A QUANTITATIVE MODULUS OF CONTINUITY FOR THE TWO-PHASE STEFAN PROBLEM

PAOLO BARONI, TUOMO KUUSI, AND JOSÉ MIGUEL URBANO

Abstract. We derive the quantitative modulus of continuity

$$\omega(r) = \left[p + \ln\left(\frac{r_0}{r}\right)\right]^{-\alpha(n,p)},$$

which we conjecture to be optimal, for solutions of the p-degenerate two-phase Stefan problem. Even in the classical case $p = 2$, this represents a twofold improvement with respect to the early 1980's state-of-the-art results by Caffarelli–Evans [5] and DiBenedetto [8], in the sense that we discard one logarithm iteration and obtain an explicit value for the exponent $\alpha(n,p)$.

1. Introduction

This paper concerns the local behaviour of bounded weak solutions of the degenerate two-phase Stefan problem

$$\partial_t\big[u + \mathcal{L}_h\, H_a(u)\big] \ni \operatorname{div}\left[|Du|^{p-2}Du\right], \qquad p \geq 2\,, \tag{1.1}$$

where H_a is the Heaviside graph centred at $a \in \mathbb{R}$, defined by

$$H_a(s) = \begin{cases} 0 & \text{if } s < a\,, \\ [0,1] & \text{if } s = a\,, \\ 1 & \text{if } s > a\,, \end{cases} \tag{1.2}$$

and $\mathcal{L}_h > 0$. Our main result is the derivation of the explicit, interior modulus of continuity

$$\omega(r) := \left[p + \ln\left(\frac{r_0}{r}\right)\right]^{-\alpha(n,p)}, \qquad 0 < r \leq r_0\,, \tag{1.3}$$

which we conjecture to be optimal.

An extensive literature, both from the theoretical and the computational points of view, is available for the classical Stefan problem

$$\partial_t\big[u + \mathcal{L}_h\, H_0(u)\big] \ni \triangle u, \tag{1.4}$$

corresponding to the case $p = 2$, which is a simplified model to describe the evolution of the configuration of a substance which is changing phase, when convective effects are neglected. The function u represents the temperature and the value $u = 0$ is the level at which the change of phase occurs; the height $\mathcal{L}_h$ of the jump of the graph $\mathcal{L}_h\, H_0(\cdot)$ corresponds to the latent heat of fusion and a selection of the graph is called the *enthalpy* of the problem. For simplicity, we consider $\mathcal{L}_h \leq 1$ from now on. The case of study of positive solutions (note we are taking $a = 0$ in

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(1.4)) is usually called one-phase Stefan problem, while if no sign assumptions are made on u we are dealing with the two-phase Stefan problem; see [15, 23] for the deduction of (1.4) from the classic formulation, which goes back to Stefan at the end of the nineteenth century [30] and has been subsequently developed in [19, 25]. We mention that the model (1.4) also finds applications in finance [26], biology related to the Lotka-Volterra model [7], and flows of solutes or gases in porous media [27].

Clearly (1.4) and (1.1) need to be understood using an appropriate notion of (weak) solution, and the one we employ is that of *differential inclusion* in the sense of graphs, see Definition 1.9; other approaches can be used and the most noticeable one is that of *viscosity solutions* in the sense introduced by Crandall and Lions, and developed by Caffarelli; see [2] and the recent survey by Salsa [29]. Notice that weak solutions are viscosity solutions once one knows they are continuous (and in fact they are, see the following lines); under an additional conditions (namely, $\{u = 0\}$ is negligible) the converse also holds true, see [20].

For the one-phase Stefan problem (1.4), continuity of weak solutions has been proved by Caffarelli and Friedman in [6], with an explicit modulus of continuity:

$$C\left[\ln\left(\frac{r_0}{r}\right)\right]^{-\epsilon}, \quad \text{if } n \geq 3; \qquad C\,2^{-\left[\ln\left(\frac{r_0}{r}\right)\right]^{\gamma}}, \quad \text{if } n = 2,$$

for a positive constant C, for any $0 < \epsilon < \frac{2}{n-2}$ and $0 < \gamma < \frac{1}{2}$. For the two-phase problem, continuity was proved, almost at the same time, by Caffarelli and Evans [5] for (1.4), and by DiBenedetto [8], who considered more general, nonlinear structures for the elliptic part, albeit with linear growth with respect to the gradient, and lower order terms depending on the temperature, which is relevant when convection is taken into account:

$$\partial_t\big[u + H_0(u)\big] \ni \operatorname{div} a(x,t,u,Du) + b(x,t,u,Du), \quad a(x,t,u,Du) \approx Du; \tag{1.5}$$

see also, respectively, [28] and [34]. More general structures, including multi-phase Stefan problem, were considered in [14]. Both in Caffarelli-Evans and DiBenedetto papers [5, 8], even if not explicitly stated, the proof yields a modulus of continuity of the type

$$\omega(r) = \left[\ln\ln\left(\frac{Ar_0}{r}\right)\right]^{-\sigma}, \quad \text{for some } A, \sigma > 0, \tag{1.6}$$

see the forthcoming subsection 1.2. We also mention DiBenedetto and Friedman who, in the first of their papers about the gradient regularity for solutions of parabolic p-Laplace equations, state that the method of the paper yields (1.6) as modulus of continuity for the solutions to (1.5), see [11, Remark 3.1].

Details are somehow pointed out in [9], where DiBenedetto shows that, in the case of Hölder continuous boundary data, the solution of the Cauchy-Dirichlet problem for equation (1.5) has modulus of continuity (1.6), giving a quantitative form to the up-to-the-boundary continuity result previously proved by Ziemer in [34]. These, to the best of our knowledge, are the last quantitative results concerning the continuity of the solutions of the classical two-phase Stefan problem.

For the degenerate case, $p > 2$ in (1.1), very little is known. Existence was obtained by one of the authors in [31] using an approximation method; subsequently he proved the continuity [32] of at least one of them, in the spirit of [8], circumventing the additional difficulties resulting from the presence of the p-Laplacian in the elliptic part. The continuity proof only leads to an implicit modulus of continuity.

Our derivation of the modulus of continuity (1.3) represents an improvement with respect to the state-of-the-art in several ways: we discard an iteration of the logarithm, reaching what we conjecture to be the sharp, optimal modulus of continuity for the two-phase Stefan problem; we determine the precise value of the exponent α in terms of the data of the problem; we cover the degenerate case $p > 2$ and we provide a comprehensive proof, which we tried to keep as self-contained as possible.

1.1. **Statement of the problem and main result.** More generally, we shall consider the following extension of (1.1):

$$\partial_t\big[\beta(u) + \mathcal{L}_h\, H_a(\beta(u))\big] \ni \operatorname{div} \mathcal{A}(x,t,u,Du) \qquad \text{in } \ \Omega_T := \Omega \times (0,T)\,, \tag{1.7}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$. H_a is defined in (1.2), $\beta : \mathbb{R} \to \mathbb{R}$ is a C^1-diffeomorphism such that $\beta(0) = 0$ and satisfying the bi-Lipschitz condition

$$\Lambda^{-1}|u-v| \leq |\beta(u) - \beta(v)| \leq \Lambda|u-v|$$

for some given $\Lambda \geq 1$ and the vector field $\mathcal{A}$ is measurable with respect to the first two variables and continuous with respect to the last two, satisfying, in addition, the following growth and ellipticity assumptions:

$$|\mathcal{A}(x,t,u,\xi)| \leq \Lambda|\xi|^{p-1}\,, \qquad \langle \mathcal{A}(x,t,u,\xi), \xi\rangle \geq \Lambda^{-1}|\xi|^p\,; \tag{1.8}$$

the previous inequalities are intended to hold for almost any $(x,t) \in \Omega_T$ and for all $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$. We consider the bi-Lipschitz function β in order to include thermal properties of the medium, which may slightly change with respect to the temperature, as already done in [8, 24].

Definition 1.9. A *local weak solution* to equation (1.7) is a function

$$u \in L^\infty_{\rm loc}(0,T; L^2_{\rm loc}(\Omega)) \cap L^p_{\rm loc}(0,T; W^{1,p}_{\rm loc}(\Omega)) =: V^{2,p}_{\rm loc}(\Omega_T)$$

such that a selection $v \in \beta(u) + \mathcal{L}_h H_a(\beta(u))$ satisfies the integral identity

$$\int_{\mathcal{K}} [v\,\varphi](\cdot,\tau)\,dx\bigg|_{\tau=t_1}^{t_2} + \int_{\mathcal{K}\times[t_1,t_2]} \big[-v\,\partial_t\varphi + \langle \mathcal{A}(\cdot,\cdot,u,Du), D\varphi\rangle\big]\,dx\,dt = 0$$

for all $\mathcal{K} \Subset \Omega$, almost every $t_1, t_2 \in \mathbb{R}$ such that $[t_1,t_2] \Subset (0,T)$ and for every test function $\varphi \in L^p_{\rm loc}(0,T; W^{1,p}_0(\mathcal{K}))$ such that $\partial_t\varphi \in L^2(\mathcal{K} \times [t_1,t_2])$.

We assume in this paper that a *local weak solution* can be obtained as a locally uniform limit of locally Hölder continuous solutions to (1.7) for a regularized graph, see Section 2.1. In [4] we construct such a solution for the Cauchy-Dirichlet problem with continuous boundary datum; we derive, in addition, an explicit modulus of continuity up to the boundary. We refer to [16, 23] for the existence of weak solutions for bounded Cauchy-Dirichlet data in the case $p = 2$. For the case $p > 2$ the solution, whose existence can be retrieved from [32], is known to be unique just for homogeneous Dirichlet data, see [18].

Our main result is the derivation of a quantitative modulus of continuity for local weak solutions to (1.7).

Theorem 1.1. *Let u be a local weak solution to (1.7), obtained by approximation, and let $\alpha \equiv \alpha(n,p) \in (0,1/2]$ defined by*

$$\alpha := \begin{cases} \dfrac{p}{n+p} & \text{for } p < n\,, \\ \text{any number} < \dfrac{1}{2} & \text{for } p = n\,, \\ \dfrac{1}{2} & \text{for } p > n\,. \end{cases} \tag{1.10}$$

Then there exist constants M, L and c_*, larger than one and depending only on n, p, Λ and α, such that, considering the modulus of continuity

$$\omega(r) = L\left[p + \ln\Big(\frac{r_0}{r}\Big)\right]^{-\alpha} \tag{1.11}$$

and cylinders

$$Q_r^{\omega(\cdot)}(x_0,t_0) := B_r(x_0) \times (t_0 - M\max\{\operatorname*{osc}_{\Omega_T} u, 1\}^{2-p}[\omega(r)]^{(2-p)(1+1/\alpha)} r^p, t_0)\,, \tag{1.12}$$

we have that if $Q_{r_0}^{\omega(\cdot)} \equiv Q_{r_0}^{\omega(\cdot)}(x_0,t_0) \subset \Omega_T$, then

$$\operatorname*{osc}_{Q_r^{\omega(\cdot)}} u \le c_*\,\omega(r)\max\{\operatorname*{osc}_{\Omega_T} u, 1\} \tag{1.13}$$

holds for all $r \in (0, r_0]$.

Remark 1.14. It is rather straightforward to see that the above defined space-time cylinders $Q_r^{\omega(\cdot)}$ satisfy

$$Q_{r_1}^{\omega(\cdot)} \subset Q_{r_2}^{\omega(\cdot)} \subset Q_{r_0}^{\omega(\cdot)}$$

whenever $0 < r_1 \le r_2 \le r_0$; moreover, $r \mapsto \omega(r)$ is concave for $0 < r \le r_0$. For details, see Section 4.

Remark 1.15. Observe that, in the above theorem, α can be taken arbitrarily close to $1/2$ in the case $p = n$; however, the constants c_* and M in Theorem 1.1 blow up as $\alpha \uparrow 1/2$.

1.2. **Some notes about the proof.** We explain here, briefly and formally, the main ideas behind the continuity proofs, which can perhaps be blurred by the technical details. We shall indeed work with approximate solutions u_ε and show ultimately that

$$\operatorname*{osc}_{Q_r^{\omega(\cdot)}} u_\varepsilon \le c_*\,\omega(r)\max\{\operatorname*{osc}_{\Omega_T} u, 1\} + c\,\varepsilon\,,$$

where $\omega(\cdot)$ and $Q_r^{\omega(\cdot)}$ are defined in (1.11)-(1.12), u_ε is the solution to the approximating equation (2.1) and c does not depend on ε. From this it will be easy to deduce Theorem 1.1 simply by taking the limit as $\varepsilon \downarrow 0$ and using the convergence of u_ε to u. For simplicity, we shall directly describe, in the following lines, the formal argument for u.

Usually continuity estimates are based on estimating the rate of decrease of the oscillation of the solution in a family of nested cylinders $\{Q_j\}$ in terms of a vanishing sequence $\{\omega_j\}$: $\operatorname{osc}_{Q_j} u \le c\,\omega_j$. The sequence $\{\omega_j\}$ can be taken satisfying a recursive expression of the type

$$\omega_{j+1} = \eta(\omega_j)\omega_j, \qquad \eta \text{ non-increasing, concave,} \tag{1.16}$$

such that $\eta(0) = 1$ and $\eta(1) \in (0,1)$. If η has the form

$$\eta(\omega) = 1 - e^{-\vartheta_1 \omega^{-\vartheta_2}}$$

where ϑ_1, ϑ_2 are given positive constants, this yields the modulus of continuity (1.6) and this can be see from easy modifications of the argument of subsection 4.1. Concerning the Stefan problem, see [5, Proposition 4.1], [8, Lemma 4.1], [9, End of Section 3].

In this paper we manage to prove that for the two-phase Stefan problem, in (1.16) we can take

$$\eta(\omega) = 1 - \vartheta\, \omega^{1/\alpha}$$

$\vartheta \in (0,1)$ and α as in (1.10), see Theorem 4.1. The main contribution for this improvement is the use of the weak Harnack inequality for supersolutions, whose use is allowed by the simple Lemma 2.3.

We sketch now the proof of the reduction of the oscillation. After fixing a cylinder $Q_j \equiv Q_{r_j}^{\omega_j}$ as above, we can suppose, up to translation and rescaling, that $\sup u = \operatorname{osc} u \leq 1$ on Q_j. Moreover, we can clearly suppose that $\operatorname{osc} u > \omega_j = \omega(r_j)$ and also that the jump is in the interval $[\operatorname{osc} u/2, \operatorname{osc} u]$ (note that if the jump is outside $[0, \operatorname{osc} u]$ there is nothing to prove, since we are dealing with the parabolic p-Laplace equation in Q_j), see subsection 3.1. Next we fix a classical alternative: either $\sup u = \operatorname{osc} u$ is greater than $\omega_j/4$ in a large portion of the cylinder $\widetilde{Q}_j \subset Q_j$ (Alt. 1), or this does not hold (Alt. 2). Here, $\widetilde{Q}_j$ is a suitable, smaller cylinder, whose time-scale differs from that used for Q_j.

In the case that (Alt. 1) holds true, we truncate the solution below the jump, obtaining a weak supersolution to the parabolic p-Laplace equation, and we use the weak Harnack inequality, together with (Alt. 1) to lift up the infimum of u, therefore reducing the oscillation. Note that here we shall use that the jump belongs to the interval $[\operatorname{osc} u/2, \operatorname{osc} u]$ in order to have enough room to make the truncation possible.

In the second case, we use Caccioppoli's inequality to perform a De Giorgi iteration, starting from the smallness in measure information of (Alt. 2). We have to use two tools in order to rebalance the high degeneracy of the problem, caused both by the jump (which produces an L^1 term on the right-hand side of the energy estimate, see (2.8)) and by the presence of the p-Laplacian: the latter is rebalanced by the size of the cylinder, which depends on ω_j, see $\widetilde{T}_{r_j}^{\omega_j}$ in (2.5), while the former is rebalanced by the fact that we introduce ω_j in the size conditions of the alternatives, see again (Alt. 1)-(Alt. 2). Notice that, in the case $p = 2$, the cylinders we consider are the standard parabolic ones, $B_{r_j}(x_0) \times (t_0 - M{r_j}^2, t_0)$, for a large but universal constant M; hence, for the logarithmic continuity for the classical Stefan problem (1.4), the trick essentially consists in rebalancing the presence of the jump with an alternative involving the modulus of continuity itself. Having reduced the supremum of u on a part of the cylinder (see (3.14)), using the time scale given by $\widetilde{T}_{r_j}^{\omega_j}$, we forward this information in time using a logarithmic estimate and then perform another De Giorgi iteration, this time using the second time scale $T_{r_j}^{\omega_j}$ in (2.5) to rebalance the eventual degeneracy due to the p-Laplacian operator. Connecting the two alternatives and iterating the resulting information we infer (1.16) with $\eta(\omega) = 1 - \vartheta\, \omega^{1/\alpha}$ and now it is easy to deduce the explicit form of the modulus of continuity (1.3), see Subsection 4.1.

Finally, we would like to highlight the points of contact of our paper with the recent work [22], where sharp continuity results are proved for obstacle problems involving the evolutionary p-Laplacian operator. There, it is shown that, once considering obstacles with modulus of continuity $\omega(\cdot)$ (where here this expression has to be understood in an appropriate, intrinsic way), the solution has the same regularity, in the sense that it has the same modulus of continuity. In order to get such result, the authors have to deal with particular cylinders of the form (take as the center the origin, for simplicity)

$$Q_r^{\lambda\omega(\cdot)} := B_r \times (-[\lambda\omega(r)]^{2-p} r^p, 0), \qquad \text{with} \qquad \lambda \approx \frac{\operatorname{osc}_{Q_r^{\lambda\omega(\cdot)}} u}{\omega(r)}$$

and where u is the solution they are considering; these cylinders are the ones involved also in the intrinsic definition of the modulus of continuity and they allow to rebalance the inhomogeneity of the problem. This is an extension of the classic approach to regularity for the parabolic p-Laplacian, see [1, 10, 33], where results are recovered as extremal cases of a family of general interpolative intrinsic geometries. Notice the similitude with the cylinders defined in (1.12) and the fact that we also have to deal with the further inhomogeneity given by the jump; this precisely reflects in the presence of the exponent $1 + 1/\alpha$ in (1.12).

1.3. **Notation.** Our notation will be mostly self-explanatory; we mention here some noticeable facts. We shall follow the usual convention of denoting by c a generic constant, always greater or equal than one, that may vary from line to line; constants we shall need to recall will be denoted with special symbols, such as $\tilde{c}, c_*, c_1$ or the like. Dependencies of constants will be emphasised between parentheses: $c(n, p, \Lambda)$ will mean that c depends only on n, p, Λ; they will often be indicated just after displays. The dependence of constants upon α (and on κ, see (3.4)) will be meaningful only in the case $p = n$; in the case $p < n$ this would just add a dependence on n, p – see also Remark 1.15. Unless otherwise stated, we shall avoid to indicate the centre of the ball when it will be the zero vector: $B_r := B_r(0)$.

Being $A \in \mathbb{R}^k$ a measurable set with positive measure and $f : A \to \mathbb{R}^m$ an integrable map, with $k, m \geq 1$, we shall denote with $(f)_A$ the averaged integral

$$(f)_A := \fint_A f(\xi)\, d\xi := \frac{1}{|A|} \int_A f(\xi)\, d\xi\,.$$

We stress that with the statement "a vector field with the same structure as $\mathcal{A}$" (or "structurally similar to $\mathcal{A}$", or similar expressions) we shall mean that the vector field $\mathcal{A}$ satisfies (1.8), possibly with Λ replaced by a constant depending only on n, p and Λ, and continuous with respect to the last two variables.

Finally, by $\ln \ln x$, for $x > 1$, we will mean $\ln(\ln x)$; $\mathbb{N}$ will be the set $\{1, 2, \dots\}$, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{R}^+ := [0, \infty)$.

2. Collecting tools

2.1. **Approximation of the problem.** Let ρ_ε be the standard symmetric, positive one dimensional mollifier supported in $(-\varepsilon, \varepsilon)$. Set

$$H_{a,\varepsilon}(s) := (\rho_\varepsilon * H_a)(s) \qquad \text{for } s \in \mathbb{R}\,;$$

then $H_{a,\varepsilon}$ is smooth. Moreover, the support of $H'_{a,\varepsilon}$ is contained in $(a - \varepsilon, a + \varepsilon)$. Let $\{u_\varepsilon\}$ be a sequence converging locally uniformly to u as $\varepsilon \downarrow 0$, where u_ε is a weak solution to the approximate equation

$$\partial_t \big[\beta(u_\varepsilon) + \mathcal{L}_h\, H_{a,\varepsilon}(\beta(u_\varepsilon))\big] - \operatorname{div} \mathcal{A}(x, t, u_\varepsilon, Du_\varepsilon) = 0 \qquad \text{in } \Omega_T. \tag{2.1}$$

Now, setting

$$w := \beta(u_\varepsilon)\,, \tag{2.2}$$

we arrive at the regularized equation

$$\partial_t w - \operatorname{div} \widetilde{\mathcal{A}}(x,t,w,Dw) = -\mathcal{L}_h\, \partial_t H_{a,\varepsilon}(w)\,, \tag{2.3}$$

where

$$\widetilde{\mathcal{A}}(x,t,w,Dw) := \mathcal{A}(x,t,\beta^{-1}(w),[\beta'(\beta^{-1}(w))]^{-1}Dw)\,.$$

Observe that the growth and ellipticity bounds for $\widetilde{\mathcal{A}}$ are inherited from $\mathcal{A}$ and from the two-sided bound for β': indeed, we in particular get that

$$|\widetilde{\mathcal{A}}(x,t,u,\xi)| \le \Lambda^p|\xi|^{p-1}\,, \qquad \langle \widetilde{\mathcal{A}}(x,t,u,\xi),\xi\rangle \ge \Lambda^{-p}|\xi|^p \tag{2.4}$$

for almost every $(x,t) \in \Omega_T$ and for all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^n$. Moreover, $\widetilde{\mathcal{A}}$ is clearly continuous with respect to the last two variables since β is C^1-diffeomorphism. Note that we dropped ε from the notation; it will be recovered in Section 4.

By regularity theory for evolutionary p-Laplace type equations, see [10, 33], we actually have that the solution w is Hölder continuous since $\beta(u_\varepsilon)+\mathcal{L}_h H_{a,\varepsilon}(\beta(u_\varepsilon))$ is a diffeomorphism. However, this kind of regularity depends on the regularization and, in particular, will deteriorate as $\varepsilon \downarrow 0$. Nonetheless, we may assume that the solution w to the regularized equation is continuous having pointwise values.

2.2. **Scaling of the equation.** Once given a function z solving (2.1) or (2.3) in $B_r(x_0)\times(t_0-\lambda^{2-p}\overline{T},t_0)$, for some $\overline{T}>0, \lambda\ge 1$, if we consider the function

$$\bar z(y,s) := \lambda^{-1} z(x_0+y, t_0-\lambda^{2-p}(T_0+\overline{T})+\lambda^{2-p}s), \quad (y,s)\in B_r\times(T_0,T_0+\overline{T})\,,$$

it is easy to see that $\bar z$ solves an equation which is structurally similar to the one solved by z, but with a multiplier $\lambda^{-1}\mathcal{L}_h \in [0,1]$ for the phase-transition term.

2.3. **Space-time geometry.** We set

$$\widetilde{T}_r^\omega := \omega^{2-p} r^p \qquad \text{and} \qquad T_r^\omega := M\omega^{(2-p)(1+1/\alpha)} r^p\,, \tag{2.5}$$

for any number $\omega>0$; accordingly, we define

$$\widetilde{Q}_r^\omega = B_{r/4}\times\big(0,\widetilde{T}_r^\omega\big) \qquad \text{and} \qquad Q_r^\omega := B_r\times\big(0,T_r^\omega\big)\,.$$

Note that later on we shall choose $\omega=\omega(r)$, after having chosen appropriately the value of L, see subsection 4.1; for the moment, it will be enough to think to ω as a free parameter.

2.4. **Energy estimates.** We consider in this subsection continuous weak solutions to the following equation

$$\partial_t v - \operatorname{div} \widetilde{\mathcal{A}}(x,t,v,Dv) = -\widetilde{\mathcal{L}_h}\, \partial_t H_{b,\varepsilon}(v)\,, \tag{2.6}$$

where $\widetilde{\mathcal{A}}$ has the same structure of $\mathcal{A}$, $b\in\mathbb{R}$, and $\widetilde{\mathcal{L}_h}\in[0,1]$; we shall, in particular, use the next results for equation (3.2), with b defined in (3.1). The following is a Caccioppoli's inequality for (2.6); for ease of notation we shall denote, from now on,

$$\mathcal{H}(s) := s + \widetilde{\mathcal{L}_h}\, H_{b,\varepsilon}(s)\,, \qquad s\in\mathbb{R}\,. \tag{2.7}$$

Lemma 2.1. *There exists a constant c, depending only on n, p and Λ, such that, if v is a solution to (2.6) in a cylinder $Q = B \times \Gamma$, then*

$$\sup_{\tau\in\Gamma} \frac{\widetilde{\mathcal{L}_h}}{|\Gamma|} \fint_B \Big[\int_k^v H'_{b,\varepsilon}(\xi)(\xi-k)_+\,d\xi\,\phi^p\Big](\cdot,\tau)\,dx + \sup_{\tau\in\Gamma}\frac{1}{|\Gamma|}\fint_B \big[(v-k)_+^2\phi^p\big](\cdot,\tau)\,dx + \fint_Q \big|D[(v-k)_+\phi]\big|^p\,dx\,dt \leq c\fint_Q \Big[(v-k)_+^p|D\phi|^p + (v-k)_+^2\,(\partial_t\phi^p)_+\Big]\,dx\,dt + c\,\widetilde{\mathcal{L}_h}\fint_Q\int_k^v H'_{b,\varepsilon}(\xi)(\xi-k)_+\,d\xi\,(\partial_t\phi^p)_+\,dx\,dt \tag{2.8}$$

for any $k \in \mathbb{R}$ and any test function $\phi \in C^\infty(Q)$, such that $(v-k)_+\phi^p$ vanishes on the parabolic boundary of Q.

Proof. In order to get (2.8), we test, in the weak formulation of (2.6), with $(v-k)_+\phi^p\chi_{\Gamma\cap(-\infty,\tau)}$ for $\tau \in \Gamma$. The calculations are standard; we only show here how to formally treat the parabolic term (see also the proof of Lemma 2.3): being $\hat{Q} := Q \cap [B \times (-\infty, \tau)]$,

$$\int_{\hat{Q}} \partial_t v\mathcal{H}'(v)(v-k)_+\phi^p\,dx\,dt = \int_{\hat{Q}}\partial_t\Big[\int_k^v \mathcal{H}'(\xi)(\xi-k)_+\,d\xi\Big]\phi^p\,dx\,dt = \int_B\int_k^{v(\cdot,\tau)}\mathcal{H}'(\xi)(\xi-k)_+\,d\xi\,\phi^p\,dx - \int_{\hat{Q}}\int_k^v\mathcal{H}'(\xi)(\xi-k)_+\,d\xi\,\partial_t\phi^p\,dx\,dt\,.$$

□

The next lemma allows to forward information in time. The result in the case of evolutionary p-Laplace type equations is a standard "Logarithmic Lemma", see for example the proof in [10, Chapter II].

Lemma 2.2. *Let $\overline{T} \in (0, T_r^\omega)$, for T_r^ω as in (2.5). Suppose that $v \in C(\overline{Q_r^\omega})$ solves (2.6) in Q_r^ω and*

$$v(x,\overline{T}) \leq \operatorname{osc} v - \frac{\omega}{4}, \qquad \forall\, x \in B_{r/8}\,;$$

let moreover $\nu^* \in (0,1)$. Then there exists a constant $\varsigma \in (0,1/2)$, depending only on n, p, Λ, M and ν^*, such that, if $\hat{Q} := B_{r/16} \times (\overline{T}, T_r^\omega)$, then

$$\frac{\Big|\hat{Q}\cap\Big\{v \geq \operatorname{osc} v - \varsigma\,\omega^{1+1/\alpha}\Big\}\Big|}{|\hat{Q}|} \leq \nu^*. \tag{2.9}$$

Proof. Denote, in short, $\widetilde{\mathcal{A}}(Dv) := \widetilde{\mathcal{A}}(x,t,v,Dv)$ and recall the definition of $\mathcal{H}$ in (2.7). Consider a time independent cut-off function $\phi \in C_0^\infty(B_r)$, $0 \leq \phi \leq 1$, such that

$$\phi \equiv 1 \quad \text{in } B_{r/16} \quad \text{and} \quad \phi = 0 \quad \text{on } \partial B_{r/8} \quad \text{with} \quad |D\phi| \leq 32/r\,.$$

Take

$$0 < S^+ := \frac{\omega^{1+1/\alpha}}{8} \leq \frac{\omega}{4} \qquad \text{and} \qquad k = \operatorname{osc} v - S^+,$$

and define the logarithmic function

$$\Psi(v)=\left[\ln\left(\frac{S^+}{S^+-(v-k)_++\varsigma S^+}\right)\right]_+,\qquad \varsigma\in(0,1/2)\ \text{ to be fixed.}$$

We only have $\Psi(v)\neq 0$ when

$$S^+>S^+-(v-k)_++\varsigma S^+\quad\iff\quad v>\operatorname{osc} v-\frac{1-\varsigma}{8}\omega^{1+1/\alpha}=:v_-\,.$$

Note, in particular, that $v_->\operatorname{osc} v-\omega/4$ and that $v_--k=\varsigma S^+$. We have, formally,

$$\Psi'(v)=\chi_{\{v>v_-\}}\frac{1}{S^+-(v-k)_++\varsigma S^+}$$

and

$$\begin{aligned}\Psi''(v)&=\delta_{v-v_-}\frac{1}{S^+-(v-k)_++\varsigma S^+}+\chi_{\{v>v_-\}}\frac{1}{(S^+-(v-k)_++\varsigma S^+)^2}\\&=\delta_{v-v_-}\frac{1}{S^+}+\chi_{\{v>v_-\}}\frac{1}{(S^+-(v-k)_++\varsigma S^+)^2}=\frac{\delta_{v-v_-}}{S^+}+\big[\Psi'(v)\big]^2,\end{aligned}$$

where δ_{v-v_-} is the Dirac delta centered in $v-v_-$. Testing formally the equation with $\eta=\Psi'(v)\Psi(v)\phi^p\chi_{(\overline{T},\tau)}(t)$, for $\tau\in(\overline{T},T_r^\omega]$, we have

$$-\int_{B_{r/8}\times(\overline{T},\tau)}\langle\widetilde{\mathcal{A}}(Dv),D\eta\rangle\,dx\,dt=\int_{B_{r/8}\times(\overline{T},\tau)}\partial_t\mathcal{H}(v)\eta\,dx\,dt\,.$$

The choice of the test function is admissible after a suitable mollification in time, following the same steps as in the end of the proof of Lemma 2.3, when treating the first integral. For the time term, we have

$$\partial_t\mathcal{H}(v)\Psi'(v)\Psi(v)=\partial_t\int_{v_-}^{v}\mathcal{H}'(\xi)\Psi'(\xi)\Psi(\xi)\,d\xi$$

and integration by parts gives that

$$\int_{B_{r/8}\times(\overline{T},\tau)}\partial_t\mathcal{H}(v)\Psi'(v)\Psi(v)\phi^p\,dx\,dt=\int_{B_{r/8}}\int_{v_-}^{v(\cdot,\tau)}\mathcal{H}'(\xi)\Psi'(\xi)\Psi(\xi)\,d\xi\ \phi^p\,dx\Bigg|_{t=\overline{T}}^{\tau},$$

since ϕ is time independent; here, we have also used the fact that $v\in C(\overline{Q_r^\omega})$. Since $v\le v_-$ on $B_{r/8}\times\{\overline{T}\}$, we have that

$$\int_{B_{r/8}}\int_{v_-}^{v(\cdot,\overline{T})}\mathcal{H}'(\xi)\Psi'(\xi)\Psi(\xi)\,d\xi\ \phi^p\,dx=0\,.$$

Therefore

$$\int_{B_{r/8}\times(\overline{T},\tau)}\partial_t\mathcal{H}(v)\Psi'(v)\Psi(v)\phi^p\,dx\,dt=\int_{B_{r/8}}\int_{v_-}^{v(\cdot,\tau)}\mathcal{H}'(\xi)\Psi'(\xi)\Psi(\xi)\,d\xi\ \phi^p\,dx$$

and since $\mathcal{H}'\ge 1$ and $\Psi(v_-)=0$, we obtain that

$$\int_{B_{r/8}}\Psi^2(v(x,\tau))\phi^p\,dx\le 2\int_{B_{r/8}\times(\overline{T},\tau)}\partial_t\mathcal{H}(v)\Psi'(v)\Psi(v)\phi^p\,dx\,dt.$$

As for the elliptic term, we get, from (2.4), since $\Psi(v)\delta_{v-v_-}=0$,

$$-\int_{B_{r/8}\times(\overline{T},\tau)}\langle\widetilde{\mathcal{A}}(Dv),D\eta\rangle\,dx\,dt=-\int_{B_{r/8}\times(\overline{T},\tau)}\langle\widetilde{\mathcal{A}}(Dv),D\phi^p\rangle\,\Psi'(v)\Psi(v)\,dx\,dt$$

$$
\begin{aligned}
&- \int_{B_{r/8}\times(\overline{T},\tau)} \langle \widetilde{\mathcal{A}}(Dv), Dv\rangle \big(1+\Psi(v)\big)\,[\Psi'(v)]^2\,\phi^p\,dx\,dt \\
&\le c(p,\Lambda) \int_{B_{r/8}\times(0,T_r^\omega)} \Psi(v)\,[\Psi'(v)]^{2-p}\,|D\phi|^p\,dx\,dt \\
&\quad - c(p,\Lambda) \int_{B_r\times(\overline{T},\tau)} |Dv|^p(1+\Psi(v))\,[\Psi'(v)]^2\,\phi^p\,dx\,dt\,,
\end{aligned}
$$

using Young's inequality. We thus obtain, discarding the negative term on the right-hand side,

$$
\int_{B_{r/8}} \Psi^2(v(\cdot,\tau))\phi^p\,dx \le c\int_{Q_r^\omega} \Psi(v)\,[\Psi'(v)]^{2-p}\,|D\phi|^p\,dx\,dt\,;
$$

this holds for all $\tau \in (\overline{T}, T_r^\omega]$. The very definitions of Ψ and T_r^ω then imply

$$
\int_{B_{r/16}} [\Psi(v(\cdot,\tau))]^2\,dx \le c\,\frac{|B_{r/8}|\,T_r^\omega}{r^p}\ln\frac{1}{\varsigma}\,(2S^+)^{p-2} \le c\,M|B_{r/16}|\ln\frac{1}{\varsigma}\,,
$$

since $(v-k)_+ \le S^+$ and

$$
r^{-p}\,T_r^\omega\,(2S^+)^{p-2} = 2^{p-2}M\omega^{(2-p)(1+1/\alpha)}\left(\frac{\omega^{1+1/\alpha}}{8}\right)^{p-2} = 4^{2-p}M\,.
$$

Moreover, the left-hand side can be bounded below as

$$
\int_{B_{r/16}} [\Psi(v(\cdot,\tau))]^2\,dx \ge \big|B_{r/16}\cap\{v(\cdot,\tau)\ge \operatorname{osc} v - \varsigma S^+\}\big|\left(\ln\frac{1}{2\varsigma}\right)^2
$$

and we conclude, recalling the definition of S^+, that

$$
\frac{\big|B_{r/16}\cap\{v(\cdot,\tau)\ge \operatorname{osc} v - \varsigma\,\omega^{1+1/\alpha}\}\big|}{|B_{r/16}|} \le cM\,\frac{\ln\frac{1}{\varsigma}}{\ln\frac{1}{2\varsigma}} = \nu^*\,,
$$

for a convenient choice of ς. Finally, integrate in time to obtain (2.9) and complete the proof. □

2.5. **Supersolutions of evolutionary p-Laplace equations.** We recall that a weak supersolution to

$$
\partial_t v - \operatorname{div}\hat{\mathcal{A}}(x,t,v,Dv) = 0 \qquad \text{in } B\times\Gamma\,, \tag{2.10}
$$

B open set and Γ open interval, where $\hat{\mathcal{A}}$ has the same structure of $\widetilde{\mathcal{A}}$ (and $\mathcal{A}$), is a function $w \in V^{2,p}(B\times\Gamma)$ satisfying

$$
\int_{\mathcal{K}} [w\,\varphi](\cdot,\tau)\,dx\Big|_{\tau=t_1}^{t_2} + \int_{\mathcal{K}\times[t_1,t_2]} \big[-w\,\partial_t\varphi + \langle \mathcal{A}(\cdot,\cdot,w,Dw), D\varphi\rangle\big]\,dx\,dt \ge 0
$$

for all $\mathcal{K} \Subset B$, almost every $t_1, t_2 \in \mathbb{R}$ such that $[t_1,t_2] \Subset \Gamma$ and for every test function $\varphi \in L^p_{\mathrm{loc}}(\Gamma; W_0^{1,p}(\mathcal{K}))$ such that $\partial_t\varphi \in L^2(\mathcal{K}\times[t_1,t_2])$ and $\varphi \ge 0$. Analogously, w is a weak subsolution if the quantity on the left-hand side in (2.10) is non-positive for any such test function. The following simple lemma is one of the keys in our proof of the interior continuity.

Lemma 2.3. *If $k < b-\varepsilon$ and v is a weak solution of (2.6) in Q_r^ω, then $(k-v)_+$ is a weak subsolution and $\min(k,v) = k-(k-v)_+$ is a weak supersolution of (2.10) in Q_r^ω, where $\hat{\mathcal{A}}$ has the same structure of $\mathcal{A}$.*

Proof. Let $\mathcal{K} \Subset B_r$, $[t_1, t_2] \Subset (0, T_r^\omega)$, call $\mathcal{Q} := \mathcal{K} \times [t_1, t_2]$ and let φ be a test function as above, in particular non-negative; in order to simplify the proof we suppose $\varphi \equiv 0$ in $\mathcal{K} \times \{t_1, t_2\}$, it will be easy to deduce the proof also in the general case. Set

$$\phi_{k,\epsilon}(\xi) = \min\left\{\frac{(k-\xi)_+}{\epsilon}, 1\right\}, \qquad \text{for } \epsilon \in (0,1)\,,$$

and test equation (2.6) with $\phi_{k,\epsilon}(v)\,\varphi$. Formally, the time derivative terms give

$$\begin{aligned}\int_{\mathcal{Q}} \partial_t v \phi_{k,\epsilon}(v)\,\varphi\,dxdt &= -\int_{\mathcal{Q}} \partial_t \int_v^k \phi_{k,\epsilon}(\xi)\,d\xi\,\varphi\,dx\,dt\\ &= \int_{\mathcal{Q}} \int_v^k \phi_{k,\epsilon}(\xi)\,d\xi\,\partial_t\varphi\,dx\,dt\\ &\overset{\epsilon\downarrow 0}{\longrightarrow} \int_{\mathcal{Q}} (k-v)_+\,\partial_t\varphi\,dx\,dt\,, \end{aligned} \tag{2.11}$$

by the dominated convergence theorem, and

$$\begin{aligned}-\int_{\mathcal{Q}} \partial_t v H'_{b,\varepsilon}(v)\phi_{k,\epsilon}(v)\,\varphi\,dxdt &= \int_{\mathcal{Q}} \partial_t \int_v^k H'_{b,\varepsilon}(\xi)\phi_{k,\epsilon}(\xi)\,d\xi\,\varphi\,dx\,dt\\ &= -\int_{\mathcal{Q}} \int_v^k H'_{b,\varepsilon}(\xi)\phi_{k,\epsilon}(\xi)\,d\xi\,\partial_t\varphi\,dx\,dt\\ &= 0\,,\end{aligned}$$

since $\operatorname{supp} H'_{b,\varepsilon} \subset (b-\varepsilon, b+\varepsilon)$ does not intersect the integration interval (v,k) due to the fact that we assume $k < b-\varepsilon$. As for the elliptic part, noting that $\phi'_{k,\epsilon}(v) = -\frac{1}{\epsilon}\chi_{\{k-\epsilon<v<k\}} \le 0$ and hence

$$\int_{\mathcal{Q}} \langle \widetilde{\mathcal{A}}(x,t,v,Dv), D\phi_{k,\epsilon}(v)\rangle \varphi\,dx\,dt \le 0\,,$$

we obtain

$$\begin{aligned}&\int_{\mathcal{Q}} \big\langle \widetilde{\mathcal{A}}(x,t,v,Dv), D\big[\phi_{k,\epsilon}(v)\,\varphi\big]\big\rangle\,dx\,dt\\ &\qquad \le \int_{\mathcal{Q}} \langle \widetilde{\mathcal{A}}(x,t,v,Dv), D\varphi\rangle \phi_{k,\epsilon}(v)\,dx\,dt\\ &\qquad \overset{\epsilon\downarrow 0}{\longrightarrow} \int_{\mathcal{Q}} \langle \widetilde{\mathcal{A}}(x,t,v,Dv), D\varphi\rangle \chi_{\{v<k\}}\,dx\,dt\,,\end{aligned}$$

yielding the conclusion for $(k-v)_+$, once we define $\hat{\mathcal{A}}(x,t,w,\xi) := -\widetilde{\mathcal{A}}(x,t,k-w,-\xi)$. The second result follows immediately from this one.

To justify the above calculations, we demonstrate how to rigorously test equation (2.6) with a test function depending on v itself; indeed, there is a well recognized difficulty concerning the time regularity of solutions and one has to suitably mollify the test function in time. To this end, take $\rho_h(s)$, for $h \in (0,1)$, the standard symmetric positive mollifier, with support in $(-h,h)$ and denote, for any function $\theta : \mathbb{R} \to \mathbb{R}$, its mollification by $\theta_h := \theta * \rho_h$. If θ is not defined over $\mathbb{R}$, extend it to zero elsewhere before mollifying. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be any Lipschitz function; note that, for $\mathcal{H}(\cdot)$ defined in (2.7), we have that $v \mapsto \mathcal{H}(v)$ is an increasing function.

Therefore, consider as a test function in (2.6) the function

$$\phi \equiv \phi_h := \Big[f\big(\mathcal{H}^{-1}([\mathcal{H}(v)]_h)\big)\varphi \Big]_h ,$$

for $h > 0$ small, where $[\mathcal{H}(v)]_h$ is the convolution of $\mathcal{H}(v)$ with respect to the time variable and φ is as in the beginning of the proof. Note finally that since ρ_h is symmetric, then $\int f g_h \, dt = \int f_h g \, dt$ by Fubini's theorem; therefore, first using this fact and subsequently integrating by parts, we get

$$\begin{aligned} -\int_B \int_\Gamma \mathcal{H}(v) \partial_t \phi_h \, dt \, dx &= \int_B \int_\Gamma \partial_t [\mathcal{H}(v)]_h f\big(\mathcal{H}^{-1}([\mathcal{H}(v)]_h)\big) \varphi \, dt \, dx \\ &= -\int_B \int_\Gamma \partial_t \int_{[\mathcal{H}(v)]_h}^{\mathcal{H}(k)} f\big(\mathcal{H}^{-1}(\zeta)\big) \, d\zeta \, \varphi \, dt \, dx \\ &= \int_B \int_\Gamma \int_{[\mathcal{H}(v)]_h}^{\mathcal{H}(k)} f\big(\mathcal{H}^{-1}(\zeta)\big) \, d\zeta \, \partial_t \varphi \, dt \, dx \\ &\overset{h \downarrow 0}{\longrightarrow} \int_{\mathcal{Q}} \int_{\mathcal{H}(v)}^{\mathcal{H}(k)} f\big(\mathcal{H}^{-1}(\zeta)\big) \, d\zeta \, \partial_t \varphi \, dx \, dt \\ &= \int_{\mathcal{Q}} \int_v^k f(\xi)\big(1 + H'_{b,\varepsilon}(\xi)\big) \, d\xi \, \partial_t \varphi \, dx \, dt , \end{aligned}$$

recalling the definition of $\mathcal{H}$. In the case $f(\xi) = \phi_{k,\epsilon}(\xi)$, for $\epsilon \in (0,1)$, we then also take the limit for $\epsilon \downarrow 0$ as in (2.11) and we discard the remaining null term. As for the elliptic part we may use dominated convergence, together with the fact that $v \in L^p(t_1, t_2; W^{1,p}(\mathcal{K}))$, and send first h and then ϵ to zero to follow the formal calculation in the beginning of the proof. □

2.6. **Harnack estimates.** The following weak Harnack inequality for supersolutions is Theorem 1.1 of [21].

Theorem 2.4 (Weak Harnack inequality)**.** *Let v be a non-negative continuous weak supersolution to*

$$\partial_t v - \operatorname{div} \mathcal{A}(x,t,v,Dv) = 0 \qquad in \quad B_{4R_0}(x_0) \times (0,T) , \tag{2.12}$$

with $\mathcal{A}$ satisfying (1.8). Then there exist constants c_1 and c_2, both depending only on n, p and Λ, such that for every $0 < t_1 < T$ we have

$$\fint_{B_{R_0}(x_0)} v(x,t_1) \, dx \le \frac{1}{2} \left(\frac{c_1 R_0^p}{T - t_1} \right)^{1/(p-2)} + c_2 \inf_{\mathcal{Q}} v , \tag{2.13}$$

where $\mathcal{Q} := B_{2R_0}(x_0) \times (t_1 + \tau/2, t_1 + \tau)$ and

$$\tau := \min \left\{ T - t_1, c_1 R_0^p \left(\fint_{B_{R_0}(x_0)} v(x,t_1) \, dx \right)^{2-p} \right\} . \tag{2.14}$$

The factor $1/2$ in the above theorem is not present in the formulation of [21]. Nonetheless, this constant is insignificant as it only increases the value of the constants c_1 and c_2, a fact that can be easily deduced from the proof in [21]. For related results, see the recent interesting monograph by DiBenedetto, Gianazza and Vespri [13], and also [12], by the same authors, about the Harnack inequality for *weak solutions*.

The next proposition, which encodes the decay rate of supersolutions, follows from the iteration of the previous theorem; see [17, Corollary 3.4] for a very similar statement.

Proposition 2.5 (Decay of positivity)**.** *Let v be a non-negative continuous weak supersolution to (2.12) in $B_{4R_0}(x_0)\times(t_0,t_0+T)$. Then there exists a constant c_3, depending only on n, p and Λ, such that, if*

$$\inf_{x\in B_{2R_0}(x_0)} v(x,t_0) \ge k \tag{2.15}$$

for some level $k>0$, then

$$\inf_{x\in B_{2R_0}(x_0)} v(x,t) \ge \lambda(t) := \frac{k}{c_3}\left(1+c_3(p-2)k^{p-2}\frac{t-t_0}{R_0^p}\right)^{-\frac{1}{p-2}}$$

for all $t\in(t_0,t_0+T]$.

Proof. Suppose, without loss of generality, that $t_0=0$. Define inductively

$$\tau_0 := t_0 = 0, \qquad \tau_j := c_1R_0^p\sum_{\ell=1}^{j}\left(\fint_{B_{R_0}(x_0)}\min\left\{v(\cdot,\tau_{\ell-1}),(2c_2)^{-\ell}k\right\}dx\right)^{2-p},$$

for all indices j such that $\tau_j\le T$, say $j\in\{1,\dots,\bar{\jmath}\}$, and where c_2 is the constant of Theorem 2.4. Note that, for $i\in\{1,\dots,\bar{\jmath}\}$, there holds

$$\left(\frac{c_1R_0^p}{\tau_i-\tau_{i-1}}\right)^{\frac{1}{p-2}} = \fint_{B_{R_0}(x_0)}\min\left\{v(\cdot,\tau_{i-1}),(2c_2)^{-i}k\right\}dx\,;$$

hence, since τ in (2.14) turns out to be, in our case, exactly $\tau_i-\tau_{i-1}$, Harnack estimate (2.13) applied to the supersolution $v_i:=\min\{v,(2c_2)^{-i}k\}$ gives

$$\inf_{B_{2R_0}(x_0)\times((\tau_{i-1}+\tau_i)/2,\tau_i)} v_i \ge \frac{1}{2c_2}\fint_{B_{R_0}(x_0)} v_i(\cdot,\tau_{i-1})\,dx \ge \frac{k}{(2c_2)^i}\,, \tag{2.16}$$

and the last inequality holds if $\inf_{B_{R_0}(x_0)} v_i(\cdot,\tau_{i-1})\ge(2c_2)^{-(i-1)}k$. Using an iterative argument, starting from (2.15), we see that (2.16) holds for any $j\in\{1,\dots,\bar{\jmath}\}$. This means that, for such a j, we have $v_j(x,\tau_j)=(2c_2)^{-j}k$ in $B_{R_0}(x_0)$ and $\tau_j=c_1k^{2-p}R_0^p\sum_{\ell=1}^{j}(2c_2)^{\ell(p-2)}$. Therefore,

$$\int_0^j(2c_2)^{s(p-2)}\,ds \le \frac{\tau_j}{c_1k^{2-p}R_0^p} \le \int_1^{j+1}(2c_2)^{s(p-2)}\,ds$$

and we thus obtain a lower and an upper bound for τ_j:

$$\frac{(2c_2)^{j(p-2)}-1}{(p-2)\ln(2c_2)} \le \frac{\tau_j}{c_1k^{2-p}R_0^p} \le 2c_2\frac{(2c_2)^{j(p-2)}-1}{(p-2)\ln(2c_2)}\,.$$

The bound from below gives

$$(2c_2)^{-j} \ge \left(1+(p-2)\frac{\ln(2c_2)}{c_1}\frac{\tau_j}{k^{2-p}R_0^p}\right)^{-1/(p-2)} \ge \frac{c_3}{k}\lambda(\tau_j)\,,$$

provided that $c_3\ge\ln(2c_2)/c_1$. Finally, taking into account that $v_i\le v$, another application of (2.16), for an appropriate R_0, and with starting time τ_{j-1}, together with a simple covering argument, shows that

$$\inf_{B_{2R_0}(x_0)\times(\tau_{j-1},\tau_j)} v \ge \frac{1}{2c_2}c_3\lambda(\tau_{j-1}) \ge \lambda(\tau)$$

whenever $\tau \in (\tau_{j-1}, \tau_j)$, provided that $c_3 \geq 2c_2$. Clearly, at this point, taking $c_3 := 2c_2 \geq \ln(2c_2)/c_1$ finishes the proof. □

3. Reducing the oscillation

Recalling now the definitions of $\widetilde{Q}_r^\omega$ and Q_r^ω from subsection 2.3, we suppose that w is a weak solution to (2.3) in Q_r^ω. Note that one can also take as ω the value $\omega(r)$; this would mean essentially proving Theorem 1.1 without passing trough Theorem 4.1, see the somehow different proof we give in [3].

3.1. **Basic reductions.** Define

$$v(x,t) := w(x,t) - \inf_{Q_r^\omega} w \qquad \text{and} \qquad b := a - \inf_{Q_r^\omega} w\,. \tag{3.1}$$

Then $\sup v = \operatorname{osc} v = \operatorname{osc} w$, $\inf v = 0$, these quantities being meant over Q_r^ω, and

$$\partial_t v - \operatorname{div} \widetilde{\mathcal{A}}(x,t,v + \inf_{Q_r^\omega} w, Dv) = -\widetilde{\mathcal{L}_h}\, \partial_t H_{b,\varepsilon}(v)\,, \tag{3.2}$$

$\widetilde{\mathcal{L}_h} \in [0,1]$. From now on we shall also suppose that

$$\operatorname{osc} v := \operatorname*{osc}_{Q_r^\omega} v \geq \omega \qquad \text{and} \qquad \varepsilon < \frac{\omega}{8}\,. \tag{3.3}$$

Note that if $b \notin [0, \operatorname{osc} v]$, we then have

$$\partial_t v - \operatorname{div} \widetilde{\mathcal{A}}(x,t,v + \inf_{Q_r^\omega} w, Dv) = 0 \qquad \text{in } Q_r^\omega$$

for ε small enough, and the oscillation reduction follows by the well-known argument of DiBenedetto, see [10, 33]. In this case, even if the modulus of continuity is Hölder, we will not make use of this information since the intrinsic geometry we are using does not allow us to reproduce the estimates of [10, 33]. We, instead, observe that our reasoning also works in the case of evolutionary p-Laplace type equations since the phase transition term $\widetilde{\mathcal{L}_h}\, \partial_t H_{b,\varepsilon}$ only appears as an inhomogeneous term in our calculations, and in particular it works for $\widetilde{\mathcal{L}_h} = 0$.

Thus we may assume from now on $b \in [0, \operatorname{osc} v]$. If $b \in \left[0, \frac{\operatorname{osc} v}{2}\right]$, we can consider $\bar{v} = \operatorname{osc} v - v$ and $\bar{b} = \operatorname{osc} v - b$ instead, and then

$$\partial_t \bar{v} - \operatorname{div} \bar{\mathcal{A}}(x,t,\bar{v}, D\bar{v}) = -\widetilde{\mathcal{L}_h}\, \partial_t H_{\bar{b},\varepsilon}(\bar{v})$$

with $\bar{b} \in \left[\frac{\operatorname{osc} \bar{v}}{2}, \operatorname{osc} \bar{v}\right]$. Here

$$\bar{\mathcal{A}}(x,t,\bar{v}, D\bar{v}) = -\widetilde{\mathcal{A}}(x,t,-\bar{v} + \sup_{Q_r^\omega} w, -D\bar{v})\,,$$

which has the same structure as $\mathcal{A}$. Consequently we can further assume that

$$b \in \left[\frac{\operatorname{osc} v}{2}, \operatorname{osc} v\right].$$

Let us, finally, introduce the Sobolev conjugate exponent of p, κp, where

$$\kappa := \begin{cases} \frac{n}{n-p} & \text{for } p < n\,, \\ \text{any number} > 1 & \text{for } p = n\,, \\ +\infty & \text{for } p > n\,; \end{cases} \tag{3.4}$$

α, appearing in (1.10), will be related to κ in the following way:

$$\frac{1}{\alpha} = 1 + \frac{\kappa}{\kappa - 1}\,. \tag{3.5}$$

From now on, it will be more convenient for our purposes to work with κ.

Now we fix the classical alternative. Clearly one of the following two options must hold: for ε_1 a free parameter, to be fixed in due course, either

$$\left|\widetilde{Q}_r^\omega \cap \left\{v \geq \frac{\operatorname{osc} v}{4}\right\}\right| > \varepsilon_1 \omega^{1+\frac{\kappa}{\kappa-1}} \left|\widetilde{Q}_r^\omega\right| \tag{Alt. 1}$$

or

$$\left|\widetilde{Q}_r^\omega \cap \left\{v \geq \frac{\operatorname{osc} v}{4}\right\}\right| \leq \varepsilon_1 \omega^{1+\frac{\kappa}{\kappa-1}} \left|\widetilde{Q}_r^\omega\right| \tag{Alt. 2}$$

holds true. We analyze separately the two different cases.

3.2. **The first alternative.** Consider first the case where (Alt. 1) holds. Then there exists $t_r^1 \in (0, \widetilde{T}_r^\omega)$ such that

$$\left|B_{r/4} \cap \left\{v(\cdot, t_r^1) \geq \frac{\operatorname{osc} v}{4}\right\}\right| > \varepsilon_1 \omega^{1+\frac{\kappa}{\kappa-1}} \left|B_{r/4}\right|; \tag{3.6}$$

otherwise, just integrate to get a contradiction.

Observing that, due to (3.3),

$$\frac{\operatorname{osc} v}{4} < \frac{\operatorname{osc} v}{2} - \frac{\operatorname{osc} v}{8} \leq b - \frac{\omega}{8} < b - \varepsilon,$$

we can use the weak Harnack estimate on the supersolution $\hat{v} := \min\{v, \operatorname{osc} v/4\}$. Thus, Lemma 2.3, and hence Theorem 2.4, apply to $\hat{v}$:

$$⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx \leq \frac{1}{2}\left(\frac{c_1 (r/4)^p}{T_r^\omega - t_r^1}\right)^{\frac{1}{p-2}} + c_2 \inf_{B_{r/2} \times (t_r^1 + \tau/2, t_r^1 + \tau)} \hat{v}\,, \tag{3.7}$$

where

$$\tau = \min\left\{T_r^\omega - t_r^1, c_1 \left(\frac{r}{4}\right)^p \left(⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx\right)^{2-p}\right\}.$$

Due to (3.6),

$$⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx \geq \varepsilon_1 \omega^{1+\frac{\kappa}{\kappa-1}} \frac{\operatorname{osc} v}{4} \geq \frac{\varepsilon_1}{4} \omega^{2+\frac{\kappa}{\kappa-1}}, \tag{3.8}$$

where the last inequality follows from (3.3). Now, if

$$T_r^\omega - t_r^1 \geq c_1 \left(\frac{r}{4}\right)^p \left(⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx\right)^{2-p}, \tag{3.9}$$

then

$$\tau = c_1 \left(\frac{r}{4}\right)^p \left(⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx\right)^{2-p}$$

and

$$\left(\frac{c_1 \left(\frac{r}{4}\right)^p}{T_r^\omega - t_r^1}\right)^{\frac{1}{p-2}} \leq \left(\frac{c_1 \left(\frac{r}{4}\right)^p}{c_1 \left(\frac{r}{4}\right)^p \left(⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx\right)^{2-p}}\right)^{\frac{1}{p-2}} = ⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx\,.$$

So (3.7) reads

$$⨍_{B_{r/4}} \hat{v}(x, t_r^1)\, dx \leq 2c_2 \inf_{B_{r/2} \times (t_r^1 + \tau/2, t_r^1 + \tau)} \hat{v}$$

and consequently, combining the previous display with (3.8), we get

$$\frac{\varepsilon_1}{8c_2}\omega^{2+\frac{\kappa}{\kappa-1}} \le \inf_{B_{r/2}\times(t_r^1+\tau/2,t_r^1+\tau)} \hat{v}\,. \tag{3.10}$$

Hence if (3.9) holds, then we infer (3.10). Note now that, in particular, if we fix

$$M := 1 + \frac{\varepsilon_1^{2-p}c_1}{16} \ge 2(2\Lambda)^{p-2} \ge 2 \tag{3.11}$$

in the definition of T_r^ω, provided that $\varepsilon_1^{p-2} \le c_1(2\Lambda)^{p-2}/16$, then

$$T_r^\omega - \widetilde{T}_r^\omega \ge T_r^\omega - \omega^{(2-p)(2+\frac{\kappa}{\kappa-1})}r^p = \varepsilon_1^{2-p}c_1 r^p\frac{\omega^{(2-p)(2+\frac{\kappa}{\kappa-1})}}{16}\,.$$

Thus we have, by (3.8), that

$$\begin{aligned} T_r^\omega - t_r^1 \ge T_r^\omega - \widetilde{T}_r^\omega &= c_1\left(\frac{r}{4}\right)^p\left(\frac{\varepsilon_1}{4}\omega^{2+\frac{\kappa}{\kappa-1}}\right)^{2-p} \\ &\ge c_1\left(\frac{r}{4}\right)^p\left(⨍_{B_{r/4}}\hat{v}(x,t_r^1)\,dx\right)^{2-p} = \tau \end{aligned}$$

and hence (3.9) is satisfied.

Now the goal is to push positivity at time $t_r^1+\tau$ up to time T_r^ω; note that by (3.9) and subsequent lines, $t_r^1+\tau \le T_r^\omega$. To do this, we use Proposition 2.5, with $k = \varepsilon_1\omega^{2+\frac{\kappa}{\kappa-1}}/(8c_2)$, to obtain

$$\begin{aligned} \inf_{B_{r/2}\times(t_r^1+\tau/2,T_r^\omega)}\hat{v} &\ge \frac{k}{c_3}\left(1+c_3(p-2)k^{p-2}\frac{T_r^\omega-(t_r^1+\tau/2)}{(r/4)^p}\right)^{-\frac{1}{p-2}} \\ &\ge \frac{\varepsilon_1}{8c_2c_3}\omega^{2+\frac{\kappa}{\kappa-1}}\left(1+\tilde{c}\,c_3(p-2)\right)^{-\frac{1}{p-2}}, \end{aligned}$$

since

$$T_r^\omega - \left(t_r^1+\frac{\tau}{2}\right) \le T_r^\omega \le \frac{c_1}{8\varepsilon_1^{p-2}}\omega^{(2-p)(2+\frac{\kappa}{\kappa-1})}r^p = \tilde{c}\,k^{2-p}r^p\,,$$

$\tilde{c}$ depending on p, c_1, c_2 and hence, ultimately, only on n, p and Λ. Recalling that, clearly, $\hat{v} \le v$, and noting that $\tau \le T_r - \widetilde{T}_r^\omega$ and $\widetilde{T}_r^\omega \le T_r^\omega/2$, by (3.9) and (3.11), we conclude that the infimum of v has been lifted and thus we have reduced the oscillation: we have indeed proved that

$$\text{(3.3) and (Alt. 1)} \quad \Longrightarrow \quad \operatorname*{osc}_{B_{r/4}\times(3T_r^\omega/4,T_r^\omega)} v \le \operatorname*{osc}_{Q_r^\omega} v - \theta_1\omega^{2+\frac{\kappa}{\kappa-1}}\,, \tag{3.12}$$

with $\theta_1 \equiv \theta_1(n,p,\Lambda,\varepsilon_1) \in (0,1)$.

3.3. **The second alternative.** Let us now consider the case when the second alternative (Alt. 2) holds:

$$\left|\widetilde{Q}_r^\omega \cap \left\{v \ge \frac{\operatorname{osc} v}{4}\right\}\right| \le \varepsilon_1\omega^{1+\frac{\kappa}{\kappa-1}}\left|\widetilde{Q}_r^\omega\right|.$$

We shall use this information as a starting point for a De Giorgi-type iteration, where we fix the sequence of nested cylinders as

$$U_j = B_j \times \Gamma_j := B_{(1+2^{-j})r/8} \times \left(\frac{1-2^{-j}}{2}\,\widetilde{T}_r^\omega, \widetilde{T}_r^\omega\right),$$

and we consider cut-off functions ϕ_j such that

$$\phi_j \equiv 1 \quad \text{in } U_{j+1} \qquad \text{and} \qquad \phi_j = 0 \quad \text{on } \partial_p U_j\,,$$

with

$$\left(\partial_t \phi_j^p\right)_+ \leq \frac{c\,2^j}{\widetilde{T}_r^\omega} \qquad \text{and} \qquad |D\phi_j| \leq \frac{c\,2^j}{r}\,. \tag{3.13}$$

Using then the energy estimate (2.8), with κ defined in (3.4) (with the formal agreement that when $\kappa = \infty$, $1 - 1/\kappa = 0$ and

$$\left(\fint_{B_j} [(v-k)_+\phi_j]^{\kappa p}\, dx \right)^{1/\kappa} := \left\|(v-k)_+\phi_j\right\|^p_{L^\infty(B_j)}),$$

we infer

$$\begin{aligned}
&\fint_{U_{j+1}} (v-k)_+^{2(1-1/\kappa)+p}\, dx\, dt \\
&\qquad \leq \fint_{U_j} \left[(v-k)_+^2\phi_j^p\right]^{(1-1/\kappa)} (v-k)_+^p \phi_j^p\, dx\, dt \\
&\qquad \leq \fint_{\Gamma_j} \left[\fint_{B_j} (v-k)_+^2\phi_j^p\, dx\right]^{1-1/\kappa} \left[\fint_{B_j} [(v-k)_+\phi_j]^{\kappa p}\, dx\right]^{1/\kappa} dt \\
&\qquad \leq c\left[\widetilde{T}_r^\omega\right]^{1-1/\kappa} \left[\sup_{t\in\Gamma_j} \frac{1}{\widetilde{T}_r^\omega} \fint_{B_j} \left[(v-k)_+^2\phi_j^p\right](\cdot,t)\, dx\right]^{1-1/\kappa} \times \\
&\qquad\qquad \times r^p \fint_{U_j} \left|D\left[(v-k)_+\phi_j\right]\right|^p dx\, dt \\
&\qquad \leq c\, r^p \left[\widetilde{T}_r^\omega\right]^{1-1/\kappa} \Bigg[\fint_{U_j} \Big((v-k)_+^p |D\phi_j|^p \\
&\qquad\qquad + \left[(v-k)_+^2 + \widetilde{\mathcal{L}_h}\,(b+\varepsilon-k)_+\chi_{\{v\geq k\}}\right] \left(\partial_t\phi_j^p\right)_+ \Big)\, dx\, dt\Bigg]^{2-1/\kappa},
\end{aligned}$$

using Hölder's inequality and Sobolev's embedding. The next step is to choose the levels

$$k_j := \operatorname{osc} v - \frac{1+2^{-j}}{4}\omega\,.$$

We have $k_j > \frac{\operatorname{osc} v}{4}$, since $\omega \leq \operatorname{osc} v$, and the relations

$$\begin{aligned}
&(v-k_j)_+ \geq (k_{j+1}-k_j)\chi_{\{v\geq k_{j+1}\}} = 2^{-j-3}\omega\chi_{\{v\geq k_{j+1}\}}\,, \\
&(v-k_j)_+ \leq \omega\chi_{\{v\geq k_j\}}\,, \\
&(b+\varepsilon-k_j)_+ \leq \omega \qquad (\text{since } b \leq \operatorname{osc} v \text{ and } \varepsilon \leq \omega/8)\,.
\end{aligned}$$

We go back to the iteration inequality, with the notation

$$A_j := \frac{|U_j \cap \{v \geq k_j\}|}{|U_j|}\,,$$

to obtain, using the definition of T_r^ω (2.5) and (3.13)

$$\begin{aligned}
&\left(2^{-j-3}\omega\right)^{2(1-1/\kappa)+p} A_{j+1} \\
&\qquad \leq c\, r^p \left[\widetilde{T}_r^\omega\right]^{1-1/\kappa} \left[2^j \frac{\omega}{\widetilde{T}_r^\omega} + 2^j \frac{\omega^2}{\widetilde{T}_r^\omega} + 2^{jp}\frac{\omega(r)^p}{r^p}\right]^{2-1/\kappa} A_j^{2-1/\kappa}
\end{aligned}$$

$$
\begin{aligned}
&\le c^j r^p \big[r^p \omega^{2-p}\big]^{1-1/\kappa}\left[\frac{\omega^{p-1}}{r^p}+\frac{\omega^p}{r^p}\right]^{2-1/\kappa} A_j^{2-1/\kappa}\\
&\le c^j \omega^{(2-p)(1-1/\kappa)+(p-1)(2-1/\kappa)} A_j^{2-1/\kappa}\,.
\end{aligned}
$$

Note here that we also appealed to the fact that $0 \le \widetilde{\mathcal{L}_h} \le 1$. Thus,

$$
\begin{aligned}
A_{j+1} &\le c_0^j\, \omega^{(2-p)(1-1/\kappa)+(p-1)(2-1/\kappa)-p-2(1-1/\kappa)} A_j^{2-1/\kappa}\\
&= c_0^j\, \omega^{-(2-1/\kappa)} A_j^{2-1/\kappa}\,,
\end{aligned}
$$

where the constant c_0 depends only on n, p, Λ and κ. The lemma on the fast convergence of sequences asserts that $A_j \to 0$ if

$$
A_0 \le c_0^{-(1-1/\kappa)^{-2}} \omega^{\frac{2\kappa-1}{\kappa-1}}\,,
$$

which is exactly our assumption (Alt. 2), once we fix the value of ε_1 as

$$
\varepsilon_1 := \min\left\{c_0^{-(1-1/\kappa)^{-2}}, \frac{1}{2\Lambda}\Big(\frac{c_1}{16}\Big)^{1/(p-2)}\right\}.
$$

We conclude that

$$
v \le \operatorname{osc} v - \frac{\omega}{4} \qquad \text{in } \; B_{r/8} \times \big(\widetilde{T}_r^\omega/2, \widetilde{T}_r^\omega\big)\,. \tag{3.14}
$$

Note that ε_1 is a quantity depending only on n, p, Λ and κ through the dependencies of c_0 and c_1. This, via (3.11), fixes also the value of M as a constant depending only on n, p, Λ and possibly on κ.

We next need to forward this information in time, and to do this we first use the logarithmic Lemma 2.2 and then another De Giorgi iteration. Note, indeed, that now $M \equiv M(n, p, \Lambda, \kappa)$ is fixed; hence, for $\nu^* \in (0,1)$ to be chosen, (3.14) together with Lemma 2.2 yields

$$
\frac{\Big|\big(B_{r/16} \times (\widetilde{T}_r^\omega/2, T_r^\omega)\big) \cap \big\{v \ge \operatorname{osc} v - \varsigma\, \omega^{2+\frac{\kappa}{\kappa-1}}\big\}\Big|}{\big|B_{r/16} \times (\widetilde{T}_r^\omega/2, T_r^\omega)\big|} \le \nu^*,
$$

for a constant $\varsigma \equiv \varsigma(n, p, \Lambda, \kappa, \nu^*) \in (0,1)$; this will be the starting point of our second iteration. Let indeed

$$
V_j := B_{(1+2^{-j})r/32} \times \big(\widetilde{T}_r^\omega, T_r^\omega\big), \qquad B_j := B_{(1+2^{-j})r/32}\,,
$$

and consider smooth cut-off functions ϕ_j, depending only on the spatial variables, such that

$$
\phi_j \equiv 1 \quad \text{in } B_{j+1} \quad \text{and} \quad \phi_j = 0 \quad \text{on } \partial B_j, \quad \text{with} \quad |D\phi_j| \le \frac{c\, 2^j}{r}\,.
$$

If we choose a level such that $k \ge \operatorname{osc} v - \omega/4$, then

$$
(v-k)_+ \phi^p = 0 \qquad \text{on } \partial_p V_j \tag{3.15}
$$

by (3.14), so recalling that $1/\alpha = 1 + \kappa/(\kappa-1)$, we put

$$
k_j = \operatorname{osc} v - \frac{(1+2^{-j})}{8} \varsigma\, \omega^{2+\frac{\kappa}{\kappa-1}} = \operatorname{osc} v - \frac{(1+2^{-j})}{8} \varsigma\, \omega^{\frac{\alpha+1}{\alpha}}\,;
$$

note that $k_j \ge \operatorname{osc} v - \omega/4$. We redefine

$$
A_j := \frac{|V_j \cap \{v \ge k_j\}|}{|V_j|}
$$

and observe that

$$(v-k_j)_+ \le \varsigma\,\omega^{\frac{\alpha+1}{\alpha}} \quad \text{and} \quad (v-k_j)_+ \ge 2^{-j-4}\varsigma\,\omega^{\frac{\alpha+1}{\alpha}}\chi_{\{v\ge k_{j+1}\}}\,.$$

Using again Caccioppoli's estimate,

$$\Big[2^{-j-4}\varsigma\,\omega^{\frac{\alpha+1}{\alpha}}\Big]^{p+\frac{2\alpha}{1-\alpha}} A_{j+1} \le c\,r^p\big[T_r^\omega\big]^{\frac{\alpha}{1-\alpha}}\left[2^{jp}\frac{\big[\varsigma\,\omega^{\frac{\alpha+1}{\alpha}}\big]^p}{r^p}\right]^{\frac{1}{1-\alpha}} A_j^{\frac{1}{1-\alpha}}$$

because of (3.15) and the fact that ϕ is time independent. This implies

$$\begin{aligned}
A_{j+1} &\le c^j M^{\frac{\alpha}{1-\alpha}} r^{\frac{p}{1-\alpha}}\varsigma^{\frac{p}{1-\alpha}-p-\frac{2\alpha}{1-\alpha}} \\
&\qquad\times\frac{\omega^{(2-p)(\frac{\alpha+1}{\alpha})\frac{\alpha}{1-\alpha}+(\frac{\alpha+1}{\alpha})[\frac{p}{1-\alpha}-(p+\frac{2\alpha}{1-\alpha})]}}{r^{\frac{p}{1-\alpha}}}A_j^{\frac{1}{1-\alpha}} \\
&= c^j M^{\frac{\alpha}{1-\alpha}}\varsigma^{\frac{\alpha}{1-\alpha}(p-2)}A_j^{\frac{1}{1-\alpha}} \\
&\le \tilde c^j M^{\frac{\alpha}{1-\alpha}}A_j^{\frac{1}{1-\alpha}}\,,
\end{aligned}$$

since $\varsigma<1$, and for $\tilde c$ depending on n,p,Λ and κ; recall indeed again that $M\equiv M(n,p,\Lambda,\kappa)$ has already been fixed. The sequence A_j is then infinitesimal if

$$A_0 \le \tilde c^{-(\frac{1-\alpha}{\alpha})^2}M^{-1} =: \nu^*\,;$$

this fixes the value of ς and also in this case we can conclude

$$\text{(3.3) and (Alt. 2)} \quad\Longrightarrow$$

$$\operatorname*{osc}_{B_{r/32}\times(\widetilde T_r^\omega,T_r^\omega)} v = \sup_{B_{r/32}\times(\widetilde T_r^\omega,T_r^\omega)} v \le \operatorname*{osc}_{Q_r^\omega} v - \theta_2\omega^{2+\frac{\kappa}{\kappa-1}}\,, \tag{3.16}$$

if we call $\theta_2\equiv\theta_2(n,p,\Lambda,\kappa) := \varsigma/8 \in (0,1)$; recall that $\widetilde T_r^\omega \le T_r^\omega/2$. We have succeeded yet again to reduce the oscillation.

4. Deriving the modulus of continuity

Theorem 1.1 will follow essentially as corollary of the following theorem, whose proof consists in the iteration of the argument of the previous section.

Theorem 4.1. *Let u be a local weak solution to (1.7), obtained by approximation. There exists a constant $\tilde M$, depending on n,p,Λ,α, with α defined in (1.10), such that if*

$$\operatorname*{osc}_{Q_0} u \le 1, \qquad Q_0 = B_R(x_0)\times(t_0-\tilde M R^p, t_0)\subset\Omega_T$$

then

$$\operatorname*{osc}_{Q_j} u \le c(p,\Lambda)\,\omega_j, \qquad j\in\mathbb{N}_0\,, \tag{4.1}$$

where

$$Q_j := B_{R_j}(x_0)\times(t_0-\tilde M\omega_j^{(2-p)(1+1/\alpha)}R_j^p, t_0), \qquad R_j := 32^{-j}R \tag{4.2}$$

and the sequence ω_j is defined by

$$\omega_{j+1} := \max\Big\{\omega_j(1-\vartheta\,\omega_j^{1/\alpha}), 2^{-\frac{25p}{p-2}}\omega_j\Big\}, \quad \textit{for } j\in\mathbb{N}, \qquad \omega_0 = 1\,, \tag{4.3}$$

for a constant $\vartheta\in(0,1)$ depending only on n,p,Λ,α.

Remark 4.4. The requirement that $\omega_{j+1} \geq 2^{-\frac{25p}{p-2}}\omega_j$ is necessary in order to ensure that $\omega_j^{(2-p)(1+1/\alpha)} R_j^p/4 \geq \omega_{j+1}^{(2-p)(1+1/\alpha)} R_{j+1}^p$ and in particular $Q_{j+1} \subset Q_j$; if $p=2$, such requirement is clearly not needed (in this sense, $2^{-\frac{25p}{p-2}} = 0$ by definition if $p=2$). In the case $p \neq 2$ it is still not really relevant, since, for ω_j small, ω_{j+1} clearly equals $\omega_j(1-\vartheta\,\omega_j^{1/\alpha})$.

Proof. Take $\tilde{M} := (2\Lambda)^{2-p}M \geq 1$, where $M \geq (2\Lambda)^{p-2}$ is the constant being fixed in (3.11) and take $\tilde{\varepsilon}$ small enough so that $\operatorname{osc}_{Q_0} u_\varepsilon \leq 2$ for any $\varepsilon \leq \tilde{\varepsilon}$; notice now that Q_0 is fixed and u_ε is the approximating solution, solving (2.1), that we suppose to locally uniformly converge to u; $\tilde{\varepsilon}$ could depend on the starting cylinder in (4.11) but this is not a problem here. Now scale u_ε as described in subsection 2.2, with $r=R$, $T_0 = t_0 - MR^p$, $\overline{T} = MR^p$ and $\lambda = 2\Lambda$; this allows to obtain solutions $\bar{u}_\varepsilon$ in

$$\hat{Q}_0 = B_R \times (t_0 - MR^p, t_0) = B_R \times (t_0 - T_R^1, t_0)\,,$$

T_R^1 as in subsection 2.3 with $\omega = 1$. Note that $\operatorname{osc}_{\hat{Q}_0} \bar{u}_\varepsilon \leq [\operatorname{osc}_{Q_0} u_\varepsilon]/(2\Lambda) \leq 1/\Lambda$. We shall further consider $w = \beta(\bar{u}_\varepsilon)$ as in (2.2); observe that, by the Lipschitz regularity of β, we have

$$\operatorname*{osc}_{\hat{Q}_0} w = \operatorname*{osc}_{\hat{Q}_0} \beta(\bar{u}_\varepsilon) \leq \Lambda \operatorname*{osc}_{\hat{Q}_0} \bar{u}_\varepsilon \leq 1\,.$$

Finally, we shall also translate our solution w to v as in (3.1); notice that also $\operatorname{osc}_{\hat{Q}_0} v \leq 1$. We now fix $\vartheta := \min\{\theta_1,\theta_2\}/32$ (see (3.12) and (3.16)) and, since now $\{\omega_j\}$ is given, we shall show that if $\varepsilon < \omega_{\bar{\imath}}/8$ for some $\bar{\imath} \in \mathbb{N}_0$, then

$$\operatorname*{osc}_{\hat{Q}_i} v \leq 32\,\omega_i \tag{4.5}$$

for all $i \in \{0,1,\ldots,\bar{\imath}+1\}$, where $\hat{Q}_i = B_{R_i}(x_0) \times (t_0 - M\omega_i^{(2-p)(1+1/\alpha)} R_i^p, t_0)$ - note we are considering here cylinders whose length depends on M, not on $\tilde{M}$; this is done for scaling reasons. Incidentally, observe also that, by direct computation, the "doubling" property $\omega_i \leq 32\omega_{i+1}$ holds true for any $i \in \mathbb{N}_0$. From the analysis of Section 3, we get that if $\omega_i \leq \operatorname{osc}_{\hat{Q}_i} v$ and $\varepsilon < \omega_i/8$, then

$$\operatorname*{osc}_{\hat{Q}_{i+1}} v \leq \operatorname*{osc}_{\hat{Q}_i} v - 32\vartheta\,\omega_i^{2+\frac{\kappa}{\kappa-1}}\,. \tag{4.6}$$

Indeed, following again subsection 2.2, rescale v defined in $\hat{Q}_i$ to $\bar{v}$ in $B_{r_i} \times (0, T_{r_i}^{\omega_i})$ (take simply $\lambda = 1$ now); since $\omega_i \leq \operatorname{osc}_{B_{r_i}\times(0,T_{r_i}^{\omega_i})} \bar{v}$, (3.12) and (3.16) give

$$\operatorname*{osc}_{B_{r_i/32}\times(\frac{3}{4}T_{r_i}^{\omega},T_{r_i}^{\omega})} \bar{v} \leq \operatorname*{osc}_{B_{r_i}\times(0,T_{r_i}^{\omega_i})} \bar{v} - 32\vartheta\,\omega_i^{2+\frac{\kappa}{\kappa-1}}$$

and, after scaling back, (4.6) is a consequence of the fact that $T_{r_{i+1}}^{\omega_{i+1}} \leq \frac{1}{4}T_{r_i}^{\omega_i}$, see Remark 4.4.

Suppose then that (4.5) holds for $i \in \{0,1,\ldots,j\}$, with $j \leq \bar{\imath}$ and let us prove that it holds for $j+1$; note that, by the monotonicity of ω_i, we have $\varepsilon < \omega_i/8$ for $i \in \{0,1,\ldots,j\}$. Let now i^* be the largest integer in $\{0,1,\ldots,j\}$ such that $\operatorname{osc}_{\hat{Q}_{i^*}} v \leq \omega_{i^*}$ holds; note that such an index exists since $\operatorname{osc}_{\hat{Q}_0} v \leq 1 = \omega_0$, and this fixes the inductive starting step. If $i^* = j$, then the induction step follows from the "doubling" property of ω_i. Assume then that $i^* < j$ so that, by the induction assumption, we have

$$\omega_i < \operatorname*{osc}_{\hat{Q}_i} v \le 32\omega_i, \qquad \forall\, i \in \{i^*+1, \dots, j\}\,.$$

Therefore, (4.6) is at our disposal for any such index (recall $\varepsilon < \omega_i/8$, for all $i \le j$) and it leads to

$$\operatorname*{osc}_{\hat{Q}_{i+1}} v \le \operatorname*{osc}_{\hat{Q}_i} v - 32\vartheta\, \omega_i^{2+\frac{\kappa}{\kappa-1}} \le \left(1 - \vartheta\, \omega_i^{1+\frac{\kappa}{\kappa-1}}\right) \operatorname*{osc}_{Q_i} v,$$

for $i \in \{i^*+1, \dots, j\}$. Iterating and using (4.3), which implies

$$\omega_{j+1} \ge \prod_{i=k}^{j} \big(1 - \vartheta\, \omega_i^{1+\frac{\kappa}{\kappa-1}}\big)\omega_k, \qquad \text{for any } j+1 \ge k \ge 0$$

(recall that $1/\alpha = 1 + \kappa/(\kappa-1)$), and the fact that $\operatorname{osc}_{Q_{i^*+1}} v \le \operatorname{osc}_{Q_{i^*}} v \le \omega_{i^*} \le 32\omega_{i^*+1}$, we get

$$\operatorname*{osc}_{\hat{Q}_{j+1}} v \le \prod_{i=i^*+1}^{j} \left(1 - \vartheta\, \omega_i^{1+\frac{\kappa}{\kappa-1}}\right) \omega_{i^*} \le \frac{\omega_{j+1}}{\omega_{i^*+1}} \omega_{i^*} \le 32\omega_{j+1}$$

and the (finite) induction is complete. Now (4.5) yields

$$\operatorname*{osc}_{\hat{Q}_i} v \le 32\omega_i + 2^8\varepsilon, \qquad \text{for all } i \in \mathbb{N}_0,$$

since if $i > \bar{\imath} + 1$ we simply have $\operatorname{osc}_{\hat{Q}_i} v \le \operatorname{osc}_{Q_{\bar{\imath}+1}} v \le 32\omega_{\bar{\imath}+1} < 2^8\varepsilon$. It remains to translate this information back to $w = \beta(\bar{u}_\varepsilon)$, and then back again to u_ε on the cylinders Q_i defined in (4.2), and this yields a multiplicative factor $\Lambda^{1-p}2^{2-p}$ on the right-hand side. Finally, we can let $\varepsilon \downarrow 0$ to conclude with (4.1). □

Remark 4.7. We stress that the previous theorem holds for any choice of numbers $\{\omega_j\}$ such that

$$\omega_{j+1} \ge \omega_j(1 - \vartheta\, \omega_j^{1/\alpha}), \qquad \text{for } j \in \mathbb{N}, \qquad \omega_0 = 1\,, \tag{4.8}$$

ϑ as in the statement, provided that the time scales are monotone, that is

$$\omega_{j+1}^{(2-p)(1+1/\alpha)} R_{j+1}^p \le \frac{1}{4} \omega_j^{(2-p)(1+1/\alpha)} R_j^p\,.$$

Indeed, by enlarging ω_j, we also reduce the size of the cylinder Q_j.

4.1. **An explicit modulus.** We now show how the results of the previous section can lead to Theorem 1.1; firstly, we fix the value of L as follows:

$$L := \max\left\{\left(\frac{\alpha \ln 32}{\vartheta}\right)^{\alpha}, 2p^{\alpha}\right\}, \tag{4.9}$$

for α defined in (1.10) and ϑ as above. We stress that this in particular gives

$$\omega(r) \ge \left(\frac{\alpha \ln 32}{\vartheta}\right)^{\alpha} \left[p + \ln\left(\frac{r_0}{r}\right)\right]^{-\alpha}. \tag{4.10}$$

Now we consider a cylinder $Q_{r_0}^{\omega(\cdot)} \subset \Omega_T$, where

$$Q_r^{\omega(\cdot)} := B_r(x_0) \times (t_0 - \max\{\operatorname*{osc}_{\Omega_T} u, 1\}^{2-p} T_r^{\omega(\cdot)}, t_0)\,; \tag{4.11}$$

$T_r^{\omega(\cdot)} = M[\omega(r)]^{(2-p)(1+1/\alpha)} r^p$, with M being fixed in (3.11) and $\omega(\cdot)$ now is defined according to the choice of L performed above. These will be the cylinders considered in (1.12).

We shall now show how to use Theorem 4.1 to deduce Theorem 1.1 with the choice $M = \tilde{M}$. Consider a solution over $Q_{r_0}^{\omega(\cdot)}$ as in the statement of Theorem 1.1 and, following subsection 2.2, rescale it with $T_0 = t_0 - T_{r_0}^{\omega(\cdot)}$, $\overline{T} = T_{r_0}^{\omega(\cdot)}$ and $\lambda := \max\{\operatorname{osc}_{\Omega_T} u, 1\}$, to a solution, which, with a small abuse of notation, we still call u, in

$$B_{r_0}(x_0) \times (t_0 - \tilde{M} r_0^p, t_0) \qquad \text{with} \qquad \operatorname*{osc}_{B_{r_0}(x_0)\times(t_0-\tilde{M}r_0^p, t_0)} u \leq 1\,. \tag{4.12}$$

We put $\overline{Q}_r^{\omega(\cdot)}(x_0,t_0) := B_r(x_0) \times (t_0 - \tilde{M} r^p, t_0)$. Now we choose the initial cylinder of Theorem 4.1 as follows: noting that $\omega(r_0) \geq 1$, $\omega(\varrho) \to 0$ as $\varrho \downarrow 0$ and $\omega(\cdot)$ is continuous and increasing, we take the largest (and unique) radius $\tilde{r} \in (0, r_0]$ such that $\omega(\tilde{r}) = 1$. The radius $\tilde{r}$ can be written as $r_0/\tilde{c}$, where $\tilde{c}$ depends only on n, p, Λ and α, and

$$\operatorname*{osc}_{Q_0} u := \operatorname*{osc}_{\overline{Q}_{\tilde{r}}^{\omega(\cdot)}(x_0,t_0)} u \leq 1\,.$$

We let, for $i \in \mathbb{N}_0$, $r_j := 32^{-j}\tilde{r}$ and we claim that $\omega_j := \omega(r_j)$ is a legitimate choice in Theorem 4.1, also in light of Remark 4.7. Monotonicity of time scales is a consequence of the fact that $T_{r_{i+1}}^{\omega(\cdot)} \leq \frac{1}{4} T_{r_i}^{\omega(\cdot)}$: a direct calculation shows that

$$\frac{\omega'(\varrho)\varrho}{\omega(\varrho)} \leq \frac{\alpha}{p} \quad \text{for } 0 < \varrho \leq r_0 \quad \Longrightarrow \quad \frac{\omega(\varrho_2)}{\omega(\varrho_1)} \leq \left(\frac{\varrho_2}{\varrho_1}\right)^{\frac{\alpha}{p}} \quad \text{for } \varrho_1 \leq \varrho_2 \leq \varrho_0\,. \tag{4.13}$$

Moreover, $\omega''(\rho) \leq 0$ for $\rho \leq r_0$ if $p + \ln(r_0/\rho) \geq \alpha + 1$, and hence ω is concave. Then we have at hand the shrinking sequence of cylinders $\{Q_j\}$, with $Q_0 = \overline{Q}_{\tilde{r}}^{\omega(\cdot)}(x_0,t_0)$; observe also that $\overline{Q}_{r_j}^{\omega(\cdot)}(x_0,t_0) = Q_j$. We check that (4.8) holds in the following way: using the elementary estimate $1 - x \leq e^{-x}$, we see that

$$\begin{aligned}
1 - \vartheta\big[\omega(r_j)\big]^{1/\alpha} &\leq \exp\Big(-\vartheta[\omega(r_j)]^{1/\alpha}\Big) \\
&\leq \exp\left(-\alpha \int_{r_{j+1}}^{r_j} \frac{1}{p + \ln\left(\frac{r_0}{\rho}\right)} \frac{d\rho}{\rho}\right) \\
&= \exp\left(-\alpha\left[\ln\ln\Big(\frac{e^p r_0}{r_{j+1}}\Big) - \ln\ln\Big(\frac{e^p r_0}{r_j}\Big)\right]\right) \\
&= \exp\left(-\ln\left[\frac{p + \ln\left(\frac{r_0}{r_{j+1}}\right)}{p + \ln\left(\frac{r_0}{r_j}\right)}\right]^{\alpha}\right) = \frac{\omega_{j+1}}{\omega_j}\,;
\end{aligned}$$

we also used the fact that, by the monotonicity of $\omega(\cdot)$ and (4.10), we have

$$\big[\omega(r_j)\big]^{1/\alpha} \geq \frac{1}{\ln 32} \int_{r_{j+1}}^{r_j} \big[\omega(\rho)\big]^{1/\alpha} \frac{d\rho}{\rho} \geq \frac{\alpha}{\vartheta} \int_{r_{j+1}}^{r_j} \Big[p + \ln\Big(\frac{r_0}{\rho}\Big)\Big]^{-1} \frac{d\rho}{\rho},$$

for all $j \in \mathbb{N}_0$. Finally, by our definition, $\omega_0 = \omega(r_0) = \omega(\tilde{r}) = 1$.

To conclude, for the convenience of the reader, we briefly show how (4.1) implies (1.13). Indeed, for a radius $r \in (0, \tilde{r}]$, call $\hat{\imath} \in \mathbb{N}_0$ the index such that $r_{\hat{\imath}+1} < r \leq r_{\hat{\imath}}$.

We have, by $T_r^{\omega(\cdot)} \leq T_{r_{\hat{\imath}}}^{\omega(\cdot)}$ and the monotonicity and the doubling property of $\omega(\cdot)$,

$$\operatorname*{osc}_{Q_r^{\omega(\cdot)}} v \leq \operatorname*{osc}_{Q_{\hat{\imath}}} v \leq c(p,\Lambda)\,\omega(r_{\hat{\imath}}) \leq 32c(p,\Lambda)\,\omega(r_{\hat{\imath}+1}) \leq c\,\omega(r)\,;$$

on the other hand, if $r \in (\tilde{r}, r_0]$, we simply use (4.13) in the following way:

$$\operatorname*{osc}_{Q_r^{\omega(\cdot)}} v \leq \omega(r_0) \leq \left(\frac{r_0}{\tilde{r}}\right)^{\frac{\alpha}{p}} \omega(\tilde{r}) \leq c\,\omega(\tilde{r}) \leq c\,\omega(r)\,,$$

recalling that $\tilde{r} \equiv r_0/\tilde{c}(n,p,\Lambda,\alpha)$. Theorem 1.1 now follows taking into account how in (4.12) we scaled our original solution u.

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Uppsala University, Department of Mathematics, Lägerhyddsvägen 1, SE-751 06 Uppsala, Sweden.
E-mail address: paolo.baroni@math.uu.se

Aalto University, Institute of Mathematics, P.O. Box 11100, FI-00076 Aalto, Finland.
E-mail address: tuomo.kuusi@tkk.fi

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal.
E-mail address: jmurb@mat.uc.pt