Some remarks on the Cassinian metric

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Contents of the talk

- Definitions
- Inequalities between the Cassinian and other metrics
- A formula for $s_{\mathbb{B}^2}$

This talk is based on joint paper, with

- R. Klén, M. Vuorinen, X. Zhang.

Motivation

We are interested in metrics (in subdomains of $\mathbb{R}^n$), which behave like the hyperbolic metric in the case $n = 2$.

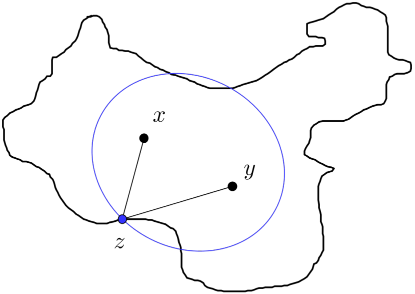
We

- study "Cassinian metric" which was introduced by Z. Ibragimov in [I].
- compare it with different hyperbolic type metrics
- study particular case of triangular ratio metric in the unit disc.

Definitions I

Let G be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$.
The triangular ratio metric in G for $x, y \in G$ is defined by

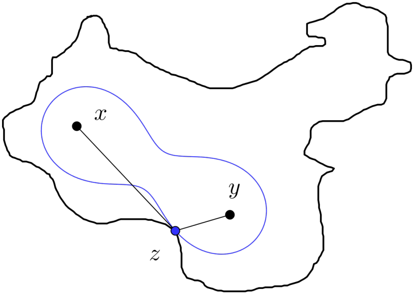
$$s_G(x,y) = \sup_{z \in \partial G} \frac{|x-y|}{|x-z|+|z-y|} \in [0,1].$$



Definitions II

Let G be a proper subdomain of $\mathbb{R}^n$, $n \geq 2$.
The Cassinian metric in G for $x, y \in G$ is defined by

$$c_G(x,y) = \sup_{z \in \partial G} \frac{|x-y|}{|x-z||z-y|}.$$



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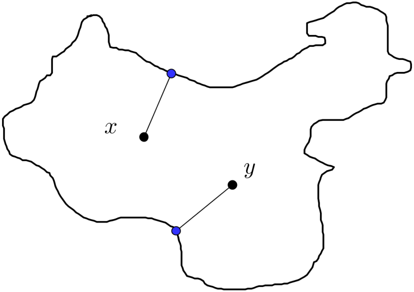
Definitions III

The distance ratio metric j_G for $x, y \in G$ is defined as

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right),$$

where $d(x) = d(x, \partial G)$ is the Euclidean distance from x to ∂G.



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Inequalities between the Cassinian and other metrics I

Lemma

(1) The function $f(x) = x^{-1}\log(1+x)$ is decreasing on $(0,\infty)$.
(2) Let $a > 0$. The function

$$g(x) = \frac{\log ax}{a - \frac{1}{x}}$$

is increasing on $(0,\infty)$.

Inequalities between the Cassinian and other metrics II

(3) The function

$$h(x) = \frac{\log \frac{1+x}{1-x}}{\frac{1}{1-x} - \frac{1}{1+x}}$$

is decreasing on $(0, 1)$.

(4) Let $x \in (0, 1)$. The function

$$f(b) = \frac{\log\left(1 + \frac{b}{1-x}\right)}{\log\left(1 + \frac{b}{(1-x)(b+1-x)}\right)},$$

is increasing on $(0, 2)$.

Inequalities between the Cassinian and other metrics III

Theorem

For all $x, y \in \mathbb{B}^n$ we have

$$j_{\mathbb{B}^n}(x,y) \le a \log(1 + c_{\mathbb{B}^n}(x,y)),$$

where

$$a = \frac{\log\left(\frac{1+\alpha}{1-\alpha}\right)}{\log\left(\frac{1+2\alpha-\alpha^2}{(1-\alpha^2)}\right)} \approx 1.3152$$

and $\alpha \in (0,1)$ is the solution of the equation

$$(1+t^2)\log\frac{1+t}{1-t} + (t^2-2t-1)\log\frac{1+2t-t^2}{1-t^2} = 0.$$

Inequalities between the Cassinian and other metrics IV

For a domain $G \subsetneq \mathbb{R}^n$ we define the quantity

$$\hat{c}_G(x,y) = \frac{|x-y|}{|x-z||z-y|},$$

where $x, y \in G \subsetneq \mathbb{R}^n$ and

$$\begin{cases} z \in \partial G \cap S^{n-1}(x, d(x)) \text{ s.t } |z-y| \text{ is minimal}, & \text{if } d(x) \leq d(y), \\ z \in \partial G \cap S^{n-1}(y, d(y)) \text{ s.t } |z-x| \text{ is minimal}, & \text{if } d(y) < d(x). \end{cases}$$

Clearly for all domains G and for all points $x, y \in G$ there holds $\hat{c}_G(x,y) \leq c_G(x,y)$.

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Inequalities between the Cassinian and other metrics V

Theorem

For all $x, y \in \mathbb{B}^n$ we have

$$j_{\mathbb{B}^n}(x,y) \le \hat{c}_{\mathbb{B}^n}(x,y)\,.$$

Moreover, the right hand side cannot be replaced with $\lambda \hat{c}_{\mathbb{B}^n}(x,y)$ for any $\lambda \in (0,1)$.

Inequalities between the Cassinian and other metrics VI

Corollary

For all $x, y \in \mathbb{B}^n$ we have

$$j_{\mathbb{B}^n}(x, y) \leq c_{\mathbb{B}^n}(x, y)\,.$$

Moreover, the right hand side cannot be replaced with $\lambda c_{\mathbb{B}^n}(x, y)$ for any $\lambda \in (0, 1)$.

A formula for $s_{\mathbb{B}^2}$ I

Theorem

Let $a = \alpha + i\beta$, $\alpha, \beta > 0$, be a point in the unit disk. If $|a - 1/2| > 1/2$, then $s_{\mathbb{B}^2}(a, \bar{a}) = |a|$ and otherwise

$$s_{\mathbb{B}^2}(a, \bar{a}) = \frac{\beta}{\sqrt{(1-\alpha)^2 + \beta^2}}.$$

Proof. From the definition of the triangular ratio metric it follows that

$$s_{\mathbb{B}^2}(a, \bar{a}) = \frac{|a - \bar{a}|}{|a - z| + |\bar{a} - z|} = \frac{2\mathrm{Im}\,(a)}{|a - z| + |\bar{a} - z|}$$

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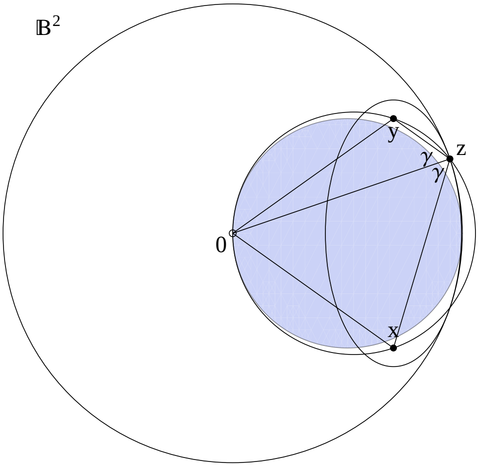
A formula for $s_{\mathbb{B}^2}$ II

For $y = a, x = \bar{a}$



for some point $z = u + iv$. In order to find z we consider the ellipse

$$E(c) = \{w : |a - w| + |\bar{a} - w| = c\}$$

and require that (1) $E(c) \subset \overline{\mathbb{B}}^2$, (2) $E(c) \cap \partial\mathbb{B}^2 \neq \emptyset$ and the x- coordinate of the point of contact of $E(c)$ and the unit circle is unique. Both requirements (1) and (2) can be met for a suitable choice of c. The major and minor semiaxes of the ellipse are $c/2$ and $\sqrt{(c/2)^2 - \beta^2}$, respectively. The point of contact can be obtained by solving the system

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A formula for $s_{\mathbb{B}^2}$ IV

$$\begin{cases} x^2+y^2=1 \\ \frac{(x-\alpha)^2}{(c/2)^2-\beta^2}+\frac{y^2}{(c/2)^2}=1. \end{cases}$$

Solving this system yields a quadratic equation for x with the discriminant

$$D=64(c^2-4\beta^2)(\alpha^2c^2+\beta^2(c^2-4)).$$

The uniqueness requirement for x requires that $D=0$ and hence

$$c=\frac{2\beta}{\sqrt{\alpha^2+\beta^2}}.$$

In this case

$$x = \frac{1}{32\beta^2} 8\alpha c^2 = \frac{\alpha}{\alpha^2+\beta^2}.$$

Consider first the case when $\frac{\alpha}{\alpha^2+\beta^2} = 1$. These points define the circle $|w - 1/2| = 1/2$ and we have $\frac{\alpha}{\alpha^2+\beta^2} > 1$ if and only if $|w - 1/2| < 1/2$. In the case $\frac{\alpha}{\alpha^2+\beta^2} > 1$ the contact point is $z = (1, 0)$, by symmetry, whereas in the case $\frac{\alpha}{\alpha^2+\beta^2} < 1$ the point is

$$z = (x, \sqrt{1 - x^2}) = \left(\frac{\alpha}{\alpha^2+\beta^2}, \frac{\sqrt{(\alpha^2+\beta^2)^2 - \alpha^2}}{\alpha^2+\beta^2} \right).$$

We now compute the focal sum c in both cases

A formula for $s_{\mathbb{B}^2}$ VI

$$\begin{cases} c = \frac{2\beta}{\sqrt{\alpha^2+\beta^2}} = \frac{2\,\mathrm{Im}\,a}{|a|}, & \text{if} \quad |a-1/2| \geq 1/2, \\ c = 2|a-(1,0)| = 2\sqrt{\beta^2+(1-\alpha)^2}, & \text{if} \quad |a-1/2| \leq 1/2. \end{cases}$$

Finally we see that

$$s_{\mathbb{B}^2}(a,\bar{a}) = \frac{|a-\bar{a}|}{c} = |a|, \quad \text{if } |a-1/2| \geq 1/2,$$

otherwise

$$s_{\mathbb{B}^2}(a,\bar{a}) = \frac{|a-\bar{a}|}{c} = \frac{\beta}{\sqrt{\beta^2+(1-\alpha)^2}} = \frac{\mathrm{Im}\,a}{\sqrt{(1-\mathrm{Re}\,(a))^2+(\mathrm{Im}\,(a))^2}}$$

Theorem

Let $x, y \in \mathbb{B}^n$ with $|x| = |y|$ and $z \in \partial\mathbb{B}^n$ such that $|y - z| < |x - z|$ and

$$\measuredangle(y, z, 0) = \measuredangle(0, z, x) = \gamma\,.$$

Then $\cos\gamma = (|x - z| + |y - z|)/2$ and hence $|y - z| < 1$. Moreover, $0, x, y, z$ are concyclic.

A formula for $s_{\mathbb{B}^2}$ VIII

Corollary

Let $D \subset \mathbb{B}^n$ be a domain and let $x, -x \in D$. Then

$$s_D(x, -x) \geq |x|.$$

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A formula for $s_{\mathbb{B}^2}$ IX

Lemma

Let B_1 be a disk with center $(1/2, 0)$ and radius $1/2$ and $x, y \in B_1$. Then

$$s_{\mathbb{B}^2}(x,y) \geq \frac{|x-y|}{\sqrt{1-|x|^2}+\sqrt{1-|y|^2}}\,.$$

Here equality holds for $x, y \in \partial B_1$, $|x| = |y|$.

A formula for $s_{\mathbb{B}^2}$ X

Lemma

Let $x, y \in \mathbb{B}^n$, $x \neq \pm y$ and $z = (x+y)/|x+y| \in \partial\mathbb{B}^n$. Let $x_1, y_1 \in \partial B^2((x+y)/2, |x-y|/2)$ be points with $|x_1 - y_1| = |x-y|$ and $|x_1| = |y_1|$. Then $x_1, y_1 \in \mathbb{B}^n$ and

$$|x-z|+|y-z| \leq |x_1-z|+|y_1-z| = \sqrt{4 + 2(|x|^2+|y|^2) - 4|x+y|}$$

and

$$|x-z||y-z| \leq |x_1-z||y_1-z| = 1 + \frac{|x|^2+|y|^2}{2} - |x+y|.$$

A formula for $s_{\mathbb{B}^2}$ XI

Theorem

Let $x, y \in \mathbb{B}^n$, $x \neq \pm y$ and $z = (x+y)/|x+y| \in \partial\mathbb{B}^n$. Let $x_1, y_1 \in \partial B^2((x+y)/2, |x-y|/2)$ be points with $|x_1 - y_1| = |x-y|$ and $|x_1| = |y_1|$. Then

$$\begin{aligned} s_{\mathbb{B}^n}(x,y) &\geq s_{\mathbb{B}^n}(x_1,y_1) = \frac{|x-y|}{\sqrt{4 + 2(|x|^2+|y|^2) - 4|x+y|}} \\ &= \frac{|x-y|}{\sqrt{|x-y|^2 + (2-|x+y|)^2}} \end{aligned}$$

and

$$c_{\mathbb{B}^n}(x,y) \geq c_{\mathbb{B}^n}(x_1,y_1) = \frac{|x-y|}{1 + \frac{|x|^2+|y|^2}{2} - |x+y|}.$$

A formula for $s_{\mathbb{B}^2}$ XII

Theorem

Let $x \in (0,1)$, $y \in \mathbb{B}^2 \setminus \{0\}$, $\operatorname{Im} y \geq 0$, with $|y| = |x|$ and denote $\omega = \measuredangle(x, 0, y)$. Then the supremum in (**??**) is attained at $z = e^{i\theta}$ for

$$\theta = \begin{cases} \frac{\omega}{2}, & \text{if } \sin\frac{\pi-\omega}{2} \geq |x|, \\ \frac{\omega-\pi}{2} + \arcsin\frac{\sin\frac{\pi-\omega}{2}}{|x|}, & \text{if } \sin\frac{\pi-\omega}{2} < |x|. \end{cases}$$

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Main Result I

Theorem

Suppose that D is a subdomain of $\mathbb{B}^n$. Then for $x, y \in D$ we have

$$2s_D(x,y) \leq c_D(x,y).$$

In the case $D = \mathbb{B}^n$, the constant 2 in the left-hand side is best possible.

Proof.

By a simple geometric observation we see that

$$\inf_{w\in\partial\mathbb{B}^n} |x-w||w-y| \le 1. \qquad (1)$$

In fact, for given $x, y \in \mathbb{B}^n$, let $x', y' \in \mathbb{B}^n$ be the points such that $y' - x' = y - x$ and $y' = -x'$. Then the size of the maximal Cassinian oval $C(x, y)$ with foci x, y which is contained in the closed unit ball is not greater than that of the maximal Cassinian oval $C(x', y')$ with foci x', y', see the Figure 2.

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Main Result III

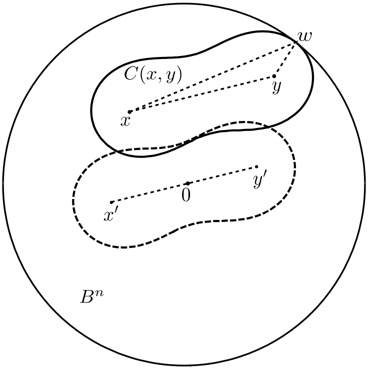

Figure : The maximal Cassinian oval $C(x, y)$ is not larger than the maximal Cassinian oval $C(x', y')$.

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This implies that

$$\inf_{w\in\partial\mathbb{B}^n}|x-w||w-y| \le \inf_{w\in\partial\mathbb{B}^n}|x'-w||w-y'|$$
$$= 1-\left(\frac{|x-y|}{2}\right)^2 \le 1.$$

Therefore, for $x, y \in D \subset \mathbb{B}^n$, we have that

$$\inf_{w\in\partial D}|x-w||w-y| \le \inf_{w\in\partial\mathbb{B}^n}|x-w||w-y| \le 1. \qquad (2)$$

For $x = y \in D$, the desired inequality is trivial. For $x, y \in D$ with $x \neq y$, it follows from the inequality of

Main Result V

arithmetic and geometric means and the inequality (2) that

$$
\begin{aligned}
\frac{c_D(x,y)}{2s_D(x,y)} &= \frac{\inf\limits_{w\in\partial D}(|x-w|+|w-y|)}{2\inf\limits_{w\in\partial D}(|x-w||w-y|)} \\
&\geq \frac{\inf\limits_{w\in\partial D}\sqrt{|x-w||w-y|}}{\inf\limits_{w\in\partial D}(|x-w||w-y|)} \\
&= \frac{\sqrt{\inf\limits_{w\in\partial D}(|x-w||w-y|)}}{\inf\limits_{w\in\partial D}(|x-w||w-y|)} \\
&\geq 1.
\end{aligned}
$$

Main Result VI

For the sharpness of the constant in the case of the unit ball, let $y = -x \to 0$. It is easy to see that both the inequality of arithmetic and geometric means and the inequality (1) will asymptotically become equalities. This completes the proof. $\square$

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Main Result VII

Corollary

Let $D \subset \mathbb{R}^n$ be a bounded domain. Then, for $x, y \in D$,

$$c_D(x,y) \geq \frac{2}{\sqrt{n/(2n+2)}\mathrm{diam(D)}} s_D(x,y).$$

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