Time integration: splitting and exponential methods

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Introduction

The modeling of many interesting physical phenomena leads to evolutionary partial differential equations with solutions containing highly oscillatory components. The numerical simulation with standard integrators requires time-steps which are restricted by the inverse of the highest frequency.

In this talk we discuss the origin of limitations of standard schemes for such problems and we present alternative methods which overcome the step-size restrictions.

Splitting methods

Splitting methods can be applied to problems of the type

$$y' = f(y) + g(y), \qquad y(t_0) = y_0.$$

The idea is to replace the exact flow of the complete system by decomposing the flows, i.e. by solving $y' = f(y)$ and $y' = g(y)$ separately. This idea is promising whenever the integration of the single flows is computationally easier than that of the complete flow. If we denote the flows by φ_f and φ_g, respectively, then a symmetric splitting method is given by

$$y(t+h) \approx \Big(\varphi_f(h/2) \circ \varphi_g(h) \circ \varphi_f(h/2)\Big)\, y(t).$$

Typical applications are Schrödinger equations in quantum dynamics, astrophysics, or plasma dynamics.

In this talk we discuss splitting methods for linear and cubically nonlinear Schrödinger equations.

Exponential methods

In the second part of the talk we will give an overview on the construction, analysis, and implementation of various exponential integrators.

After a short motivation on the construction of general purpose exponential integrators, special methods for highly oscillatory problems including Schrödinger equations with time-dependent Hamiltonian and second order equations will be discussed. It will be shown that these integrators admit error bounds which are independent of the product of the step size with the frequencies.

Finally, we present new numerical schemes for the solution of nonlinear Klein-Gordon equations arising in modeling the interaction of relativistically intense electromagnetic waves with a plasma.

Possible topics for informal sessions

1. Basics on numerical time integration
2. Implementation of exponential integrators
3. Numerical methods for time-dependent Schrödinger equations
4. Numerical methods for nonlinear wave equations
5. Numerical methods for parabolic problems

References

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