Solvability Complexity Index ($\equiv$SCI) and Towers of Algorithms

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Goal of the talk

- Is $\sigma(A)$ computable for $A \in B(\ell_2(\mathbb{N}))$?
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- To explain what different theories say about it
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- To explain what different theories say about it.

- This is a simplified layman overview.

- Then I focus on Towers of Algorithms and on the Solvability Complexity Index.

- J. Ben-Artzi, A. Hansen, O. Nevanlinna, M. Seidel
Definition of a Tower

**PROBLEM**

Ω: primary set, e.g. $B(\ell^2(\mathbb{N}))$
Λ: evaluation set, e.g. $f_{ij}: A \mapsto \langle Ae_i, e_j \rangle$ for $A \in B(\ell^2(\mathbb{N}))$
\(\mathcal{M}\): metric space
Ξ: problem function $\Omega \to \mathcal{M}$, such as $\sigma(A)$ for $A \in B(\ell^2(\mathbb{N}))$

**TOWER**

$$\Xi(A) = \lim_{n_k \to \infty} \Gamma_{n_k}(A)$$
$$\Gamma_{n_k}(A) := \lim_{n_{k-1} \to \infty} \Gamma_{n_k,n_{k-1}}(A)$$

$$\Gamma_{n_k,,n_2}(A) := \lim_{n_1 \to \infty} \Gamma_{n_k,,n_2,n_1}(A)$$
Matrices first

\[ A \in B(\mathbb{C}^n) \quad \text{solve for } \pi_A(z) = 0 \]

- \( n \leq 3 \): generally convergent rational iteration exists (McMullen 1987)
- \( n > 3 \): no such towers (Doyle, McMullen 1989)
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Matrices continues

radicals, \( z \mapsto |z| \) available, then convergent iterations exist for solving roots of polynomials

input finite: the complex coefficients of the polynomial
Computabilities...

"Turing view": problem computable if a computing device exists which solves the problem

Computation in the limit and higher hierarchies

BSS (Blum, Shub, Smale) $\mathbb{R}$-machine model

IBC (information based complexity) uses BSS, "tractability"

constructivism, computability on $\mathbb{Z}$ and within computable numbers
Any compact can be spectrum

Represent compact $K \subset \mathbb{C}$ from outside:

$$\mathcal{K} = \bigcap K_n$$

where

$$\cdots \subset K_{n+1} \subset K_n \subset \cdots$$

and testing $z \notin K_n$ "easy"
Any compact can be spectrum, so look at Julia sets

We first look at the Julia set $\mathcal{J}$ for the quadratic polynomial $z^2 + 4$.

Consider the question

$$z \in \mathcal{J} \ ?$$

Then the corresponding question for the spectrum $\sigma(A)$ is

$$\lambda \in \sigma(A) \ ?$$

The natural formulation of these questions is, can you decide whether the answer is yes or no?
Let

\[ p(z) = z^2 + 4 \]

Iterate

\[ z_{n+1} = p(z_n) \]

If \( z_n \to \infty \) then \( z_0 \notin \mathcal{J} \).

Note that if \( |z_k| > 1 + \sqrt{5} \) for some \( k \), then \( |z_{k+1}| > 2|z_k| \) and then \( z_n \to \infty \).

For this \( p(z) \) the Julia set is \textit{homeomorphic to a Cantor set}.

Observe that \( \mathbb{C} \setminus \mathcal{J} \) is open.

S. Smale and coworkers: \( \mathcal{J} \) is \textit{not decidable} ("semidecidable")
Computation in the limit...

Output as follows:

if $|z_k| \leq 1 + \sqrt{5}$, then $Out(k) = 1$
if $|z_k| > 1 + \sqrt{5}$, then $Out(k) = 0$.

So depending on the initial value we obtain sequences of the form

1, 1, ..., 1, 0, 0, 0...

and

1, 1, 1, ...

In either case the limit exists; and then you (would) know
Similar question for the spectrum in abstract Banach algebra

Consider the subalgebra generated by just one element $a$ (say, in Banach algebra $\mathcal{A}$). Then the spectrum within the subalgebra is $\text{fill}(\sigma(a))$.

If we are allowed to produce polynomials of $a$ and compute their norms but inverting is not allowed, then:

The question

$$\lambda \notin \text{fill}(\sigma(a))$$

is semidecidable as follows:

If answer positive: finite termination with sure answer, while

if negative, you will never know (the one you look after does not exist)
What exists is easier to find!

**Conclude:** Think positive, construct the resolvent

\[ \mathbb{C} \setminus \text{fill}(\sigma(A)) \rightarrow B(X) \]

\[ \lambda \mapsto (\lambda - A)^{-1} \]

instead!

Get a **multicentric holomorphic calculus** - but not during this talk...
Computation in the limit

Example
Let $A$ be diagonal operator in $\ell_2(\mathbb{N})$ such that $a_{ii} \in \{0, 1\}$. Input information: read one diagonal element in time, in a fixed enumeration. Then

$\sigma(A) \in \{0, 1\}$: this we can build in the ”machine” based on the problem description
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- $1 \in \sigma_{ess}(A)$ this needs ”two limits”, i.e. a ”tower”
1 ∈ σ(A)

- define function for each \( n \)

\[
\Gamma_n(A) = 1, \quad \text{if} \quad \sum_{i=1}^{n} a_{ii} > 0,
\]

\[0, \quad \text{otherwise}\]

and set

\[
\Gamma(A) = \lim_{n \to \infty} \Gamma_n(A).
\]

Then, answer is "yes", when \( \Gamma(A) = 1 \)
How to get the answers

1 ∈ σ(A)

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Then, answer is ”yes”, when \( \Gamma(A) = 1 \)

▶ Using quantifiers: \( \exists n \ (\sum_{i=1}^{n} a_{ii} > 0) \)
How to get the answers

1 ∈ σ_{ess}(A)

- this needs "two limits", i.e. a "tower" of height 2

\[ \Gamma_{m,n}(A) = 1, \text{ if } \sum_{i=1}^{n} a_{ii} > m, \]

\[ 0, \text{ otherwise} \]

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Again, answer is "yes", when $\Gamma(A) = 1$

- With two quantifiers: $\forall m \exists n \left( \sum_{i=1}^{n} a_{ii} > m \right)$
Another example

We define \( A \in B(\ell_2(\mathbb{N})) \) using diagonal blocks:

\[
A = \bigoplus_{j=1}^{\infty} A_{k(j)}
\]

where \( A_k \) are \( k \times k \)-matrices with number 1’s in the corners, like

\[
A_3 = \begin{pmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]

and \( k(j) \geq 2 \) is some sequence. Thus, \( A \) is algebraic, \( \sigma(A) = \sigma_{\text{ess}}(A) = \{0, 2\} \).
Constructivism, computability

The operator

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is effectively determined if one can determine the sequence \( \{k(j)\} \) recursively.
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But,
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- The operator

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is **effectively determined** if one can determine the sequence \( \{k(j)\} \) recursively.

- But,

- then one can ”tailor” a computing machine which computes the spectrum in a finite number of operations.
The operator

\[ B = \bigoplus_{j=1}^{\infty} \beta_j A_{k(j)} \]

is *effectively determined* if one can determine the sequence \( \{k(j)\} \) recursively and the coefficient sequence \( \{\beta_j\} \) is a computable sequence of reals.
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Constructivism, computability 2

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Then,

the spectrum is computable.
In this theory effectively described bounded self-adjoint operators have computable spectra
Constructivism, computability 3

- In this theory effectively described bounded self-adjoint operators have computable spectra

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In this theory effectively described bounded self-adjoint operators have computable spectra.

but

there exists an effectively determined bounded non-selfadjoint operator which has a noncomputable real as an eigenvalue.
We assume:

- algorithm given for a class of operators $A = (a_{ij}) \in B(\ell_2(\mathbb{N}))$
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Computability; towers

We assume:
- algorithm given for a class of operators $A = (a_{ij}) \in B(\ell_2(\mathbb{N}))$
- can be adaptive but only based on what it has already computed
- input enters by reading one element $a_{ij}$ at a time

Example
Then for each such fixed algorithm one can ”tailor” a sequence \{k(j)\} such that the algorithm keeps the number 1 as a candidate for the spectrum for the operator

$$A = \bigoplus_{j=1}^{\infty} A_{k(j)}$$
In fact, the algorithm would be made to see a finite matrix consisting of diagonal blocks $A_{k(j)}$ and a block having just one nonzero element

\[
\begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
\vdots & & \ddots \\
\end{pmatrix}
\]

Thus,

- just *one limit* would give *wrong* answer
Example continues

In fact, the algorithm would be made to see a finite matrix consisting of diagonal blocks $A_k(j)$ and a block having just one nonzero element

$$
\begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & & & \\
\cdot & & & \\
\cdot & & & 
\end{pmatrix}
$$

Thus,

- just one limit would give wrong answer
- but limits on two levels work
Idea of a tower for the example

Let $A = A^\ast \in B(\ell_2(\mathbb{N}))$ and denote by $\gamma_{m,n}(t)$ the smallest singular value of the $n \times m$- matrix $A_{nm}(t)$ representing

$$P_n(A - tl)$$

when restricted to the range of $P_m$: $P_m\ell_2(\mathbb{N})$. 
Example continues

Applied to

\[ A = \bigoplus_{j=1}^{\infty} A_k(j) \]

the matrices \( A_{nm}(t) \) shall consist of a finite number of square blocks and possibly one rectangle which for fixed \( m \) and all large enough \( n \) is of the form

\[
\begin{pmatrix}
1 - t & 0 & 0 & \cdots \\
0 & -t & 0 & \cdots \\
0 & 0 & -t & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
1 & & & & \\
0 & & & & \\
\end{pmatrix}
\]
Proto for the tower at the Example

Since $[1]$ appears, the rectangle has full rank at $t = 1$.

- For example

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\begin{pmatrix}
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- Denote \( \Gamma_{m,n}(A) = \{ t \in \mathbb{R} : \gamma(t) = 0 \} \). Then we have with two quantifiers

\[
\forall m \exists n_m \{ n > n_m \implies \Gamma_{m,n}(A) = \{0, 2\} \}
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In particular, we may set $\Gamma_m(A) = \lim_{n \to \infty} \Gamma_{m,n}(A)$ so that
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$\Gamma(A) = \lim_{m \to \infty} \Gamma_m(A) = \{0, 2\} = \sigma(A)$. 
From Proto to a true tower one needs to have

- approximate version of $\gamma_{m,n}$ which can be performed with a finite number of arithmetic operations and radicals to give $\Gamma_{m,n}(A)$
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- Limits in the Hausdorff distance between compact sets in $\mathbb{C}$

$$\text{dist}_H(K, M) = \max\{\sup_{z \in K} \inf_{w \in M} |z - w|, \sup_{w \in M} \inf_{z \in K} |z - w|\}$$
Definition of Tower

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\( \mathcal{M} \): metric space
Ξ: problem function \( \Omega \rightarrow \mathcal{M} \), such as \( \sigma(A) \) for \( A \in \mathcal{B}(\ell^2(\mathbb{N})) \)

TOWER

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Definition of SCI

\[ k = \text{height of tower} \]

\[ \text{SCI} = \min k \text{ of towers solving the problem for arbitrary } A \in \Omega \]
SCI = 3 for bounded operators, $\Xi = \sigma(A)$

- a tower of height 3 works for all $A \in B(\ell_2(\mathbb{N}))$
SCI = 3 for bounded operators, $\Xi = \sigma(A)$

- a tower of height 3 works for all $A \in \mathcal{B}(\ell_2(\mathbb{N}))$

- we have a construction which shows that three limits are needed in general
Self-adjoint operators $A^* = A$, and further
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$A$ is similar to normal: $A = TNT^{-1}$ where $N$ is normal with a known constant $C$ such that $\|T\|\|T^{-1}\| \leq C$ (but the decomposition is not known), so that

$$\|(\lambda - A)^{-1}\| \leq \frac{C}{\text{dist}(\lambda, \sigma(A))}.$$
Self-adjoint operators $A^* = A$, and further

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there is a known function $g$ such that for $\lambda \notin \sigma(A)$

$$\|(\lambda - A)^{-1}\| \leq 1/g(\text{dist}(\lambda, \sigma(A))).$$
Dispersion known, again lowers the index

Dispersion: there is a known function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\max\{\| (I - P_{f(n)})AP_n \|, \| P_nA(I - P_{f(n)}) \| \} \rightarrow 0, \text{ as } n \rightarrow \infty$$

For example, if bandwidth $= d$ one has $f(n) = n + d$.

If $f$ is known for $A$, then $SCI = 2$

and if both resolvent control $g$ and dispersion function $f$ are known, then $SCI = 1$. 
SCI=1 for $\sigma(A)$ with $A \in \mathcal{B}(\ell_2(\mathbb{N}))$ compact

So, this is the situation in which computing eigenvalues of finite sections $A_n = (a_{ij})_{i,j \leq n}$ and studying their limit behavior is ok.
Computing the essential spectrum $\sigma_{ess}(A)$

Again $A \in \mathcal{B}(\ell^2(\mathbb{N}))$

- If we only know that $A$ is bounded, then $SCI=3$.

- If additionally both $f$ and $g$ are known, then $SCI=2$.

- If we know that $A$ is compact, then $SCI=0$, since $\sigma_{ess}(A) = \{0\}$. 
Computing the essential spectrum \( \sigma_{\text{ess}}(A) \)

Again \( A \in \mathcal{B}(\ell^2(\mathbb{N})) \)

- If we only know that \( A \) is bounded, then SCI=3.
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- If additionally both $f$ and $g$ are known, then SCI = 2.
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Schrödinger as an example

Let

\[ H = -\Delta + V \text{ where } V : \mathbb{R}^d \to \mathbb{C}. \]

- If \( V \) is bounded and in a certain total variation space. The evaluation functions are pointwise evaluations \( x \mapsto V(x) \). Then \( \text{SCI} \leq 2 \).
Schrödinger as an example

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- If \( V \) is continuous, \( |V(x)| \to \infty \) as \( \|x\| \to \infty \) and its values are in a sector with opening less than \( \pi \) and including the positive real axis, then the resolvent of \( H \) is compact and \( \text{SCI} = 1 \).
References

**Arithmetic hierarchy**


**Baire functions**

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