Minicourse

Introduction to Tauberian theory. A distributional approach

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Abstract

Tauberian theory provides striking methods to attack hard problems in analysis. The study of Tauberian type theorems has been historically stimulated by their potential applications in diverse fields of mathematics such as number theory, combinatorics, complex analysis, probability theory, and the analysis of differential operators [1, 3, 7, 8, 17, 19]. Even mathematical physics has pushed forward developments of the subject; indeed, theoretical questions in quantum field theory motivated the incorporation of generalized functions into the theory [17].

The aim of this minicourse is to give a modern introduction to Tauberian theory via distributional methods. The central topic is Tauberian theorems for the Laplace transform of (Schwartz) distributions with applications to prime number theory and PDE with constant coefficients. It will also be shown how to recover the classical Tauberian theorems from their distributional versions.

The minicourse consists of four parts. The first part is dedicated to explain the nature of Tauberian theory through classical examples and its formulation from a functional analysis perspective. Basics from distribution theory will be also recalled. In the second part, we study one-dimensional Hardy–Littlewood–Karamata type Tauberian theorems. The third part is devoted to complex Tauberian theorems and applications to the theory of (Beurling) generalized prime numbers. The final part deals with multidimensional theory; the results will be applied to asymptotics of solutions to convolution equations, in particular, hyperbolic PDE.

The minicourse aims to give a modern introduction to Tauberian theory for generalized functions and some of its applications. We will focus in Tauberian theorems for the Laplace transform and applications to number theory and PDE. It is intended for a general audience ranging from (advanced) undergraduate and graduate students to experienced researchers. It only requires knowledge of basics from functional analysis (specifically, the Hahn–Banach and Banach–Steinhaus theorems). Some familiarity with the Laplace transform and distribution theory would be helpful, though not a requirement.

The course consists of five lectures and, conceptually, it may be divided into four parts:

Tauberian theory and functional analysis methods

We will introduce the main problem in Tauberian theory through classical examples. We start with various summability procedures for divergent series and integrals and state their corresponding Abelian and Tauberian theorems; special attention will be paid to the summability methods by Abel, Cesáro, and Lambert means. We then reinterpret the particular cases within the language of functional analysis, and explain a general scheme to attack problems in Tauberian theory; the
discussion is then specialized to distributional methods. In this introductory part, we will also review the necessary background material from distribution theory.

References: [3, 5, 7, 8, 10, 12, 19]

**Hardy–Littlewood type theorems. The one-dimensional case**

We begin our incursion into Tauberian theorems for Laplace transforms by giving a simple proof of the celebrated Littlewood’s Tauberian theorem for the converse of Abel’s theorem on power series [10, 7, 9, 15]. We then proceed to show a general version of the Hardy–Littlewood Tauberian theorem for the distributional Laplace transform, we use the distributional version to easily recover several classical Tauberians of Hardy and Littlewood for power series.

References: [7, 8, 10, 14, 15, 17]

**Complex Tauberian theorems and prime number theory**

Tauberian theorems in which complex-analytic or boundary properties of the transform play an important role are usually referred as *complex Tauberians*. Many of these complex Tauberians have been inspired by number theoretic questions; for example, the classical Ikehara [8] theorem was motivated by the search for a simple proof of the prime number theory. They have also important implications in PDE theory [1]. We will study recent generalizations of the Landau–Ikehara theorem which relax the boundary requirements on the Laplace transform to a minimum [9, 13]. We shall also discuss applications to the theory of (Beurling) generalized primes [13, 16].

References: [1, 8, 9, 13, 16]

**Multidimensional theory**

This last part of the minicourse deals with multidimensional extensions of the Hardy–Littlewood–Karamata theorem [17]. The theorems to be studied are applicable to distributions and measures supported by cones. We will apply the results to obtain asymptotic properties of fundamental solutions to convolution equations, in particular, to hyperbolic operators with constant coefficients. As an illustration, we will consider several relevant operators from mathematical physics.

References: [4, 12, 17, 18]

**References**


