^{\prime}\right)} \mu(Q) \leq \frac{1}{2^{n-m}} \mu\left(Q^{\prime}\right), \quad n=m, m+1, \ldots \tag{3.15}
\end{equation*}
$$

Then, by (3.2), (3.4) and (3.15)

$$
\left\|\varphi_{i}^{n}-\varphi_{i}^{n-1}\right\|_{L^{1}(X)}=\left\|\sum_{Q \in \mathcal{G}_{n}} a_{i, Q} \chi_{Q}\right\|_{L^{1}(X)} \leq A_{N} \sum_{Q \in \mathcal{G}_{n}} \mu(Q) \leq \frac{A_{N}}{2^{n}} \mu\left(Q_{0}\right)
$$

for $n \geq 1$. Thus, each $\left\{\varphi_{i}^{n}\right\}_{n}$ is a Cauchy sequence in $L^{1}\left(Q_{0}\right)$ and the limit function $\varphi_{i} \in L^{1}\left(Q_{0}\right)$ exists. Moreover, using (3.2) recursively, we see that $\varphi_{i}$ can be written almost everywhere as

$$
\begin{equation*}
\varphi_{i}=a_{i, Q_{0}}+\sum_{n=1}^{\infty} \sum_{Q \in \mathcal{G}_{n}} a_{i, Q} \chi_{Q}, \tag{3.16}
\end{equation*}
$$

and the average of $\varphi_{i}$ over a dyadic cube $Q^{\prime}$ is

$$
\begin{equation*}
\left(\varphi_{i}\right)_{Q^{\prime}}=a_{i, Q_{0}}+\sum_{n=1}^{\infty} \sum_{\substack{Q \in \mathcal{G}_{n} \\ Q \nsupseteq Q^{\prime}}} a_{i, Q}+\sum_{n=1}^{\infty} \sum_{Q \in \mathcal{G}_{n}\left(Q^{\prime}\right)} a_{i, Q} \frac{\mu(Q)}{\mu\left(Q^{\prime}\right)} . \tag{3.17}
\end{equation*}
$$

Let $Q^{\prime} \subset Q_{0}$ be a dyadic cube. If $\mathcal{G}_{n}\left(Q^{\prime}\right)=\emptyset$ for every $n$, then $\varphi_{i}=\left(\varphi_{i}\right)_{Q^{\prime}}$ in $Q^{\prime}$, and thus

$$
f_{Q^{\prime}}\left|\varphi_{i}-\left(\varphi_{i}\right)_{Q^{\prime}}\right| d \mu=0 .
$$

If, in turn, $m$ is the smallest index such that $\mathcal{G}_{m}\left(Q^{\prime}\right) \neq \emptyset$, then

$$
\begin{aligned}
f_{Q^{\prime}}\left|\varphi_{i}-\left(\varphi_{i}\right)_{Q^{\prime}}\right| d \mu & \leq f_{Q^{\prime}} \sum_{n=1}^{\infty} \sum_{Q \in \mathcal{G}_{n}\left(Q^{\prime}\right)}\left|a_{i, Q}\right|\left|\chi_{Q}-\frac{\mu(Q)}{\mu\left(Q^{\prime}\right)}\right| d \mu \\
& \leq f_{Q^{\prime}} \sum_{n=\max \{m, 1\}} \sum_{Q \in \mathcal{G}_{n}\left(Q^{\prime}\right)}\left|a_{i, Q}\right|\left(\chi_{Q}+\frac{\mu(Q)}{\mu\left(Q^{\prime}\right)}\right) d \mu \\
& \leq 2 A_{N} \sum_{n=m}^{\infty} \sum_{Q \in \mathcal{G}_{n}\left(Q^{\prime}\right)} \frac{\mu(Q)}{\mu\left(Q^{\prime}\right)} \\
& \leq 2 A_{N} \sum_{n=m}^{\infty} \frac{1}{2^{n-m}}=: C_{N}
\end{aligned}
$$

by (3.16), (3.17), (3.4) and (3.15). Hence

$$
\begin{equation*}
\left\|\varphi_{i}\right\|_{\mathrm{BMO}_{\mathrm{d}}\left(Q_{0}\right)} \leq C_{N}, \quad i=1, \ldots, N . \tag{3.18}
\end{equation*}
$$

Set $f_{i}=\varphi_{i} / \lambda$ for $i=1, \ldots, N$. Then (3.11) follows from (3.3) and (3.13) follows from (3.18). To conclude the proof, we establish (3.12). By (2.4) and (2.5) every $x \in Q_{0}$ lies in a unique dyadic cube $Q_{k}(x)$ in each generation $k=0,1, \ldots$, where $Q_{0}(x)=Q_{0}$. For almost every such $x, Q_{k}(x) \in \bigcup_{n=0}^{\infty} \mathcal{G}_{n}$ for only finitely many $k$. To show this, denote

$$
G=\left\{x \in Q_{0}: Q_{k}(x) \in \bigcup_{n=0}^{\infty} \mathcal{G}_{n} \text { for infinitely many } k\right\} .
$$

Then by construction of $\mathcal{G}_{n}$, for every $x \in G$ and $n \in \mathbb{N}$, there exists $k$ such that $Q_{k}(x) \in \mathcal{G}_{n}$. In particular, since $x \in Q_{k}(x)$, we have $G \subset \bigcup_{Q \in \mathcal{G}_{n}} Q$ for every $n$, and consequently by (3.15)

$$
\mu(G) \leq \sum_{Q \in \mathcal{G}_{n}} \mu(Q) \leq \frac{1}{2^{n}} \mu\left(Q_{0}\right) \rightarrow 0
$$

as $n \rightarrow \infty$. Hence for almost every $x \in Q_{0}$ there exist indices $k_{x}$ and $n_{x}$ such that for every $k>k_{x}$ and $n>n_{x}$, we have $Q_{k}(x) \notin \mathcal{G}_{n}$ and each $\varphi_{i}=\varphi_{i}^{n-1}$ is constant in $Q_{k}(x)$. Thus, (3.7) fails for $Q_{k}(x)$ and

$$
\varphi_{i}(x)=\left(\varphi_{i}^{n-1}\right)_{Q_{k}(x)} \leq \log _{C_{\mu}}\left(\frac{\mu\left(Q_{k}(x)\right)}{\mu\left(Q_{k}(x) \cap E_{i}\right)}\right)
$$

for $k>k_{x}$ and $n>n_{x}$. On the other hand, by dyadic Lebesgue's theorem 2.1,

$$
\frac{\mu\left(Q_{k}(x) \cap E_{i}\right)}{\mu\left(Q_{k}(x)\right)}=f_{Q_{k}(x)} \chi_{E_{i}} d \mu \rightarrow 1
$$

for almost every $x \in E_{i}$ as $k \rightarrow \infty$, and consequently

$$
\log _{C_{\mu}}\left(\frac{\mu\left(Q_{k}(x)\right)}{\mu\left(Q_{k}(x) \cap E_{i}\right)}\right) \rightarrow 0
$$

for almost every $x \in E_{i}$ as $k \rightarrow \infty$. Hence $f_{i}=\varphi_{i}=0$ almost everywhere in $E_{i}$, which completes the proof.

We also give a proof for the global version of Theorem 3.2.
Corollary 3.3 (Global version of dyadic Uchiyama's theorem). Let $\lambda>0$ and let $E_{1}, \ldots, E_{N}$ be measurable subsets of $X$ such that

$$
\begin{equation*}
\min _{1 \leq i \leq N} \frac{\mu\left(Q \cap E_{i}\right)}{\mu(Q)} \leq C_{\mu}^{-2 \lambda} \tag{3.19}
\end{equation*}
$$

for all dyadic $Q \subset X$. Then there exist functions $f_{1}, \ldots, f_{N}$ satisfying

$$
\begin{equation*}
\sum_{i=1}^{N} f_{i}=1, \quad 0 \leq f_{i} \leq 1, \quad i=1, \ldots, N \tag{3.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|f_{i}\right\|_{\mathrm{BMO}_{\mathrm{d}}} \leq \frac{C_{N}}{\lambda}, \quad i=1, \ldots, N \tag{3.21}
\end{equation*}
$$

where the constant $C_{N}$ depends only on $N$ and the doubling constant $C_{\mu}$.
Proof. For simplicity, we assume that $X$ contains only one ancestor space. If this is not the case, we can make our deductions individually in each ancestor space and finally combine the resulting functions to obtain the desired properties in the whole $X$.

Let $Q_{1} \subset Q_{2} \subset \ldots$ be a sequence of dyadic cubes of decreasing generations in $X$ and let $E_{i, k}=E_{i} \cap Q_{k}$ for every $i$ and $k$. Then

$$
\min _{1 \leq i \leq N} \frac{\mu\left(Q \cap E_{i, k}\right)}{\mu(Q)} \leq \min _{1 \leq i \leq N} \frac{\mu\left(Q \cap E_{i}\right)}{\mu(Q)} \leq C_{\mu}^{-2 \lambda}
$$

for every dyadic $Q \subset Q_{k}$, and thus Theorem 3.2 gives us functions $f_{i, k}, i=$ $1, \ldots, N, k=1,2, \ldots$, satisfying (3.20) with $f_{i, k}=0$ almost everywhere in $E_{i, k}$ and

$$
\left\|f_{i, k}\right\|_{\mathrm{BMO}_{\mathrm{d}}\left(Q_{k}\right)} \leq \frac{C_{N}}{\lambda} .
$$

Since the functions $f_{i, k}$ are uniformly bounded, we can pick subsequences $\left\{f_{i, k_{j}}\right\}_{j}, i=1, \ldots, N$, converging weak* in $L^{\infty}(X)$. Setting $f_{i}, i=1, \ldots, N$, to be the corresponding weak* limits, they satisfy (3.20) and (3.21) by the definition of weak* convergence. To prove (3.22), let $g$ satisfy (2.13) with a dyadic cube $Q \subset X$. Then $g \in L^{1}(X)$ and

$$
\begin{aligned}
\left|\int_{X} f_{i} g d \mu\right| & =\left|\lim _{j \rightarrow \infty} \int_{X} f_{i, k_{j}} g-\left(f_{i, k_{j}}\right)_{Q} g d \mu\right| \\
& \leq \limsup _{j \rightarrow \infty}\left\|f_{i, k_{j}}\right\|_{\mathrm{BMO}_{\mathrm{d}}\left(Q_{k_{j}}\right)} \leq \frac{C_{N}}{\lambda}
\end{aligned}
$$

by the definition of weak* convergence, (2.13) and the fact that $Q \subset Q_{k_{j}}$ when $j$ is large enough. Thus, (3.22) with constant $2 C_{N}$ follows from Lemma 2.3.

## 4. Uchiyama's theorem from dyadic Uchiyama's theorem

In this section, we prove Uchiyama's theorem from its dyadic counterpart using finitely randomized dyadic systems. For simplicity, we assume that all dyadic cubes used in the section are constructed such that there is $z_{0} \in X$ which is a center point of some cube in each generation $k$, which is possible according to [8].

In [10], Hytönen and Tapiola constructed a probability space $(\Omega, \mathbb{P})$ defining a system of dyadic cubes $\mathcal{D}(\omega)=\left\{Q_{\alpha}^{k}(\omega)\right\}_{k, \alpha}$ with the properties (2.2)-(2.8) for each $\omega \in \Omega$, satisfying also the following property. There exist constants $C>0$ and $\eta>0$ such that

$$
\begin{equation*}
\mathbb{P}\left(\left\{\omega \in \Omega: x \in \bigcup_{\alpha \in \mathscr{A}_{k}} \partial_{\varepsilon} \bar{Q}_{\alpha}^{k}(\omega)\right\}\right) \leq C\left(\frac{\varepsilon}{\delta^{k}}\right)^{\eta} \tag{4.1}
\end{equation*}
$$

for every $x \in X, k \in \mathbb{Z}$ and $\varepsilon>0$. Here $\partial_{\varepsilon} A$ denotes the $\varepsilon$-boundary of a set $A \subset X$ defined by

$$
\partial_{\varepsilon} A:=\{x \in A: d(x, X \backslash A)<\varepsilon\} \cup\{x \in X \backslash A: d(x, A)<\varepsilon\} .
$$

In the construction, they use the sample space

$$
\Omega=\left\{0,1, \ldots,\left\lfloor\frac{1}{\delta}\right\rfloor\right\}^{\mathbb{Z}}
$$

and independent uniform distributions in all generations $k \in \mathbb{Z}$. The proof of (4.1) in [10, Theorem 5.2] needs randomization only in finitely many generations $k, \ldots, k+L$, where $L$ is determined by

$$
\delta^{k+L+1}<\varepsilon \leq \delta^{k+L}
$$

In particular, the randomization is needed only up to the generation

$$
\begin{equation*}
m=k+L=\left\lfloor\frac{\log \varepsilon}{\log \delta}\right\rfloor \tag{4.2}
\end{equation*}
$$

We define a finite version of the probability space $(\Omega, \mathbb{P})$ for every $m \in \mathbb{N}$ as $\left(\Omega_{m}, \mathbb{P}_{m}\right)$, where the sample space is given by

$$
\Omega_{m}=\prod_{k<-m}\{0\} \times \prod_{-m \leq k \leq m}\left\{0,1, \ldots,\left\lfloor\frac{1}{\delta}\right\rfloor\right\} \times \prod_{k>m}\{0\}
$$

and $\mathbb{P}_{m}$ has a uniform distribution in $\Omega_{m}$. From (4.2) we see that the following weaker version of (4.1) holds in $\left(\Omega_{m}, \mathbb{P}_{m}\right)$. There exist constants $C>0$ and $\eta>0$ such that

$$
\begin{equation*}
\mathbb{P}_{m}\left(\left\{\omega \in \Omega_{m}: x \in \bigcup_{\alpha \in \mathscr{\mathscr { A } _ { k }}} \partial_{\varepsilon} \bar{Q}_{\alpha}^{k}(\omega)\right\}\right) \leq C\left(\frac{\varepsilon}{\delta^{k}}\right)^{\eta} \tag{4.3}
\end{equation*}
$$

for every $m \in \mathbb{N}, x \in X, k \in\{-m, \ldots, m\}$ and $\varepsilon \geq \delta^{m}$. In addition, the properties (2.2)-(2.8) hold for $\mathcal{D}(\omega)$ for every $\omega \in \Omega_{m}$ since $\Omega_{m} \subset \Omega$.

The unique dyadic cube of generation $k$ in $\mathcal{D}(\omega)$ containing $x \in X$ is denoted by $Q_{\omega}^{k}(x)$. The averaging function of a locally integrable function $f$ is defined by

$$
E_{\omega}^{k} f(x):=f_{Q_{\omega}^{k}(x)} f d \mu
$$

and the difference function by

$$
\Delta_{\omega}^{k} f(x):=E_{\omega}^{k+1} f(x)-E_{\omega}^{k} f(x)
$$

When we consider dyadic BMO in a specified dyadic system $\mathcal{D}(\omega)$ for $\omega \in \Omega$, more precisely take the supremum in (2.11) over all $Q \in \mathcal{D}(\omega)$, we denote $\mathrm{BMO}_{\omega}$ instead of $\mathrm{BMO}_{\mathrm{d}}$.

The following is a finitely randomized version of [3, Theorem 3.1].
Theorem 4.1. Let $m \in \mathbb{N}$ and let $\left(\Omega_{m}, \mathbb{P}_{m}\right)$ be the probability space defined above. Denote $B_{m}=B\left(z_{0}, c_{0} \delta^{-m}\right)$ and $Q_{m, \omega}=Q_{\omega}^{-m}\left(z_{0}\right)$. Let $\left\{f^{\omega}\right\}_{\omega \in \Omega_{m}}$ be a family of functions with $f^{\omega} \in \mathrm{BMO}_{\omega}\left(Q_{m, \omega}\right)$ satisfying

$$
\begin{equation*}
\left\|f^{\omega}\right\|_{\mathrm{BMO}_{\omega}\left(Q_{m, \omega}\right)} \leq M_{1} \quad \text { and } \quad\left\|f^{\omega}\right\|_{L^{\infty}\left(Q_{m, \omega}\right)} \leq M_{2} \tag{4.4}
\end{equation*}
$$

for every $\omega \in \Omega_{m}$ with $M_{1}$ and $M_{2}$ independent of $\omega$. Then the function $f_{m}$ defined by the expectation

$$
f_{m}(x)=\mathbb{E}_{m}\left[f^{\omega}(x)\right]:=\int_{\Omega_{m}} f^{\omega}(x) d \mathbb{P}_{m}(\omega)
$$

satisfies

$$
\begin{equation*}
f_{B}\left|f_{m}-\left(f_{m}\right)_{B}\right| d \mu \leq C M_{1}+\frac{4 M_{2}}{\mu(B)} \mu\left(B\left(x_{0}, \delta^{m}\right) \backslash\left\{x_{0}\right\}\right) \tag{4.5}
\end{equation*}
$$

for every $B=B\left(x_{0}, r\right) \subset B_{m}$ with $r \geq \delta^{m}$, where the constant $C$ depends only on the doubling constant $C_{\mu}$.
Proof. Fix a ball $B=B\left(x_{0}, r\right) \subset B_{m}$ such that $r \geq \delta^{m}$, and let $n$ be the unique integer satisfying

$$
\delta^{n+1}<r \leq \delta^{n} .
$$

Then, in particular, $-m \leq n \leq m$. Decompose $f_{m}(x)$ into two parts

$$
f_{m}(x)=g(x)+h(x)
$$

with

$$
g(x):=\mathbb{E}_{m}\left[f^{\omega}(x)-E_{\omega}^{n} f^{\omega}(x)\right]
$$

and

$$
h(x):=\mathbb{E}_{m}\left[E_{\omega}^{n} f^{\omega}(x)\right]=\mathbb{E}_{m}\left[\sum_{-m \leq k<n} \Delta_{\omega}^{k} f^{\omega}(x)+E_{\omega}^{-m} f^{\omega}(x)\right]
$$

We have

$$
\begin{aligned}
f_{B}\left|f_{m}-\left(f_{m}\right)_{B}\right| d \mu & \leq f_{B}\left|g-g_{B}\right| d \mu+f_{B}\left|h-h_{B}\right| d \mu \\
& \leq 2 f_{B}|g| d \mu+2 f_{B}\left|h-h\left(x_{0}\right)\right| d \mu
\end{aligned}
$$

and to prove the claim (4.5), it suffices to show that

$$
\begin{gather*}
\frac{1}{\mu(B)} \int_{B}|g| d \mu \leq C M_{1},  \tag{4.6}\\
\frac{1}{\mu(B)} \int_{B \backslash B\left(x_{0}, \delta^{m}\right)}\left|h-h\left(x_{0}\right)\right| d \mu \leq C M_{1} \tag{4.7}
\end{gather*}
$$

$$
\frac{1}{\mu(B)} \int_{B\left(x_{0}, \delta^{m}\right)}\left|h-h\left(x_{0}\right)\right| d \mu \leq \frac{2 M_{2}}{\mu(B)} \mu\left(B\left(x_{0}, \delta^{m}\right) \backslash\left\{x_{0}\right\}\right) .
$$

We begin with inequality (4.6). Since

$$
\frac{1}{\mu(B)} \int_{B}|g| d \mu \leq \mathbb{E}_{m}\left[f_{B}\left|f^{\omega}-E_{\omega}^{n} f^{\omega}\right| d \mu\right],
$$

it is sufficient to prove that for each $\omega \in \Omega_{m}$,

$$
f_{B}\left|f^{\omega}-E_{\omega}^{n} f^{\omega}\right| d \mu \leq C M_{1}
$$

There is a finite index set $\mathscr{B} \subset \mathscr{A}_{n}$ such that $B \subset \bigcup_{\beta \in \mathscr{B}} Q_{\beta}^{n}(\omega)$ with $B \cap$ $Q_{\beta}^{n}(\omega) \neq \emptyset$. Moreover, the number of elements in $\mathscr{B}, \# \mathscr{B}$, is bounded by a constant depending only on the doubling constant since $r \leq \delta^{n}$ and the centers $z_{\beta}^{n}$ are $c_{0} \delta^{n}$ separated by (2.4) and (2.6). Also, by the doubling property (2.1) we have $\mu\left(Q_{\beta}^{n}(\omega)\right) \leq C \mu(B)$ for each $\beta \in \mathscr{B}$. Thus,

$$
\begin{aligned}
f_{B}\left|f^{\omega}-E_{\omega}^{n} f^{\omega}\right| d \mu & \leq \frac{1}{\mu(B)} \int_{\bigcup_{\beta \in \mathscr{R}} Q_{\beta}^{n}(\omega)}\left|f^{\omega}-E_{\omega}^{n} f^{\omega}\right| d \mu \\
& =\sum_{\beta \in \mathscr{B}} \frac{\mu\left(Q_{\beta}^{n}(\omega)\right)}{\mu(B)} f_{Q_{\beta}^{n}(\omega)}\left|f^{\omega}-\left(f^{\omega}\right)_{Q_{\beta}^{n}(\omega)}\right| d \mu \\
& \leq C \# \mathscr{B}\left\|f^{\omega}\right\|_{\operatorname{BMO}_{\omega}\left(Q_{m, \omega}\right)} \\
& \leq C M_{1}
\end{aligned}
$$

by (4.4). Notice that $Q_{\beta}^{n}(\omega) \subset Q_{m, \omega}$ since $Q_{\beta}^{n}(\omega) \cap B \neq \emptyset$ and $B \subset B_{m} \subset Q_{m, \omega}$.
Next we consider inequality (4.7). For fixed $x \in B \backslash B\left(x_{0}, \delta^{m}\right)$, denote

$$
\Lambda_{k}:=\left\{\omega \in \Omega_{m}: \text { there exists } Q \in \mathcal{D}_{k+1}(\omega) \text { with } x, x_{0} \in Q\right\}
$$

Then $\mathbb{P}_{m}\left(\Lambda_{k}\right)=1$ for every $k<-m$ since $x, x_{0} \in B \subset B_{m} \subset Q_{m, \omega}$ for every $\omega \in \Omega_{m}$. On the other hand, when $-m \leq k \leq m$,

$$
\begin{aligned}
\mathbb{P}_{m}\left(\Omega_{m} \backslash \Lambda_{k}\right) & \leq \mathbb{P}_{m}\left(\left\{\omega \in \Omega_{m}: x \in \bigcup_{\alpha \in \mathscr{A}_{k+1}} \partial_{d\left(x, x_{0}\right)} \bar{Q}_{\alpha}^{k+1}(\omega)\right\}\right) \\
& \leq C\left(\frac{d\left(x, x_{0}\right)}{\delta^{k+1}}\right)^{\eta} \leq C \delta^{(n-k) \eta}
\end{aligned}
$$

by (4.3) and the fact that $d\left(x, x_{0}\right)<r \leq \delta^{n}$. Since $E_{\omega}^{-m} f^{\omega}(x)=E_{\omega}^{-m} f^{\omega}\left(x_{0}\right)$ for each $\omega \in \Omega_{m}$ and $\Delta_{\omega}^{k} f^{\omega}(x)=\Delta_{\omega}^{k} f^{\omega}\left(x_{0}\right)$ when $\omega \in \Lambda_{k}$, we obtain

$$
\begin{aligned}
\left|h(x)-h\left(x_{0}\right)\right| & =\left|\mathbb{E}_{m}\left[\sum_{-m \leq k<n}\left(\Delta_{\omega}^{k} f^{\omega}(x)-\Delta_{\omega}^{k} f^{\omega}\left(x_{0}\right)\right)\right]\right| \\
& \leq \sum_{-m \leq k<n} \int_{\Omega_{m} \backslash \Lambda_{k}}\left|\Delta_{\omega}^{k} f^{\omega}(x)-\Delta_{\omega}^{k} f^{\omega}\left(x_{0}\right)\right| d \mathbb{P}_{m}(\omega) \\
& \leq \sum_{-m \leq k<n} \sup _{\omega \in \Omega_{m}}\left(\left|\Delta_{\omega}^{k} f^{\omega}(x)\right|+\left|\Delta_{\omega}^{k} f^{\omega}\left(x_{0}\right)\right|\right) \mathbb{P}_{m}\left(\Omega_{m} \backslash \Lambda_{k}\right) \\
& \leq C \sum_{-m \leq k<n} \sup _{\omega \in \Omega_{m}}\left(\left|\Delta_{\omega}^{k} f^{\omega}(x)\right|+\left|\Delta_{\omega}^{k} f^{\omega}\left(x_{0}\right)\right|\right) \delta^{(n-k) \eta} .
\end{aligned}
$$

Now for each $\omega \in \Omega_{m}$ and $-m \leq k<n$, we have

$$
\begin{aligned}
\left|\Delta_{\omega}^{k} f^{\omega}(x)\right| & =\left|f_{Q_{\omega}^{k+1}(x)} f^{\omega}(y)-\left(f^{\omega}\right)_{Q_{\omega}^{k}(x)} d \mu(y)\right| \\
& \leq \frac{\mu\left(Q_{\omega}^{k}(x)\right)}{\mu\left(Q_{\omega}^{k+1}(x)\right)} f_{Q_{\omega}^{k}(x)}\left|f^{\omega}(y)-\left(f^{\omega}\right)_{Q_{\omega}^{k}(x)}\right| d \mu(y) \\
& \leq C\left\|f^{\omega}\right\|_{\mathrm{BMO}_{\omega}\left(Q_{m, \omega}\right)} \leq C M_{1}
\end{aligned}
$$

by (2.9) and (4.4) since $Q_{\omega}^{k}(x) \subset Q_{m, \omega}$, and the same estimate holds also for $x_{0}$. Consequently,

$$
\frac{1}{\mu(B)} \int_{B \backslash B\left(x_{0}, \delta^{m}\right)}\left|h-h\left(x_{0}\right)\right| d \mu \leq 2 C M_{1} \sum_{k<n} \delta^{(n-k) \eta} \leq C M_{1} .
$$

Finally, inequality (4.8) is obvious since we have

$$
\left|h(x)-h\left(x_{0}\right)\right| \leq \mathbb{E}_{m}\left[\left|E_{\omega}^{n} f^{\omega}(x)-E_{\omega}^{n} f^{\omega}\left(x_{0}\right)\right|\right] \leq 2 M_{2}
$$

by the definition of $h$ and (4.4). This finishes the proof.
Notice that the last term in (4.5) goes to 0 as $m \rightarrow \infty$ when $M_{2}$ is independent of $m$. Finally, we apply our Theorem 4.1 to obtain a non-dyadic version of Uchiyama's theorem, which also occurs in [11].

Theorem 4.2 (Uchiyama's theorem). Let $\lambda>0$ and let $E_{1}, \ldots, E_{N}$ be measurable subsets of $X$ such that

$$
\begin{equation*}
\min _{1 \leq i \leq N} \frac{\mu\left(B \cap E_{i}\right)}{\mu(B)} \leq C_{\mu}^{-2 \lambda} \tag{4.9}
\end{equation*}
$$

for all balls $B \subset X$. Then there exist functions $f_{1}, \ldots, f_{N}$ satisfying

$$
\begin{equation*}
\sum_{i=1}^{N} f_{i}=1, \quad 0 \leq f_{i} \leq 1, \quad i=1, \ldots, N \tag{4.10}
\end{equation*}
$$

$$
\begin{equation*}
f_{i}=0 \quad \text { almost everywhere in } E_{i}, \quad i=1, \ldots, N \tag{4.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|f_{i}\right\|_{\mathrm{BMO}} \leq \frac{C_{N}}{\lambda}, \quad i=1, \ldots, N \tag{4.12}
\end{equation*}
$$

where the constant $C_{N}$ depends only on $N$ and the doubling constant $C_{\mu}$.
Proof. Let $m \in \mathbb{N}$ and let $\left(\Omega_{m}, \mathbb{P}_{m}\right)$ be the probability space defined in the beginning of the section. Denote $B_{m}:=B\left(z_{0}, c_{0} \delta^{-m}\right), E_{i, m}:=E_{i} \cap B_{m}$ and $Q_{m, \omega}:=Q_{\omega}^{-m}\left(z_{0}\right)$ for $\omega \in \Omega_{m}$. For any $Q=Q_{\alpha}^{k}(\omega) \in \mathcal{D}(\omega), \omega \in \Omega$, we have $B^{1} \subset Q \subset B^{2}$ with $B^{1}:=B\left(z_{\alpha}^{k}, c_{0} \delta^{k}\right)$ and $B^{2}:=B\left(z_{\alpha}^{k}, C_{0} \delta^{k}\right)$ by (2.6). Thus by the doubling condition (2.1) and (4.9),

$$
\begin{aligned}
\min _{1 \leq i \leq N} \frac{\mu\left(Q \cap E_{i, m}\right)}{\mu(Q)} & \leq \min _{1 \leq i \leq N} \frac{\mu\left(B^{2} \cap E_{i}\right)}{\mu\left(B^{1}\right)} \leq C_{1} \min _{1 \leq i \leq N} \frac{\mu\left(B^{2} \cap E_{i}\right)}{\mu\left(B^{2}\right)} \\
& \leq C_{1} C_{\mu}^{-2 \lambda}=C_{\mu}^{-2 \lambda^{\prime}}
\end{aligned}
$$

where $\lambda^{\prime}=\lambda-\frac{1}{2} \log _{C_{\mu}} C_{1}$ and $C_{1}$ depends only on the doubling constant. If $\lambda \leq \log _{C_{\mu}} C_{1}$, the claim follows as in the beginning of the proof of Theorem 3.2. Hence we can assume $\lambda>\log _{C_{\mu}} C_{1}$, which in particular implies $\lambda^{\prime}>$ $\frac{1}{2} \log _{C_{\mu}} C_{1} \geq 0$.

By dyadic Uchiyama's theorem 3.2 applied individually in each dyadic system $\mathcal{D}(\omega)$, there are families of functions $\left\{f_{i}^{\omega}\right\}_{\omega \in \Omega_{m}}, i=1, \ldots, N$, satisfying (4.10), (4.11) on $E_{i, m}$ and

$$
\begin{equation*}
\left\|f_{i}^{\omega}\right\|_{\mathrm{BMO}_{\omega}\left(Q_{m, \omega}\right)} \leq \frac{C_{N}^{\prime}}{\lambda^{\prime}} \tag{4.13}
\end{equation*}
$$

where the constant $C_{N}^{\prime}$ depends only on $N$ and the doubling constant $C_{\mu}$. For each $i$, denote $f_{i, m}(x):=\mathbb{E}_{m}\left[f_{i}^{\omega}(x)\right]$. Then the functions $f_{i, m}$ satisfy (4.10) and (4.11) on $E_{i, m}$ by the linearity of expectation, and moreover by Theorem 4.1

$$
\begin{equation*}
f_{B}\left|f_{i, m}-\left(f_{i, m}\right)_{B}\right| d \mu \leq C \frac{C_{N}^{\prime}}{\lambda^{\prime}}+\frac{4}{\mu(B)} \mu\left(B\left(x_{0}, \delta^{m}\right) \backslash\left\{x_{0}\right\}\right) \tag{4.14}
\end{equation*}
$$

for every $B=B\left(x_{0}, r\right) \subset B_{m}$ with $r \geq \delta^{m}$, where the constant $C$ depends only on the doubling constant.

Since the functions $f_{i, m}$ are uniformly bounded with respect to $m$, we can pick subsequences $\left\{f_{i, m_{j}}\right\}_{j}, i=1, \ldots, N$, converging weak* in $L^{\infty}(X)$. Setting $f_{i}$ to be the corresponding weak* limits as $j \rightarrow \infty$, they satisfy (4.10) and (4.11) by the definition of weak* convergence. To prove (4.12), let $g$ satisfy (2.12) with a ball $B \subset X$. Then $g \in L^{1}(X)$ and

$$
\begin{aligned}
\left|\int_{X} f_{i} g d \mu\right| & =\left|\lim _{j \rightarrow \infty} \int_{X} f_{i, m_{j}} g-\left(f_{i, m_{j}}\right)_{B} g d \mu\right| \\
& \leq \limsup _{j \rightarrow \infty} f_{B}\left|f_{i, m_{j}}-\left(f_{i, m_{j}}\right)_{B}\right| d \mu \leq C \frac{C_{N}^{\prime}}{\lambda^{\prime}}
\end{aligned}
$$

by the definition of weak* convergence, (2.12), (4.14) and the fact that $B \subset$ $B_{m_{j}}$ with a radius $r \geq \delta^{m_{j}}$ when $j$ is large enough. Consequently, by Lemma
2.2 we obtain

$$
\left\|f_{i}\right\|_{\mathrm{BMO}} \leq 2 C \frac{C_{N}^{\prime}}{\lambda^{\prime}}
$$

Finally, since

$$
\frac{\lambda}{\lambda^{\prime}}=\frac{\lambda^{\prime}+\frac{1}{2} \log _{C_{\mu}} C_{1}}{\lambda^{\prime}} \leq 2,
$$

the claim (4.12) follows with $C_{N}=4 C C_{N}^{\prime}$, and the proof is complete.

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