Regularity of the local fractional maximal function

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June 12 2015

Workshop on harmonic analysis and partial differential equations
Aalto University

Based on a joint work with T. Heikkinen, J. Kinnunen and H. Tuominen.

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- One might expect that similar estimates hold for the weak gradient as in the global case.
- However, there will be extra terms occurring in our estimates.

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  [E.M. Stein, 1976] ($\alpha = 0$, $n \geq 3$),
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- The same estimate also holds for the local version $\mathcal{S}_{\alpha,\Omega} u$.

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Theorem 1 (Heikkinen, Kinnunen, K, Tuominen)

Let $n \geq 2$, $p > n/(n-1)$ and let $1 \leq \alpha < \min\left\{\frac{n-1}{p}, n - \frac{2n}{(n-1)p}\right\} + 1$. If $u \in L^p(\Omega)$, then $|D\mathcal{M}_{\alpha,\Omega}u| \in L^q(\Omega)$ with $q = np/(n-(\alpha-1)p)$. Moreover,

$$|D\mathcal{M}_{\alpha,\Omega}u(x)| \leq C\left(\mathcal{M}_{\alpha-1,\Omega}u(x) + \mathcal{S}_{\alpha-1,\Omega}u(x)\right)$$

for almost every $x \in \Omega$, where the constant C depends only on n.

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- Additional term containing local spherical fractional function compared to the global case.

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$$|Du_t^\alpha(x)| \le C\big(\mathcal{M}_{\alpha-1,\Omega}u(x) + \mathcal{S}_{\alpha-1,\Omega}u(x)\big), \quad 0 < t < 1, \qquad (1)$$

for fractional average functions

$$u_t^\alpha(x) = (t\delta(x))^\alpha \fint_{B(x,t\delta(x))} u(y)\,dy, \quad \delta(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega).$$

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3. Local fractional maximal function can be approximated pointwise by $v_k = \max_{1 \leq j \leq k} |u|_{t_j}^\alpha$, where t_j is an enumeration of the rationals between 0 and 1.
4. We can extract a subsequence $\{v_{k_j}\}$ such that $|Dv_{k_j}|$ converges weakly to $|D\mathcal{M}_{\alpha,\Omega}u|$ in $L^q(\Omega)$ as $j \to \infty$.

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- Moreover, $\mathcal{M}_{\alpha,\Omega} u$ has zero boundary values in Sobolev's sense since

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Corollary 5

If Ω is bounded with a C^1-boundary, then Theorem 4 holds with a better exponent $p^* = np/(n-\alpha p)$ instead of q.

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- When $|x|$ is small enough, the maximizing radius for $\mathcal{M}_{\alpha,\Omega}u(x)$ is the largest possible, i.e. $1 - |x|$.
- For those x, $|D\mathcal{M}_{\alpha,\Omega}u(x)|$ consists of an average term over a ball and one over a sphere, where the former is uniformly bounded and the latter tends to ∞ as $|x| \to 0$.

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- Let $n \geq 2$, $\alpha \geq 1$ and $(\alpha-1)p < n$. Let

$$\Omega = \operatorname{int}\Big(\bigcup_{k=1}^{\infty} Q_k \cup C_k\Big),$$

$$Q_k = [k, k+2^{-k}] \times [0, 2^{-k}]^{n-1}, \quad C_k = [k+2^{-k}, k+1] \times [0, 2^{-3k}]^{n-1}.$$

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- There are domains $\Omega \subset \mathbb{R}^n$, for which $\mathcal{M}_{\alpha,\Omega}(W^{1,p}(\Omega)) \not\subset W^{1,\hat{q}}(\Omega)$ when $\hat{q} > q = np/(n-(\alpha-1)p)$; The exponent q in Theorem 4 is sharp.
- Let $n \geq 2$, $\alpha \geq 1$ and $(\alpha-1)p < n$. Let

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- For every $\hat{q} > q = np/(n-(\alpha-1)p)$ we define a function u such that $u = 2^{kn/\hat{p}}$ on Q_k and u increases linearly from $2^{kn/\hat{p}}$ to $2^{(k+1)n/\hat{p}}$ on C_k, where $\hat{p} = n\hat{q}/(n+(\alpha-1)\hat{q}) > p$.

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- Then $u \in W^{1,p}(\Omega)$ but

$$|D\mathcal{M}_{\alpha,\Omega}u| \notin L^{\hat{q}}(\Omega).$$

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- Then $|D\mathcal{M}_{\alpha,\Omega} u|$ for $u \equiv 1$ does not belong to $L^r(\Omega)$.

References

- **T. Heikkinen, J. Kinnunen, J. Korvenpää and H. Tuominen, Regularity of the local fractional maximal function, Ark. Mat. 53 (2015), 127-154.**
- J. Bourgain, Averages in the plane over convex curves and maximal operators, J. Anal. Math. 47 (1986), 69–85.
- J. Kinnunen, The Hardy–Littlewood maximal function of a Sobolev-function, Israel J.Math. 100 (1997), 117–124.
- J. Kinnunen and P. Lindqvist, The derivative of the maximal function, J. Reine Angew. Math. 503 (1998), 161–167.
- J. Kinnunen and E. Saksman, Regularity of the fractional maximal function, Bull. London Math. Soc. 35 (2003), no. 4, 529–535.
- W. Schlag, A generalization of Bourgain's circular maximal theorem J. Amer. Math. Soc. 10, no. 1 (1997), 103–122.
- W. Schlag and C. Sogge, Local smoothing estimates related to the circular maximal theorem, Math. Res. Lett. 4 (1997), no.1, 1–15.
- E.M. Stein, Maximal functions. I. Spherical means, Proc. Nat. Acad. Sci. U.S.A. 73 (1976), no. 7, 2174–2175.

THANK YOU

FOR YOUR ATTENTION