Stochastic relations of random variables and processes

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Fundamental problem of applied probability

$$\mathrm{E}\, f(X(t)) = ?$$

What if X is complex?

- Asymptotics
- Simulation
- Bounds

Stochastic bounds

Let X_1 and X_2 be (irreducible, positive recurrent) Markov processes with stationary distributions μ_1 and μ_2.

Problem

Can we show that $\mu_1 \leq_{\mathrm{st}} \mu_2$ without explicitly knowing μ_1 or μ_2?

Recall that μ_1 is stochastically less than μ_2, denoted $\mu_1 \leq_{\mathrm{st}} \mu_2$, if $\int f \, d\mu_1 \leq \int f \, d\mu_2$ for all positive increasing f.

Sufficient condition

Theorem (Whitt 1986; Massey 1987)

A sufficient condition for $\mu_1 \leq_{\mathrm{st}} \mu_2$ is that the transition rate kernels of X_1 and X_2 satisfy for all $x \leq y$:

- $Q_1(x, B) \leq Q_2(y, B)$ *for all upper sets B such that $x, y \notin B$*
- $Q_2(x, B) \geq Q_2(y, B)$ *for all lower sets B such that $x, y \notin B$*

The above condition is not sharp in general. Can we do any better?

Outline

Stochastic relations

Preservation of stochastic relations

Maximal subrelations

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Coupling

A coupling of random elements X and Y is a bivariate random element $(\hat{X}, \hat{Y})$ such that:

- $\hat{X}$ has the same distribution as X
- $\hat{Y}$ has the same distribution as Y

A coupling of probability measures μ on S_1 and ν on S_2 is a probability measure λ on $S_1 \times S_2$ having marginals μ and ν.

Remark

$(\hat{X}, \hat{Y})$ is a coupling of X and Y if and only if $\mathsf{P}((\hat{X}, \hat{Y}) \in \cdot)$ is a coupling of $\mathsf{P}(X \in \cdot)$ and $\mathsf{P}(Y \in \cdot)$.

Stochastic relations

Any meaningful distributional relation should have a coupling counterpart (Hermann Thorisson).

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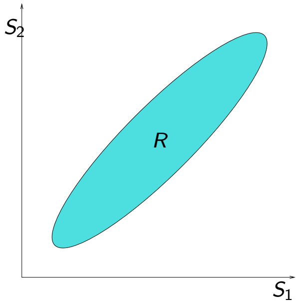


Denote

- $x \sim y$, if $(x, y) \in R$
- $X \sim_{\mathrm{st}} Y$, if there exists a coupling $(\hat{X}, \hat{Y})$ of X and Y such that $\hat{X} \sim \hat{Y}$ almost surely.
- $\mu \sim_{\mathrm{st}} \nu$, if there exists a coupling λ of μ and ν such that $\lambda(R) = 1$.

$R_{\mathrm{st}} = \{(\mu, \nu) : \mu \sim_{\mathrm{st}} \nu\}$ is the stochastic relation generated by R.

- For Dirac measures, $\delta_x \sim_{\mathrm{st}} \delta_y$ if and only if $x \sim y$.

Functional characterization

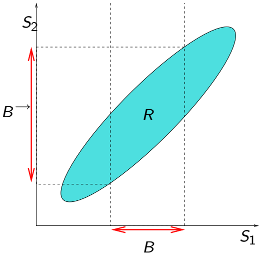
Theorem (Strassen 1965; L. 2008+)

The following are equivalent:

(i) $\mu \sim_{\mathrm{st}} \nu$

(ii) $\mu(B) \le \nu(B^{\rightarrow})$ *for all compact* $B \subset S_1$

(iii) $\int_{S_1} f\, d\mu \le \int_{S_2} f^{\rightarrow}\, d\nu$ *for all upper semicontinuous compactly supported* $f : S_1 \to \mathbb{R}_+$



$$B^{\rightarrow} = \cup_{x_1 \in B} \{x_2 \in S_2 : x_1 \sim x_2\}$$

$$f^{\rightarrow}(x_2) = \sup_{x_1 : x_1 \sim x_2} f(x_1).$$

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Remark

If R is an order (reflexive and transitive) relation on S, then conditions (ii) and (iii) are equivalent to

(ii') $\mu(B) \leq \nu(B)$ for all measurable upper sets B,

(iii') $\int_S f \, d\mu \leq \int_S f \, d\nu$ for all increasing measurable $f : S_1 \to \mathbb{R}_+$.

(Strassen 1965; Kamae, Krengel, O'Brien 1977)

Examples

- Stochastic equality. Let $=_{\mathrm{st}}$ be the stochastic relation generated by the equality $=$. Then $X =_{\mathrm{st}} Y$ if and only if X and Y have the same distribution.
- Stochastic ϵ-distance. Define $x \approx y$ by $|x - y| \leq \epsilon$. Two real random variables satisfy $X \approx_{\mathrm{st}} Y$ if and only if for all x the corresponding c.d.f.'s satisfy $F_Y(x - \epsilon) \leq F_X(x) \leq F_Y(x + \epsilon)$.
- Stochastic induced order. Define $x \leq^{f,g} y$ by $f(x) \leq g(y)$. Then $\mu \leq^{f,g}_{\mathrm{st}} \nu$ if and only if $\mu(f^{-1}((\alpha, \infty))) \leq \nu(g^{-1}((\alpha, \infty)))$ for all real numbers α (Doisy 2000).

Outline

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Preservation of stochastic relations

Maximal subrelations

Monotonicity vs. relation-preservation

Order relations $\rightsquigarrow$ monotone functions f:

$$x \leq y \implies f(x) \leq f(y)$$

General relations $\rightsquigarrow$ relation-preserving pairs of functions (f, g):

$$x \sim y \implies f(x) \sim g(y)$$

Stochastic relations $\rightsquigarrow$ stochastically relation-preserving pairs of probability kernels (random functions) (F, G):

$$x \sim y \implies F(x, \cdot) \sim_{\mathrm{st}} G(y, \cdot)$$

Preservation of stochastic relations

A pair of probability kernels (P_1, P_2) stochastically preserves a relation R, if

$$x_1 \sim x_2 \implies P_1(x_1, \cdot) \sim_{\mathrm{st}} P_2(x_2, \cdot)$$

or equivalently,

$$\mu_1 \sim_{\mathrm{st}} \mu_2 \quad \implies \mu_1 P_1 \sim_{\mathrm{st}} \mu_2 P_2.$$

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Theorem (Zhang 1998; L. 2008+)

A pair (P_1, P_2) stochastically preserves R if and only if there exists a probability kernel P on $S_1 \times S_2$ such that:

(i) *$P(x, \cdot)$ couples $P_1(x_1, \cdot)$ and $P_2(x_2, \cdot)$ for all $x = (x_1, x_2)$.*

(ii) $x \in R \implies P(x, R) = 1.$

Stochastic relations of Markov processes

A pair of Markov processes stochastically preserve a relation R, if

$$x \sim y \quad \Longrightarrow \quad X(x,t) \sim_{\mathrm{st}} Y(y,t) \text{ for all } t,$$

or equivalently,

$$\mu \sim_{\mathrm{st}} \nu \quad \Longrightarrow \quad X(\mu,t) \sim_{\mathrm{st}} Y(\nu,t) \text{ for all } t.$$

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Remark

A Markov process is stochastically monotone, if

$$x \leq y \quad \Longrightarrow \quad X(x,t) \leq_{\mathrm{st}} X(y,t) \text{ for all } t.$$

Relation-preserving Markov processes

Let X_1 and X_2 be discrete-time Markov processes with transition probability kernels P_1 and P_2.

Theorem (L. 2008+)

The following are equivalent:

(i) X_1 *and* X_2 *stochastically preserve the relation* R*.*

(ii) $P_1(x_1, B) \leq P_2(x_2, B^{\rightarrow})$ *for all* $x_1 \sim x_2$ *and compact* $B \subset S_1$*.*

(iii) *There exists a Markovian coupling of* X_1 *and* X_2 *for which* R *is invariant.*

Relation-preserving Markov processes

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(iii) *There exists a Markovian coupling of X_1 and X_2 for which R is invariant.*

Remarks

- If R is an order, (ii) can be replaced by
  (ii') $P_1(x_1, B) \le P_2(x_2, B)$ for all $x_1 \le x_2$ and upper sets B
  (Kamae, Krengel, O'Brien 1977).
- An analogous result holds for nonexplosive Markov jump processes, generalizing the result of Whitt and Massey.

Outline

Stochastic subrelations

Recall our starting point:

Problem

Can we show that the stationary distributions μ_1 *and* μ_2 *of Markov processes* X_1 *and* X_2 *satisfy* $\mu_1 \leq_{\mathrm{st}} \mu_2$ *without explicitly knowing* μ_1 *or* μ_2*?*

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Recall our starting point:

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Can we show that the stationary distributions μ_1 and μ_2 of Markov processes X_1 and X_2 satisfy $\mu_1 \leq_{\mathrm{st}} \mu_2$ without explicitly knowing μ_1 or μ_2?

- The sufficient condition of Whitt and Massey essentially says that X_1 and X_2 stochastically preserve the order relation $R_{\leq} = \{(x, y) : x \leq y\}$.

Stochastic subrelations

Recall our starting point:

Problem

Can we show that the stationary distributions μ_1 and μ_2 of Markov processes X_1 and X_2 satisfy $\mu_1 \leq_{\mathrm{st}} \mu_2$ without explicitly knowing μ_1 or μ_2?

- The sufficient condition of Whitt and Massey essentially says that X_1 and X_2 stochastically preserve the order relation $R_{\leq} = \{(x, y) : x \leq y\}$.
- A less stringent sufficient condition: Show that X_1 and X_2 stochastically preserve a nontrivial subrelation of $R_{\leq}$.

Subrelation algorithm

Given a closed relation R and continuous probability kernels P_1 and P_2, define a sequence of relations by $R^{(0)} = R$,

$$R^{(n+1)} = \left\{ (x, y) \in R^{(n)} : \ (P_1(x, \cdot), P_2(y, \cdot)) \in R^{(n)}_{\mathrm{st}} \right\},$$

and let $R^* = \bigcap_{n=0}^{\infty} R^{(n)}$.

Subrelation algorithm

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and let $R^* = \bigcap_{n=0}^{\infty} R^{(n)}$.

Theorem (L. 2008+)

The relation R^* is the maximal closed subrelation of R that is stochastically preserved by (P_1, P_2). Especially, the pair (P_1, P_2) preserves a nontrivial subrelation of R if and only if $R^* \neq \emptyset$.

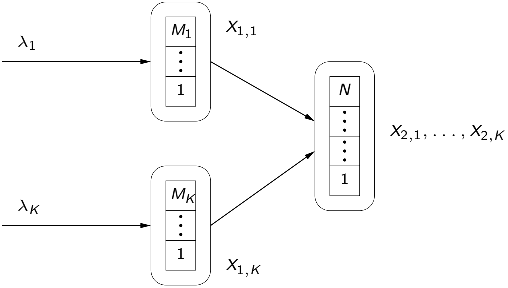
Remark

A modified algorithm works for Markov jump processes.

Application: Multilayer loss network

Multiclass loss network with

- M_k servers dedicated to class-k jobs (layer 1)
- N multiclass servers processing the overflow traffic (layer 2)



Application: Multilayer loss network

Modified system $Y = (Y_{1,1}, \ldots, Y_{1,K}; Y_{2,1}, \ldots, Y_{2,K})$

- One class-1 server replaced by a shared server
- Can we show that $\mathsf{E} \sum_{i,k} X_{i,k} \leq \mathsf{E} \sum_{i,k} Y_{i,k}$ in steady state?

Define the relation $x \sim y$ by $\sum_{i,k} x_{i,k} \leq \sum_{i,k} y_{i,k}$.

- $\sim$ is not an order (different state spaces)
- X and Y do not preserve $\sim_{\rm st}$
- But maybe (X, Y) preserves some subrelation of $\sim_{\rm st}$?

Application: Multilayer loss network

Example

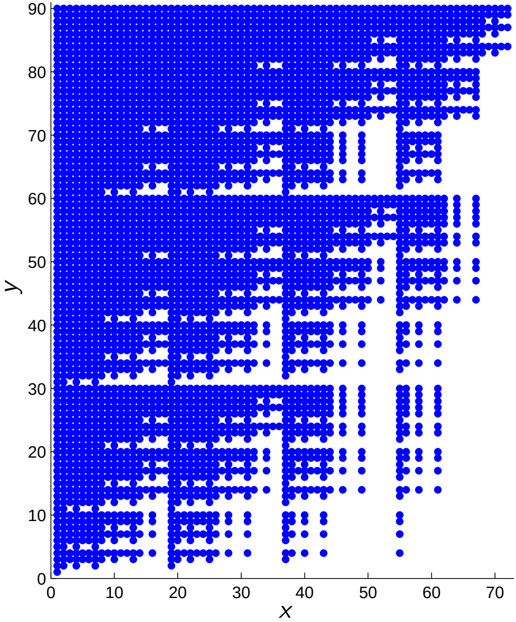
Two customer classes

- Server configuration: $M_1 = 3$, $M_2 = 2$, $N = 2$
- Arrival rates $\lambda_1 = 1$, $\lambda_2 = 2$
- Service rate $\mu = 1$

How many iterations do we need to compute R_∞?

- X has 72 possible states
- Y has 90 possible states

Application: Multilayer loss network



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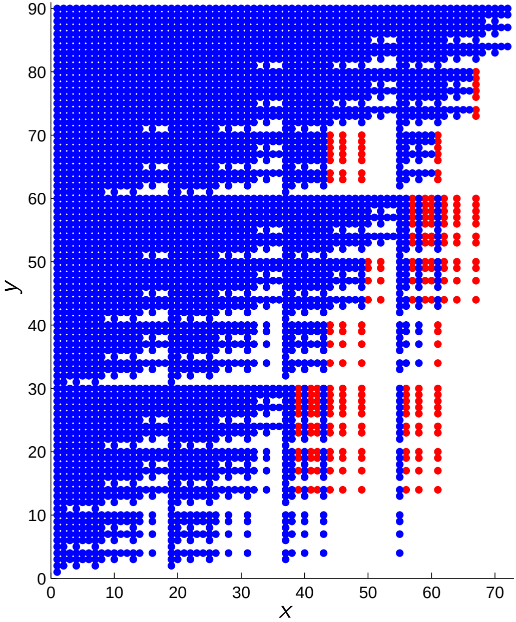

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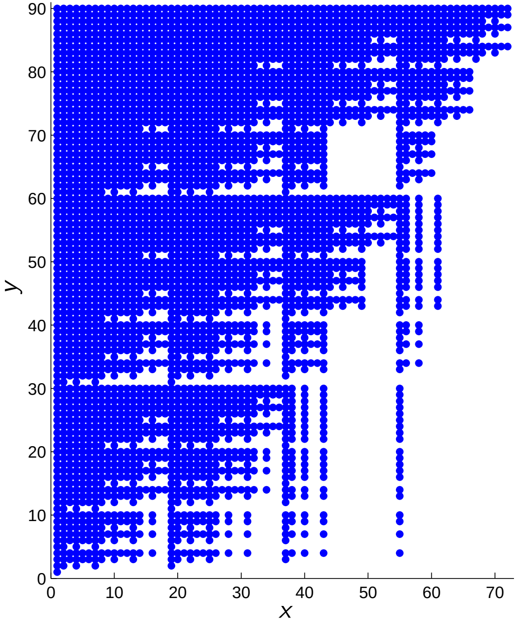

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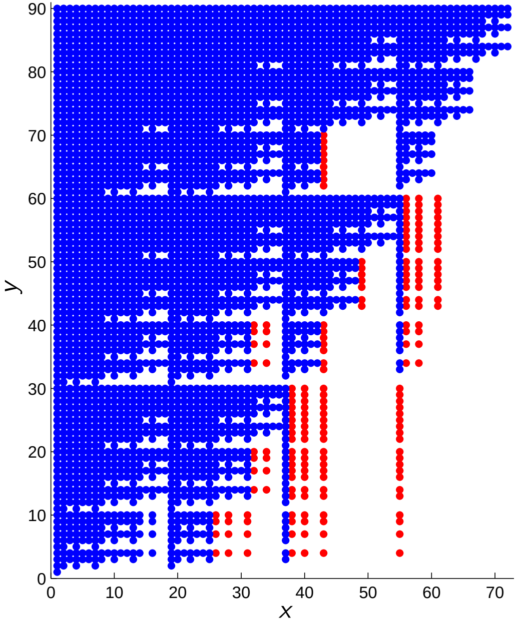

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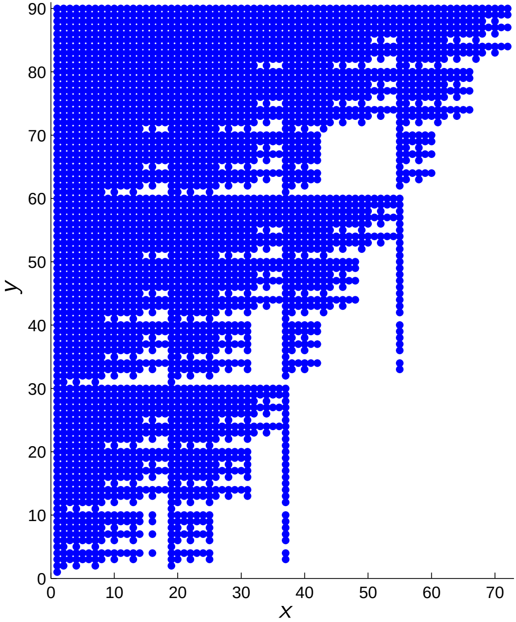

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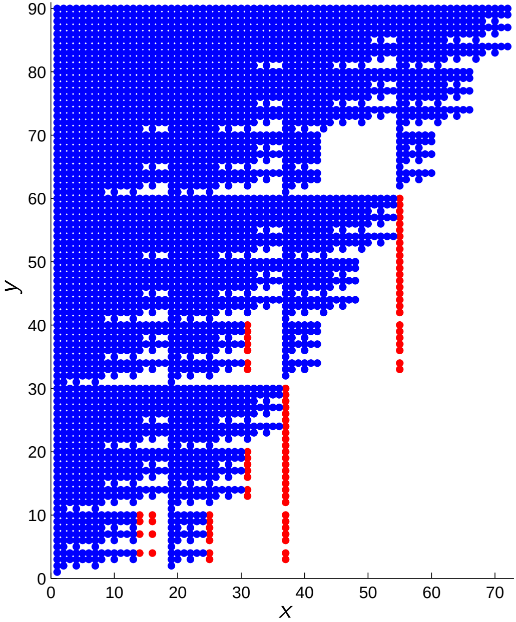

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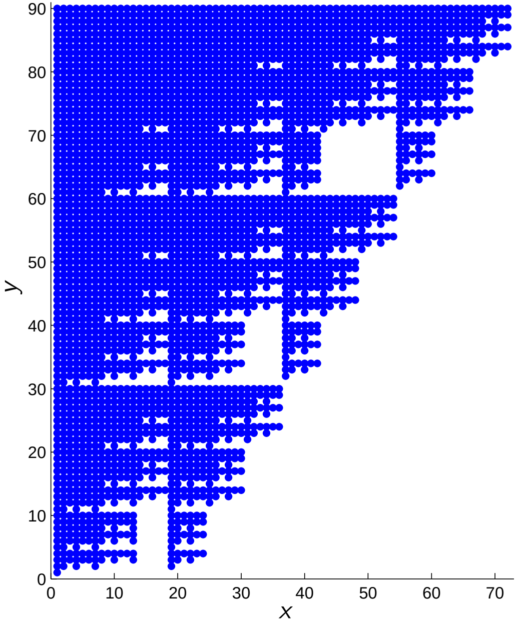

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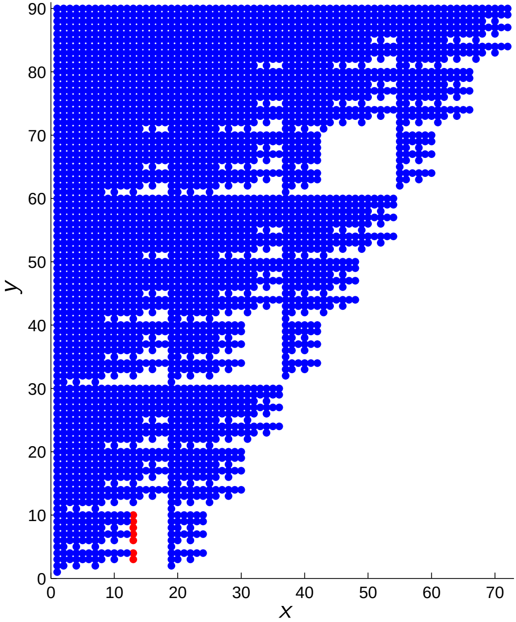

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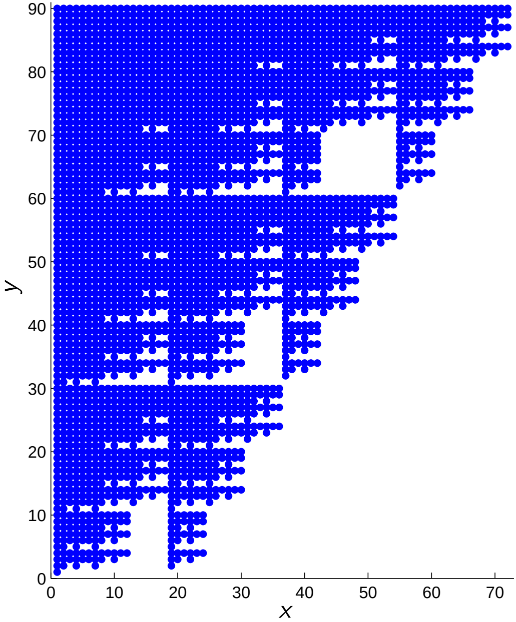


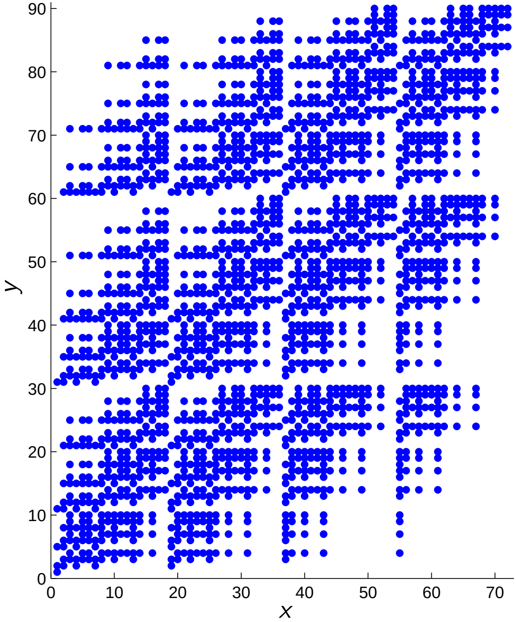
Application: Multilayer loss network

What if we started with a stricter relation?

Redefine $x \sim y$ by

$$0 \leq \sum_{i,k} y_{i,k} - \sum_{i,k} x_{i,k} \leq 1$$

Application: Multilayer loss network



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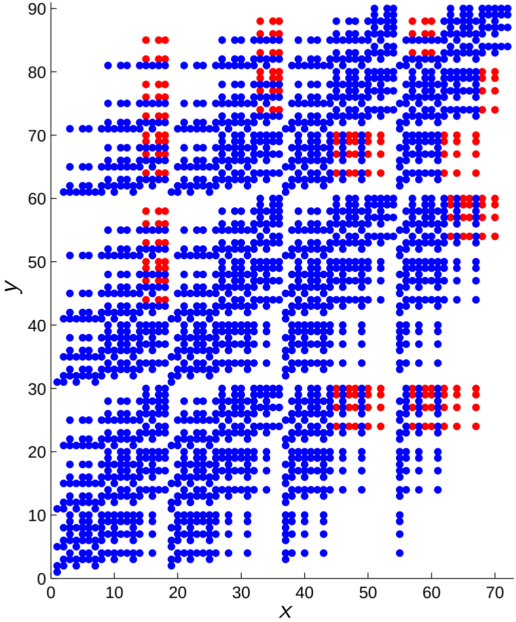

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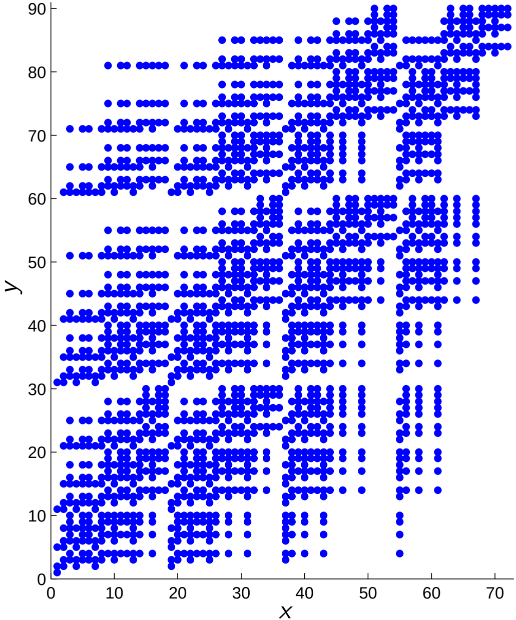

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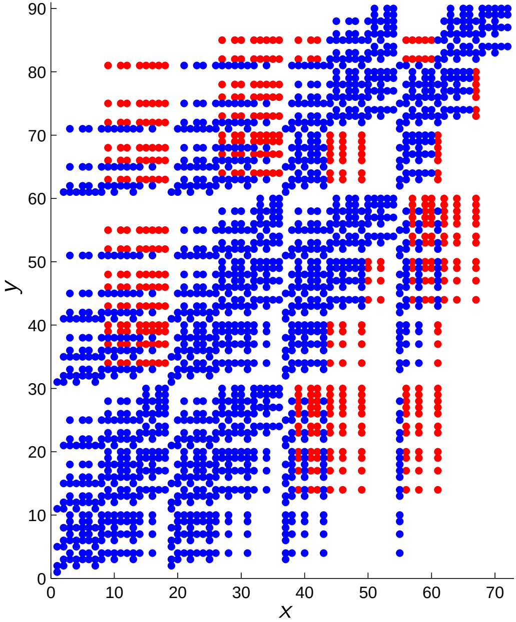

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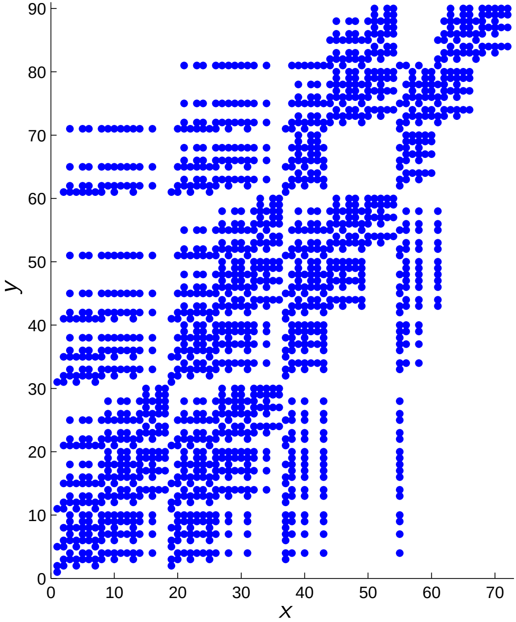

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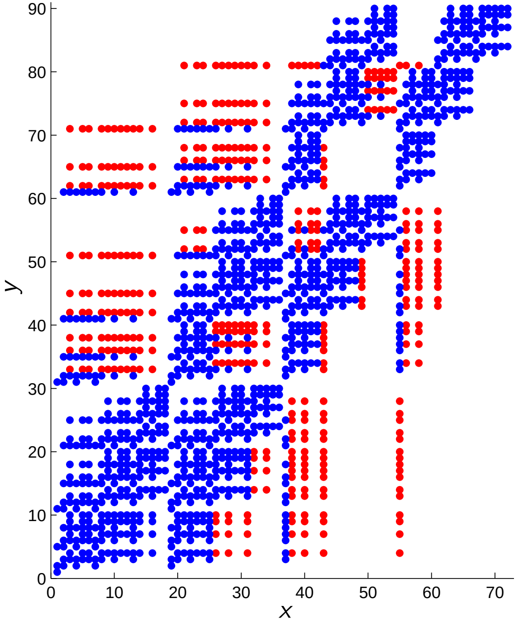

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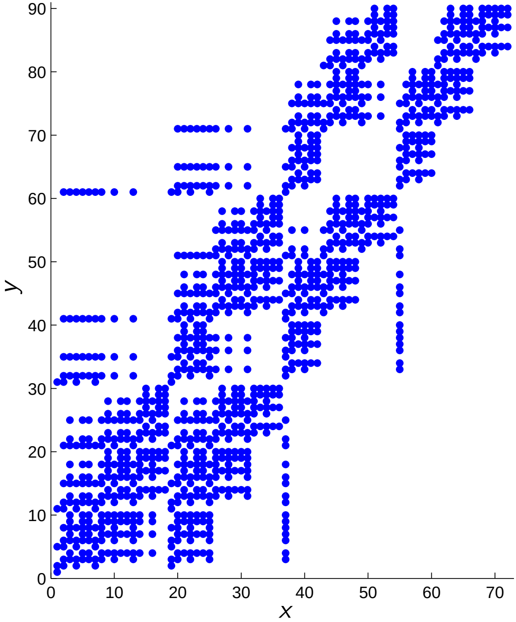

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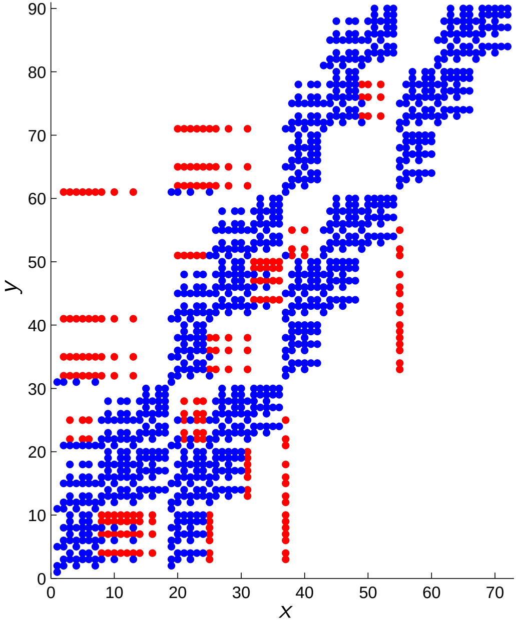

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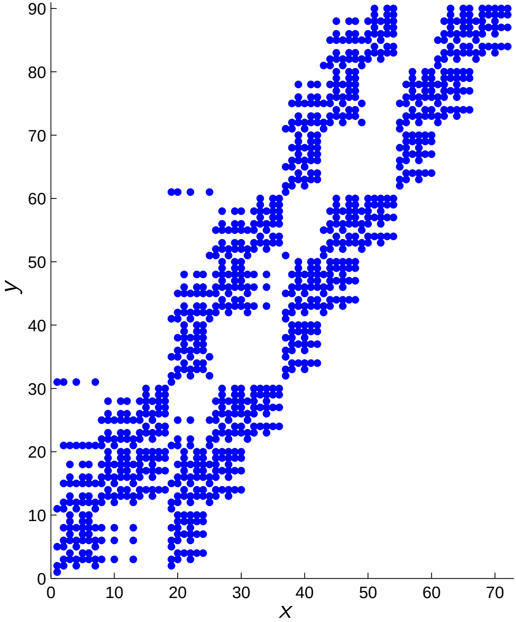

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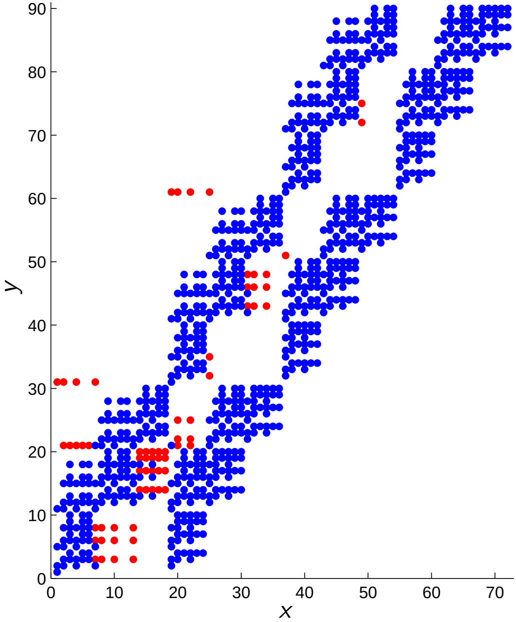

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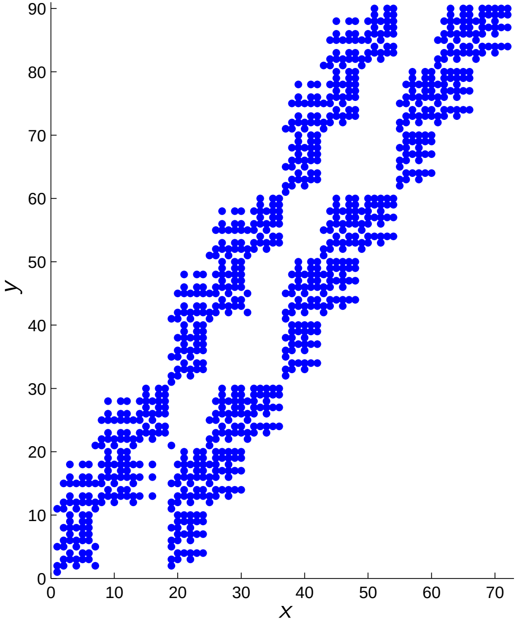

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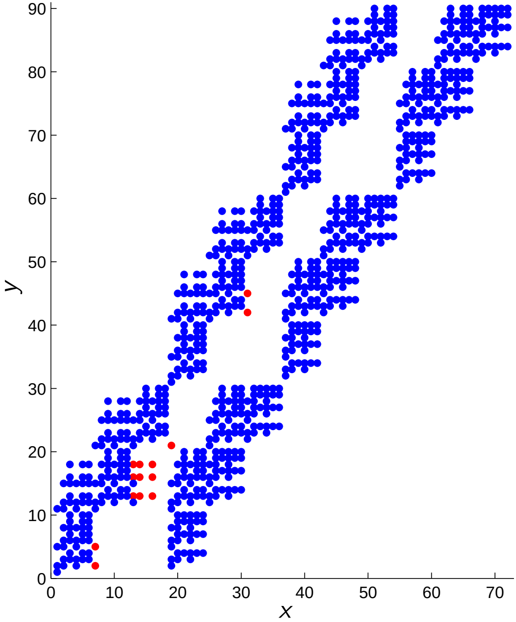

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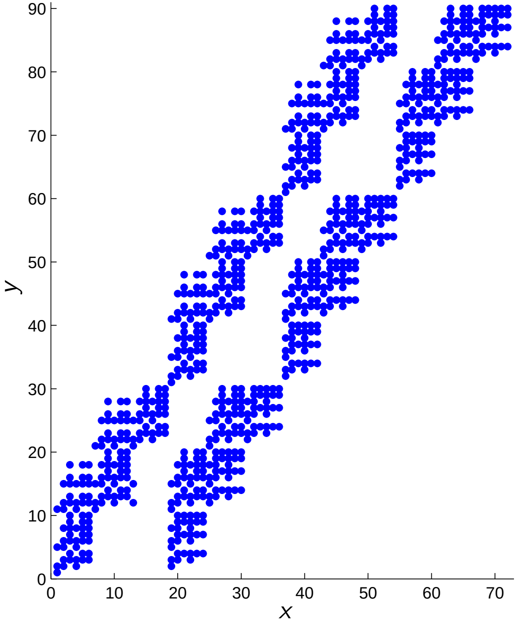

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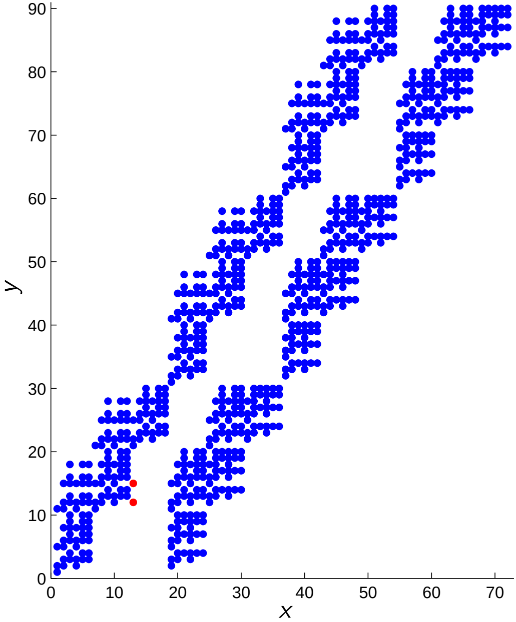

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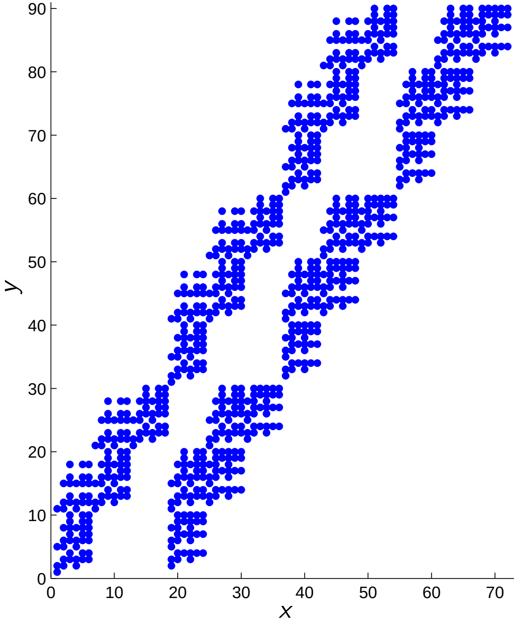


Application: Multilayer loss network

Theorem (Jonckheere & L. 2008)

The processes X and Y stochastically preserve the relation $R = \{(x, y) : |x - y| \in \Delta\}$, where

$$\Delta = \{0,\ e_2,\ e_2 - e_{1,1},\ 2e_2 - e_{1,1}\}.$$

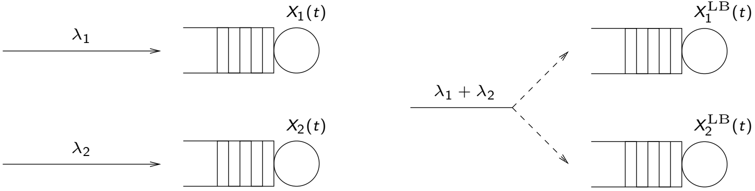
Especially, the stationary distributions of the processes satisfy

$$|Y| - 1 \le_{\mathrm{st}} |X| \le_{\mathrm{st}} |Y|,$$

and

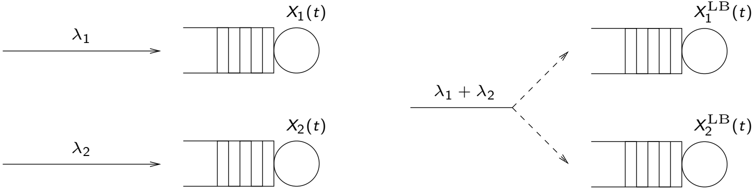
$$\begin{aligned} X_{1,1} &\ge_{\mathrm{st}} Y_{1,1}, \\ X_{1,k} &=_{\mathrm{st}} Y_{1,k} \quad \text{for all } k \neq 1, \\ \sum_k X_{2,k} &\le_{\mathrm{st}} \sum_k Y_{2,k}. \end{aligned}$$

Application: Load balancing



Common sense: $\mathsf{E}(X_1^{\mathrm{LB}}(t) + X_2^{\mathrm{LB}}(t)) \leq \mathsf{E}(X_1(t) + X_2(t))$

Application: Load balancing



Common sense: $\mathsf{E}(X_1^{\mathrm{LB}}(t) + X_2^{\mathrm{LB}}(t)) \le \mathsf{E}(X_1(t) + X_2(t))$

The rate kernel pair (Q^{LB}, Q) does not stochastically preserve:

- $R^{\mathrm{nat}} = \{(x,y) : x_1 \le y_1,\ x_2 \le y_2\}$
- $R^{\mathrm{sum}} = \{(x,y) : |x| \le |y|\}$, where $|x| = x_1 + x_2$

How about a subrelation of R^{sum}?

Application: Load balancing

Theorem (L. 2008+)

The subrelation algorithm started from R^{sum} *yields*

$$R^{(n)} = \left\{(x,y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2 + (y_1 \wedge y_2 - n)^+\right\}$$
$$\downarrow$$
$$R^* = \{(x,y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2\}.$$

Especially, (Q^{LB}, Q) *stochastically preserves the relation* R^**.*

Remark

- R^* is the weak majorization order on $\mathbb{Z}_+^2$
- $X \sim_{\mathrm{st}}^* Y$ if and only if $\mathrm{E}\, f(X) \le \mathrm{E}\, f(Y)$ for all coordinatewise increasing Schur-convex functions f (Marshall & Olkin 1979).

Conclusions

Algorithmic probability

- Computational methods for analytical results
- Comparison without ordering
- State space reduction

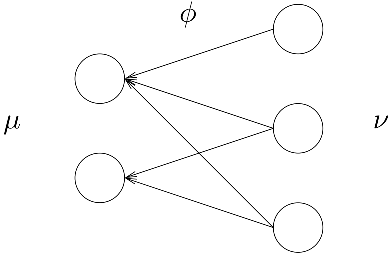
Open problems:

- Numerical methods for finite Markov chains
- Subrelations versus dependence orderings
- Diffusions, Feller processes, martingales, ...

Discussion: Coupling vs. mass transportation

$$W_\phi(\mu,\nu) = \inf_{\lambda \in K(\mu,\nu)} \int_{S_1 \times S_2} \phi(x_1, x_2)\, \lambda(dx)$$

- $K(\mu,\nu)$ is the set of couplings of μ and ν



- W_ϕ is a Wasserstein metric, if ϕ is a metric.
- $\mu \sim_{\mathrm{st}} \nu$ if and only if $W_\phi(\mu,\nu) = 0$ for $\phi(x_1, x_2) = 1(x_1 \not\sim x_2)$.

(Monge 1781, Kantorovich 1942, Wasserstein 1969, Chen 2005)

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