An Algebraic Geometric Approach to Multidimensional Symbolic Dynamics

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We study how **local constraints** enforce **global regularities**.

This is a common phenomenon in sciences. For example, formation of crystals:

Atoms attach to each other in a limited number of ways

⇒ periodic arrangement of the atoms
Our goal is to understand **fundamental underlying principles** that connect local rules to the global regularities observed in the structures.

Our setup: multidimensional symbolic dynamics (=tilings)
Configurations are infinite $d$-dimensional grids of symbols.
For a fixed finite shape $D$, we observe the $D$-patterns in the configuration.
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A quantity to measure local complexity: the **pattern complexity**

\[ P(c, D) = \# \text{ of } D\text{-patterns in } c. \]
If this quantity is small, for some $D$, global regularities ensue. The relevant **low complexity threshold**:

$$P(c, D) \leq |D|$$
Global regularity of interest is periodicity: Configuration is **periodic** if it is invariant under a non-zero translation.
Open problem 1: Nivat’s conjecture

Consider $d = 2$ and rectangular $D$.

**Conjecture (Nivat 1997)** If $P(c, D) \leq |D|$ for some rectangle $D$ then $c$ is periodic.
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This would extend the one-dimensional case $d = 1$:

Morse-Hedlund theorem: Let $c \in A^\mathbb{Z}$ and $n \in \mathbb{N}$. If $c$ has at most $n$ distinct subwords of length $n$ then $c$ is periodic.
Best known bound in 2D:

**Theorem (Cyr, Kra):** If $P(c, D) \leq \frac{1}{2} |D|$ for some rectangle $D$ then $c$ is periodic.

Case of narrow rectangles:

**Theorem (Cyr, Kra):** If $D$ is a rectangle of height at most 3 and $P(c, D) \leq |D|$ then $c$ is periodic.
In 3D and higher dimensional cases the conjecture is false.

Non-periodic $c$
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$P(c, D) = 1 + n^2 + n^2 < n^3 = |D|$ for large $n$. 
We can prove an asymptotic version in 2D:

**Theorem (Kari, Szabados):** If $P(c, D) \leq |D|$ for infinitely many different size rectangles $D$ then $c$ is periodic.
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**Theorem (Kari, Szabados):** If $P(c, D) \leq |D|$ for infinitely many different size rectangles $D$ then $c$ is periodic.

Or stated as **contrapositive:** If $c$ is not periodic then $P(c, D) > |D|$ for all sufficiently large rectangles $D$. 
Open problem 2: Periodic tiling problem

Let $T \subseteq \mathbb{Z}^d$ be finite, and call it a \textit{tile}. A \textit{tiling} is any $C \subseteq \mathbb{Z}^d$ such that

$$C \oplus T = \mathbb{Z}^d.$$
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**Graphical interpretation:** $C$ gives the positions where copies of $T$ are placed to cover $\mathbb{Z}^d$ without gaps or overlaps.
Interpret $C$ as the binary configuration $c$ with

$$c(i) = * \iff i \in C.$$
$(-T)$-patterns of $c$ contain exactly one symbol $\ast$. 
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\[ P(c, -T) = | - T| \]
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\[ P(c, -T) = | - T|\]

(Also \(P(c, T) = |T|\) as any tiling for \(T\) is also a tiling for \(-T\).)
If $X$ is the set of all tilings by $T$ then

$$P(X, T) = |T|$$

where $P(X, T)$ is the number of $T$-patterns in $c \in X$. Set $X$ is a low complexity subshift of finite type (SFT).
Periodic tiling problem (Lagarias and Wang 1996): If $T$ admits a tiling $C$, does it necessarily admit a periodic tiling?
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Known results:

- Yes if $|T|$ is a prime number (Szegedy 1998).
- Yes in 2D
  - if $T$ is 4-connected (Beauquier and Nivat 1991),
  - in general (Bhattacharya 2016).
Both the Nivat’s conjecture and the Periodic tiling problem concern periodicity under complexity constraint $P(c, D) \leq |D|$.

We are interested in analogous questions generally.

- **Algorithmic question:** given at most $|D|$ patterns of shape $D$, does there exist a configuration with only these given $D$-patterns? (=emptyness problem of a given low complexity subshift of finite type)

- **Periodicity:** If there exists a configuration whose $D$-patterns are among the given $\leq |D|$ ones, does there necessarily exist such a configuration that is periodic?
We study configurations using algebra, so we first replace symbols by integers:
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• If $P(c, D) < |D|$ then there is an (integer) vector orthogonal to all $D$-patterns of $c$.

**Indeed:** the number $P(c, D)$ of distinct vectors is less than the dimension $|D|$ of the linear space.
- If $P(c, D) < |D|$ then there is an (integer) vector orthogonal to all $D$-patterns of $c$.

- Even if $P(c, D) = |D|$ we can add a suitable rational constant to $c$ to make the vectors linearly dependent. Also then an orthogonal vector exists.
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- If \( P(c, D) < |D| \) then there is an (integer) vector orthogonal to all \( D \)-patterns of \( c \).

- Even if \( P(c, D) = |D| \) we can add a suitable rational constant to \( c \) to make the vectors linearly dependent. Also then an orthogonal vector exists.

**This is OK:** we are free to choose the numerical encoding.
\[
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\[
\begin{array}{c}
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(2, 2, 1, 1) \\
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\]

\[
\begin{array}{c}
\downarrow \\
(1, -1, 0, 0)
\end{array}
\]
The orthogonal vector is a **filter** whose convolution with \( c \) is the zero configuration. We say it **annihilates** configuration \( c \).
Conclusion: If $P(c, D) \leq |D|$ then symbols can be represented as integers in such a way that some non-trivial integer filter annihilates $c$. 
To use algebraic geometry, we next represent $c$ as a \textbf{power series} (negative exponents included).

$$c \leftrightarrow \sum_{(i_1, \ldots, i_d) \in \mathbb{Z}^d} c(i_1, \ldots, i_d)x_1^{i_1} \cdots x_d^{i_d}$$
To use algebraic geometry, we next represent $c$ as a *power series* (negative exponents included).

\[
\begin{array}{cccc}
2x^3y^2 & x^3y^2 & 2x^0y^2 & x^1y^2 & x^2y^2 \\
\tilde{x}^2y^1 & 2\tilde{x}^1y^1 & x^0y^1 & x^1y^1 & 2x^2y^1 \\
2x^2y^0 & \tilde{x}^1y^0 & x^0y^0 & 2x^1y^0 & x^2y^0 \\
x^2y^1 & \tilde{x}^1y^1 & 2x^0y^1 & x^1y^1 & 2x^2y^1 \\
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\end{array}
\]

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\]
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\[
\ldots + 2x^3y^2 + x^3y^2 + 2x^0y^2 + x^1y^2 + x^2y^2 + \ldots \\
\ldots + x^2y^1 + 2x^1y^1 + x^0y^1 + x^1y^1 + 2x^2y^1 + \ldots \\
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\[
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\]
\[ c \leftarrow \sum_{(i_1, \ldots, i_d) \in \mathbb{Z}^d} c(i_1, \ldots, i_d)x_1^{i_1} \ldots x_d^{i_d} \]

**Notations:**

- \( X = (x_1, \ldots, x_d) \)
- For \( I = (i_1, \ldots, i_d) \in \mathbb{Z}^d \) we denote by
  \[ X^I = x_1^{i_1} \ldots x_d^{i_d} \]
  the monomial that represents cell \( I \).
\[ c \longleftrightarrow \sum_{(i_1, \ldots, i_d) \in \mathbb{Z}^d} c(i_1, \ldots, i_d) x_1^{i_1} \cdots x_d^{i_d} = \sum_{I \in \mathbb{Z}^d} c(I) X^I \]

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the monomial that represents cell \( I \).
The configuration is now a power series $c(X)$ that is

- **integral** (=all coefficients are integers), and
- **finitary** (=finite number of distinct coefficients)
\[ c(X) = \sum_{I \in \mathbb{Z}^d} c(I)X^I \]

Multiplying \( c(X) \) by monomial \( X^J \) gives \textbf{translation} by \( J \in \mathbb{Z}^d \):

\[
X^J \cdot c(X) = \sum_{I \in \mathbb{Z}^d} c(I)X^{I+J}
\]

So \( c(X) \) is \textbf{\( J \)-periodic} if and only if \( X^J \cdot c(X) = c(X) \), i.e.,

\[
(X^J - 1)c(X) = 0
\]
Multiplying $c(X)$ by a (Laurent) polynomial $f(X)$ is a convolution, corresponding to \textit{filtering} operation.

We say that $f(X)$ \textit{annihilates} $c(X)$ if $f(X)c(X) = 0$. 

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- Zero polynomial \( f(X) = 0 \) annihilates every configuration – it is the **trivial annihilator**.

- Binomial \( X^I - 1 \) annihilates \( c(X) \) if and only if \( c(X) \) is \( I \)-periodic.

- Annihilators of \( c(X) \) form an **ideal**:
  - if \( f(X) \) and \( g(X) \) annihilate \( c(X) \), also \( f(X) + g(X) \) annihilates it,
  - if \( f(X) \) annihilates \( c(X) \) then also \( g(X)f(X) \) annihilates it, for all \( g(X) \).
Define

$$\text{Ann}(c) = \{ f(X) \in \mathbb{C}[X] \mid f(X)c(X) = 0 \}.$$ 

This is the polynomial ideal that contains all annihilators of $c$. 
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**Remarks:**

- We consider polynomials (not Laurent polynomials!) so that we can directly rely on polynomial algebra. **No problem:** any Laurent polynomial annihilator can be made into a proper polynomial annihilator by multiplying it with suitable monomial $X^I$.

- We allow complex coefficients because we need algebraically closed field to apply Hilbert’s Nullstellensatz.

- $\text{Ann}(c)$ is indeed an ideal of the polynomial ring $\mathbb{C}[X]$. 
Our setup (=low complexity configuration) is an integral, finitary \(c(X)\) that has some non-trivial integral annihilator

\[
f(X) \in \text{Ann}(c) \cap \mathbb{Z}[X]
\]
Ann(c) = \{ f(X) \in \mathbb{C}[X] \mid f(X)c(X) = 0 \} 

**Plugging in numbers for variables:** For any 

\[ Z = (z_1, \ldots, z_d) \in \mathbb{C}^d \]

we can compute the value \( f(Z) \in \mathbb{C} \) of any polynomial \( f(X) \in \mathbb{C}[X] \).
Ann(c) = \{ f(X) \in \mathbb{C}[X] \mid f(X)c(X) = 0 \}

To prove that Ann(c) contains “simple” polynomials we use

Nullstellensatz (Hilbert): Let \( g(X) \) be a polynomial. Suppose that \( g(Z) = 0 \) for all \( Z \) in the variety

\[ \{ Z \in \mathbb{C}^d \mid f(Z) = 0 \text{ for all } f \in \text{Ann}(c) \}. \]

Then \( g^k \in \text{Ann}(c) \) for some \( k \in \mathbb{N} \).
$c(X)$ a finitary, integral power series

$f(X) = \sum_{I \in \mathcal{I}} a_I X^I$ its non-trivial integral annihilator polynomial

$(a_I \neq 0 \text{ for all } I \in \mathcal{I})$

**Lemma:** $f(X^n) \in \text{Ann}(c)$ for every $n \in \mathbb{N}$ whose prime factors are sufficiently large.
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its non-trivial integral annihilator polynomial

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**Lemma:** $f(X^n) \in \text{Ann}(c)$ for every $n \in \mathbb{N}$ whose prime factors are sufficiently large.

**Proof:** a direct application of

\[ f(X)^p \equiv f(X^p) \pmod{p\mathbb{Z}[X]} \]

for prime factors $p$ of $n$. 
\( c(X) \) a finitary, integral power series

\[
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\( (a_I \neq 0 \text{ for all } I \in \mathcal{I}) \)

**Lemma:** \( f(X^n) \in \text{Ann}(c) \) for every \( n \in \mathbb{N} \) whose prime factors are sufficiently large.

In particular, \( f(X^{1+iM}) \) are in \( \text{Ann}(c) \) for \( i = 0, 1, 2, \ldots \), where \( M \) is the product of all small primes.
Let $Z \in \mathbb{C}^d$ be a common zero of $\text{Ann}(c)$. Then

$$f(Z^{1+iM}) = 0 \text{ for all } i = 0, 1, 2, \ldots$$
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$$f(Z^{1+iM}) = 0 \text{ for all } i = 0, 1, 2, \ldots$$

Then (proof omitted) $g(Z) = 0$ for

$$g(X) = X^1 \prod_{I,J \in \mathcal{I}, I \neq J} (X^{MI} - X^{MJ}).$$

Here:

- $M$ is the constant from the Lemma (product of small primes).
- $\mathcal{I} \subseteq \mathbb{Z}^d$ is the support of polynomial $f(X)$. 
So all elements of the variety

\[ \{ Z \in \mathbb{C}^d \mid f(Z) = 0 \text{ for all } f \in \text{Ann}(c) \} \]

are zeros of the polynomial

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So all elements of the variety

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\[ g(X) = X^1 \prod_{I,J \in \mathcal{I}, I \neq J} (X^{MI} - X^{MJ}). \]

**Nullstellensatz** \( \implies \) \( g(X)^n \in \text{Ann}(c) \) for some \( n \in \mathbb{N} \).
So all elements of the variety

$$\{ Z \in \mathbb{C}^d \mid f(Z) = 0 \text{ for all } f \in \text{Ann}(c) \}$$

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**Nullstellensatz** $\implies g(X)^n \in \text{Ann}(c)$ for some $n \in \mathbb{N}$.

Dividing $g(X)^n$ by a suitable monomial gives:

**Theorem.** Finitary, integral $c(X)$ that has a non-trivial annihilator is annihilated by a Laurent polynomial of the form

$$(1 - X^{I_1})(1 - X^{I_2}) \ldots (1 - X^{I_k}).$$
Annihilator: \((1 - X^{I_1})(1 - X^{I_2}) \ldots (1 - X^{I_k})\)

Binomials \((1 - X^I)\) correspond to difference operators that subtract from a configuration its own \(I\)-translation. The theorem states that configuration \(c(X)\) can be annihilated by a sequence of difference operations.
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The theorem states that configuration \(c(X)\) can be annihilated by a sequence of difference operations.

If \(k = 1\) then \(c(X)\) is periodic.

More generally, we can prove that \(c(X)\) is a sum of \(k\) (possibly non-finitary) integral configurations that are periodic.

**Corollary.** \(c(X) = c_1(X) + \cdots + c_k(X)\) where \(c_i(X)\) is \(I_i\)-periodic and integral (but not necessarily finitary).
Example. The 3D counter example to Nivat’s conjecture is a sum of two periodic configurations. It is annihilated by polynomial \((1 - y)(1 - x)\).
Our approach to Nivat’s conjecture.

Suppose $P(c, D) \leq |D|$ for some rectangle $D$.

Then $c$ has annihilating polynomial

$$f(X) = (1 - X^{I_1}) \ldots (1 - X^{I_k}).$$

Take the one with smallest $k$.

If $k = 1$ then $c$ is periodic, so assume that $k \geq 2$. 

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Denote $\delta_i(X) = (1 - X^{I_i})$ and $\phi_i(X) = f(X)/\delta_i(X)$.

Then $\phi_i(X)c(X)$ is annihilated by $\delta_i(X)$ so it is $I_i$-periodic. It is not doubly periodic (since otherwise $k$ could be reduced).
Viewing $c(X)$ using filter $\phi_1(X)$:

Non-periodic sequence of stripes in the direction $I_1$. 
Viewing $c(X)$ using filter $\phi_1(X)$:

W.l.g. the stripes are not horizontal

$\implies$ at least $X$ stripes are visible in every $X \times Y$ rectangle

$\implies$ more than $X$ different $X \times Y$ blocks in $\phi_1(X)c(X)$

(due to the one-dimensional Morse-Hedlund theorem)
Viewing \( c(X) \) using filter \( \phi_2(X) \):

Non-periodic sequence of stripes in a different direction \( I_2 \).
Viewing \( c(X) \) using filter \( \phi_2(X) \):

Analogously: stripes not vertical \( \Rightarrow \) more than \( Y \) different \( X \times Y \) blocks in \( \phi_2(X)c(X) \).
Pick any $X \times Y$ pattern from $\phi_1(X)c(X)\ldots$
...and any $X \times Y$ pattern from $\phi_2(X)c(X)$. 
Directions $I_1$ and $I_2$ are different
Directions $I_1$ and $I_2$ are different

so both patterns can be seen (more or less) in the same position.
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$\implies$ more than $XY$ distinct pairs of patterns in same positions
For some constant $r$ (=radius of filters $\phi_1$ and $\phi_2$), each $(X + 2r) \times (Y + 2r)$ block of $c(X)$ uniquely determines the corresponding $X \times Y$ blocks in $\phi_1(X)c(X)$ and $\phi_2(X)c(X)$.

$\implies c(X)$ has at least $XY$ patterns of size $(X + 2r) \times (Y + 2r)$. 
We get that

$$\liminf_D \frac{P(c, D)}{|D|} \geq 1.$$
We get that
\[
\liminf_D \frac{P(c, D)}{|D|} \geq 1.
\]

This can be improved further to:

**Theorem.** If \(c\) is a non-periodic 2D configuration then 
\(P(c, D) \leq |D|\) can hold only for finitely many rectangles \(D\).
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Questions:

- What can one say for other shapes than rectangles? Perhaps an analogous result for convex shapes?
- Can one use the periodic decomposition to address the periodic tiling problem? What about other low complexity subshifts of finite type?
- The original Nivat’s problem is still open...
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Thank You