A Finite Element Method for General Boundary Condition

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Summary In this note we introduce a method for handling general boundary conditions based on an approach suggested by Nitsche (1971) for the approximation of Dirichlet boundary conditions. We use Poisson's equations as a model problem and present the a priori and the a posteriori error estimates. Also, we show that conventional error estimates for Dirichlet and Neumann boundary conditions are a special case of the proposed error estimates.

Introduction

Enforcing perturbed Dirichlet boundary condition i.e. the Robin boundary condition with small coefficient in the derivative term leads to a high condition number in the system matrix. Perturbed boundary condition also plagues the adaptive mesh refinement based on the a posteriori error estimate since the straight forward formulation of the problem leads to a posteriori estimate that induces a too dense mesh on the boundary. A numerical scheme has to take these facts into account in order to produce an efficient and numerically stable method.

Perturbed boundary conditions arise for example in linear elasticity where a solid is on a very stiff but elastic support. Also, enforcing normal Dirichlet boundary condition with the penalty method is equivalent to solving a problem with perturbed Dirichlet boundary conditions since the penalty method is not consistent.

We show a method based on the Nitsche method [1] [2] [3] to circumvent the high condition number of the system matrix in the case of the perturbed boundary condition. The method is proposed in a way that it is possible to move continuously between the Neumann and the Dirichlet boundary conditions. We show the a priori error estimate that has the optimal rate of convergence. Under the saturation assumption we also show the a posteriori error estimate.

Deriving The Nitsche Method

We use the Poisson problem as a model problem.

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= \frac{1}{\epsilon}(g-u)+q && \text{on } \Gamma \\ u &= 0 && \text{on } \partial\Omega \setminus \Gamma \end{aligned} \tag{1}$$

where Ω is a bounded domain in space with polygonal boundary, $f \in L^2(\Omega)$, $g, q \in L^2(\Gamma)$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$.

Remark 1. *The value of the parameter ϵ allows to move between the Dirichlet and Neumann problems continuously i.e.*

$$\begin{aligned} \epsilon \to 0 \ &\Rightarrow \ u = g && \textit{on } \Gamma \\ \epsilon \to \infty \ &\Rightarrow \ \frac{\partial u}{\partial n} = q && \textit{on } \Gamma \ . \end{aligned} \tag{2}$$

We suppose that we have shape regular finite element partitions $\mathcal{T}_h$ of the domain $\Omega \in \mathbb{R}^N$, $N = 2, 3$. By $K \in \mathcal{T}_h$ we denote an element of the mesh and by E we denote an edge of the element. The mesh induces a partitioning also to the boundary of the domain $\partial\Omega$ and we denote

$$\mathcal{G}_h = \{E \ : \ K \cap \Gamma, \ K \in \mathcal{T}_h\} \ .$$

(The Nitsche Method). *Find $u_h \in V_h$ such that*

$$\mathcal{B}_h(u_h, v) = \mathcal{F}_h(v) \quad \forall v \in V_h \tag{3}$$

where

$$\begin{aligned}\mathcal{B}_h(u,v) = \big(\nabla u, \nabla v\big)_\Omega + \sum_{E \in \mathcal{G}_h} \Big\{ - \frac{\gamma h_E}{\epsilon + \gamma h_E} \Big[\big\langle \frac{\partial u}{\partial n}, v \big\rangle_E + \big\langle u, \frac{\partial v}{\partial n} \big\rangle_E \Big] \\ + \frac{1}{\epsilon + \gamma h_E} \big\langle u, v \big\rangle_E - \frac{\epsilon \gamma h_E}{\epsilon + \gamma h_E} \big\langle \frac{\partial u}{\partial n}, \frac{\partial v}{\partial n} \big\rangle_E \Big\}\end{aligned} \tag{4}$$

and

$$\begin{aligned}\mathcal{F}_h(v) = \big(f, v\big)_\Omega + \sum_{E \in \mathcal{G}_h} \Big\{ \frac{1}{\epsilon + \gamma h_E} \big\langle g, v \big\rangle_E - \frac{\gamma h_E}{\epsilon + \gamma h_E} \big\langle g, \frac{\partial v}{\partial n} \big\rangle_E \\ + \frac{\epsilon}{\epsilon + \gamma h_E} \big\langle q, v \big\rangle_E - \frac{\epsilon \gamma h_E}{\epsilon + \gamma h_E} \big\langle q, \frac{\partial v}{\partial n} \big\rangle_E \Big\} .\end{aligned} \tag{5}$$

Remark 2. *Setting $\gamma = 0$ in equation* (3) *yields the conventional variational formulation of the model problem* (1)*. Due to the inconsistency of the penalty method this variational form can also be seen as the variational form induced by the application of the penalty method with penalty parameter ϵ to the problem*

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g + \epsilon q && \text{on } \Gamma \ . \end{aligned}$$

Remark 3. *Setting $\epsilon = 0$ in equation* (3) *yields the variational form of the Nitsche method applied to problem [2]*

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g && \text{on } \Gamma \ . \end{aligned}$$

Remark 4. *Letting $\epsilon \to \infty$ in equation* (3) *yields the variational form of the Neumann problem*

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= q && \text{on } \Gamma \cdot \end{aligned}$$

Lemma 1 states that the proposed method is indeed consistent.

Lemma 1. *The solution u of the equations* (1) *satisfies*

$$\mathcal{B}_h(u, v) = \mathcal{F}_h(v) \quad \forall v \in V \ . \tag{6}$$

A priori error estimate

For the analysis of the method we define the following mesh-dependent norm

$$\|v\|_h^2 := \|\nabla v\|_{L^2(\Omega)}^2 + \sum_{E \in \mathcal{G}_h} \frac{1}{\epsilon + h_E} \|v\|_{L^2(E)}^2 \ . \tag{7}$$

Lemma 2. *There is a positive constant C_I such that [3]*

$$\sum_{E\in\mathcal{G}_h} h_E \left\| \frac{\partial v}{\partial n} \right\|^2_{L^2(E)} \leq C_I \|\nabla v\|^2_{L^2(\Omega)} \quad \forall v \in V_h \ . \tag{8}$$

Since it is possible to compute a value to the coefficient C_I of Lemma 2, it follows from Lemma 3 that the proposed method is always stable.

Lemma 3. *Suppose that $\gamma < 1/C_I$. Then there exists a positive constant C such that*

$$\mathcal{B}_h(v,v) \geq C\|v\|_h^2 \quad \forall v \in V_h \ . \tag{9}$$

Following interpolation estimate holds [3].

Lemma 4. *Suppose that $u \in H^s(\Omega)$, with $3/2 < s \leq p+1$. Then it holds*

$$\inf_{v\in V_h} \|u-v\|_h \leq Ch^{s-1}\|u\|_{H^s(\Omega)} \ . \tag{10}$$

Now we can formulate the a priori error estimate in the mesh dependent norm.

Theorem 1. *Suppose that $\gamma < 1/C_I$. Then it holds*

$$\|u-u_h\|_h \leq C \inf_{v\in V_h} \|u-v\|_h \tag{11}$$

and if $u \in H^s(\Omega)$ and $3/2 < s < p+1$, then

$$\|u-u_h\|_h \leq Ch^{s-1}\|u\|_{H^s(\Omega)} \ . \tag{12}$$

A posteriori error estimate

The a posteriori error estimate of the Nitsche method is based on the saturation assumption [4]. The assumption is that refining the mesh produces better solution in the mesh dependent energy norm.

Assumption 1. *Assume there exists $\beta < 1$ such that*

$$\|u-u_h\|_h \leq \beta\|u-u_{2h}\|_h \ , \tag{13}$$

where u_{2h} is a solution on a mesh size $2h$.

Theorem 2. *Suppose the saturation Assumption 1 holds and that $\gamma < 1/C_I$. Then it holds*

$$\|u-u_h\|_h \leq C\Big(\sum_{K\in\mathcal{T}_h} E_K^2(u_h)\Big)^{1/2} \ , \tag{14}$$

where

$$\begin{aligned} E_K^2(u) = {} & h_K^2\|\Delta u + f\|^2_{L^2(K)} + h_E \left\| \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \right\|^2_{L^2(\partial K\cap\mathcal{I})} \\ & + \frac{h_E}{(\epsilon+\gamma h_E)^2} \left\| \epsilon\big(\frac{\partial u}{\partial n} - q\big) + u - g \right\|^2_{L^2(\partial K\cap\Gamma)} \ , \end{aligned} \tag{15}$$

where $\mathcal{I}$ is the internal boundaries of the mesh.

Remark 5. *Setting $\epsilon = 0$ yields*

$$E_K^2(u) = h_K^2 \|\Delta u + f\|_{L^2(K)}^2 + h_E \left\| \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \right\|_{L^2(\partial K \cap \mathcal{I})}^2 + \frac{1}{h_E} \|u - g\|_{L^2(\partial K \cap \Gamma)}^2 \ , \tag{16}$$

which is the a posteriori estimate of the Nitsche method for the non-perturbed problem.

Remark 6. *Setting $\gamma = 0$ yields*

$$E_K^2(u) = h_K^2 \|\Delta u + f\|_{L^2(K)}^2 + h_E \left\| \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \right\|_{L^2(\partial K \cap \mathcal{I})}^2 + h_E \left\| \frac{\partial u}{\partial n} - q + \frac{1}{\epsilon}(u - g) \right\|_{L^2(\partial K \cap \Gamma)}^2 \ , \tag{17}$$

which is the a posteriori estimate of the penalty method or the conventional approach to the perturbed problem.

Remark 7. *Letting $\epsilon \to \infty$ yields*

$$E_K^2(u) = h_K^2 \|\Delta u + f\|_{L^2(K)}^2 + h_E \left\| \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \right\|_{L^2(\partial K \cap \mathcal{I})}^2 + h_E \left\| \frac{\partial u}{\partial n} - q \right\|_{L^2(\partial K \cap \Gamma)}^2 \ , \tag{18}$$

which is the a posteriori estimate of the Neumann problem.

These remarks show that the a posteriori estimate holds for all values of ϵ, even the limit values of ϵ yield the correct a posteriori estimate. In addition, setting $\gamma = 0$ yields the conventional approach or the penalty method, depending on the problem.
For the proofs of the error estimates check [5].

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