Appendix E

Interpolation Theorems

Mathematicians have announced the existence of a new whole number which lies between 27 and 28. "We don't know why it's there or what it does," says Cambridge mathematician, Dr. Hilliard Haliard, "we only know that it doesn't behave properly when put into equations, and that it is divisible by six, though only once."

— On The Hour

Here we present the Riesz–Thorin Interpolation Theorem, the Hausdorff–Young Theorem and similar results. In these results a function mapping $X_1 \to Y_1$ and $X_2 \to Y_2$, continuously, is shown to map "$X_r \to Y_r$ when $1 < r < 2$" when X_r, Y_r are suitable spaces (of vector-valued functions).

We use the assumptions of Chapter B in this chapter; in particular, the scalar field **K** may be either **C** or **R**.

E.1 Interpolation theorems ($\mathrm{L}^{p_1} + \mathrm{L}^{p_2} \to \mathrm{L}^{q_1} + \mathrm{L}^{q_2}$)

We start by a few auxiliary lemmas and definitions.

Lemma E.1.1 ($\mathbf{L^p + L^q}$) *Let $p \in [1,\infty]$. The space $\mathrm{L}^1(Q;B) + \mathrm{L}^\infty(Q;B)$ is a Banach space. For all measurable $f : Q \to B$, we have*

$$\frac{1}{2}\|f\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \le \|f\|_p \le \|f\|_1 + \|f\|_\infty. \tag{E.1}$$

Finally, $\mathrm{L}^q \cap \mathrm{L}^r \underset{c}{\subset} \mathrm{L}^p \underset{c}{\subset} \mathrm{L}^q + \mathrm{L}^r$, *when* $1 \le q \le p \le r \le \infty$.

Thus, L^q and L^r are sum-compatible. See Lemmas A.3.17 and A.3.18 for the norms of $X+Y$ and $X \cap Y$.

Proof: 1° $\mathrm{L}^1(Q;B) + \mathrm{L}^\infty(Q;B)$ *is a Banach space:* Let $0 \ne f \in \mathrm{L}^1 + \mathrm{L}^\infty$. Find $\varepsilon > 0$ s.t. $\mu(E_\varepsilon) > 0$, where $E_\varepsilon := \{q \in Q \,\big|\, \|f(q)\| > \varepsilon\}$. If $f = g + h$ and $\|h\|_\infty < \varepsilon$, then $\|g\|_1 > \int_{E_\varepsilon} \varepsilon \ge \varepsilon\mu(E_\varepsilon)$. Therefore $\|f\|_{\mathrm{L}^1+\mathrm{L}^\infty} \ge \min\{\varepsilon, \mu(E_\varepsilon)\varepsilon\} > 0$. Thus, $f \ne 0 \Rightarrow \|f\| > 0$.

For sums we have

$$\|f+g\|_+ \le \inf_{f=f_1+f_\infty,\ g=g_1+g_\infty} (\|f_1+g_1\|_1 + \|f_\infty+g_\infty\|_\infty) \tag{E.2}$$

$$\le \inf_{f=f_1+f_\infty,\ g=g_1+g_\infty} (\|f_1\|_1 + \|g_1\|_1 + \|f_\infty\|_\infty + \|g_\infty\|_\infty) \le \|f\|_+ + \|g\|_+. \tag{E.3}$$

Obviously, $\|\alpha f\|_+ = |\alpha|\|f\|_+$ for $\alpha \in \mathbf{K}$. Thus, we can apply by Lemma A.3.17 (set $Z := \mathrm{L}^1 + \mathrm{L}^\infty$), and deduce that $\mathrm{L}^1 + \mathrm{L}^\infty$ is complete.

2° $\|f\|_{\mathrm{L}^1+\mathrm{L}^\infty} \le 2\|f\|_p$: W.l.o.g., we assume that $1 < p < \infty$ and $\|f\|_p = 1$. Set $g := f\chi_{E_1}$, $h := f - g$. Then $\|h\|_\infty \le 1$ and

$$\mu(E_1) \le \int_{E_1} \|f\|_B^p \, d\mu \le \|f\|_p^p = 1, \tag{E.4}$$

hence $\|g\|_1 \le \|f\|_p \|\chi_{E_1}\|_q \le 1 \cdot 1 = \|f\|_p$, where $p^{-1} + q^{-1} = 1$. Thus, $\|f\| \le \|g\|_1 + \|h\|_1 \le 2$.

3° $\|f\|_p \le \|f\|_1 + \|f\|_\infty$: This follows from Lemma B.3.14.

4° *Embeddings:* By (E.1), the embeddings are continuous for $q = 1$, $r = \infty$. Thus, we can apply Lemmas A.3.17 and A.3.18 to obtain those claims in the general case. □

Given $p \in [1,\infty]$ and a measurable $f : Q \to [0,+\infty)$ s.t. $f > 0$ a.e., we define $\mathrm{L}^{p,f}(Q;B)$ to be the space of (equivalence classes of) measurable functions $g : Q \to B$ s.t.

$$\|g\|_{p,f} := \|fg\|_p < \infty. \tag{E.5}$$

Obviously, $g \mapsto fg$ is an isometric isomorphism of $\mathrm{L}^{p,f}$ onto L^p; in particular, $\mathrm{L}^{p,f}$ is a Banach space. Note also that the zero elements of all such spaces are equal. Of these spaces, we are mainly interested in spaces $\mathrm{L}^p_r := \mathrm{L}^{p,e^{-r\cdot}}$. Such spaces are sum-compatible for all p's and r's:

Lemma E.1.2 ($\mathbf{L}^p_r + \mathbf{L}^q_r$) *Let $1 \le p_1 \le p_2 \le p_3 \le \infty$, and let $f_k : Q \to [0,+\infty)$ be s.t. $f_k > 0$ a.e. ($k = 1,2,3$). Then*

Then L^{p_1,f_1} and L^{p_3,f_3} are sum-compatible. Moreover,

(a) L^{p_1,f_1} and L^{p_3,f_3} are sum-compatible.

(b1) if $f_k \ge f_2$ ($k = 1,3$), then $\mathrm{L}^{p_1,f_1} \cap \mathrm{L}^{p_3,f_3} \underset{c}{\subset} \mathrm{L}^{p_2,f_2}$.

(b2) if $f_k \le f_2$ ($k = 1,3$), then $\mathrm{L}^{p_2,f_2} \underset{c}{\subset} \mathrm{L}^{p_1,f_1} + \mathrm{L}^{p_3,f_3}$.

In particular, if $J \subset \mathbf{R}$ is an interval and $\omega_k \in \mathbf{R}$ ($k = 1,3$), then $\mathrm{L}^{p_1}_{\omega_1}$ and $\mathrm{L}^{p_3}_{\omega_3}$ are sum-compatible.

Analogously, $\ell^{p_1}_{r_1}$ and $\ell^{p_3}_{r_3}$ are sum-compatible for $r_1, r_3 > 0$ (see (13.2)).

Proof: (a) 1° *If $f_2 \ge f_1$, then $\mathrm{L}^{p,f_2} \underset{c}{\subset} \mathrm{L}^{p,f_1}$:* Obviously, $\|\cdot\|_{p,f_1} \le \|\cdot\|_{p,f_2}$.

2° *Case $p_1 = p_3$:* By 1°, we have $\mathrm{L}^{p_1,f_k} \underset{c}{\subset} \mathrm{L}^{p_1,f}$ ($k = 1,3$), where $f := \max\{f_1, f_3\}$. Thus, L^{p_1,f_1} and L^{p_1,f_3} are sum-compatible.

3° *Case* $f_1 = f_3$: Use Lemma E.1.1 and the isometric isomorphism $f_1\cdot : \mathrm{L}^{p,f_1} \to \mathrm{L}^p$ to see that $\|\cdot\|_{\mathrm{L}^{p_1,f_1}+\mathrm{L}^{p_3,f_1}}$ is a norm. Thus, $\mathrm{L}^{p_1,f_1}+\mathrm{L}^{p_3,f_1}$ is a Banach space, hence L^{p_1,f_1} and L^{p_1,f_3} are sum-compatible.

4° *General case:* Set $f := \max\{f_1, f_3\}$. By 3°, $Z := \mathrm{L}^{p_1,f}+\mathrm{L}^{p_3,f}$ is a Banach space. By 1°, $\mathrm{L}^{p_k,f} \underset{c}{\subset} Z$ $(k=1,3)$, hence $\mathrm{L}^{p_1,f}$ and $\mathrm{L}^{p_3,f}$ are sum-compatible.

(b1) Let $f_k \geq f_2$ $(k=1,3)$. Then $\mathrm{L}^{p_k,f_k} \underset{c}{\subset} \mathrm{L}^{p_k,f_2}$ $(k=1,3)$, by 1°. Therefore, $\cap_k \mathrm{L}^{p_k,f_k} \underset{c}{\subset} \cap_k \mathrm{L}^{p_k,f_2}$ $(k=1,3)$, by Lemma A.3.19(c1)&(c2). But $\cap_k \mathrm{L}^{p_k,f_2} \underset{c}{\subset} \mathrm{L}^{p_2,f_2}$, by Lemma E.1.1 (and the isomorphism, cf. 3°), hence (b1) holds.

(b2) The proof of (b2) is analogous to that of (b1) and hence omitted.

The final claim: Now $\mathrm{L}^{p_k}_{\omega_k} := \mathrm{L}^{p_k,\mathrm{e}^{-\omega_k\cdot}} \underset{c}{\subset} \mathrm{L}^{p_1,f}+\mathrm{L}^{p_3,f}$ $(k=1,3)$, by (b2), where $f := \min_{k=1,3}\mathrm{e}^{-\omega_k\cdot}$. □

The following concept is required for some interpolation results:

Definition E.1.3 ($\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$) *Let (X_1,X_2) and (Y_1,Y_2) be sum-compatible pairs of normed spaces. Then we write $T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ if $T \in \mathrm{Hom}(X_1+X_2,Y_1+Y_2)$ is s.t. $T \in \mathcal{B}(X_1,Y_1)$, $T \in \mathcal{B}(X_2,Y_2)$. We set*

$$\|T\|_{\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)} := \max\{\|T\|_{\mathcal{B}(X_1,Y_1)}, \|T\|_{\mathcal{B}(X_1,Y_1)}\}. \tag{E.6}$$

Here we do not distinguish between T and its restrictions. The above requirements force T to be continuous $X_1+X_2 \to Y_1+Y_2$ (with norm$\leq \max_{k=1,2}\|T\|_{\mathcal{B}(X_k,Y_k)}$), by 1° of the proof of Lemma E.1.4.

Note that Lemma A.3.18 provides an alternative definition for $\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$; obviously, this coincides with the one above, up to the norm (we do not consider $\mathcal{B}(X_k,Y_k)$ being a vector subspace of some vector space Z $(k=1,2)$ if $T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ does not imply $T(x_1+x_2)$ being defined and equal to Tx_1+Tx_2 for $x_k \in X_k$ $(k=1,2)$; the latter condition implies that $T \in \mathrm{Hom}(X_1+X_2,Y_1+Y_2)$).

We obviously have $\mathcal{B}(X,Y_1)\cap\mathcal{B}(X,Y_2) = \mathcal{B}(X,Y_1\cap Y_2)$, isometrically.

We now give four equivalent definitions of $\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ (the fourth on X_0):

Lemma E.1.4 *Let (X_1,X_2) be a sum-compatible pair of normed spaces, and let (Y_1,Y_2) a sum-compatible pair of Banach spaces. Then (i)⇔(ii)⇔(iii), where*

(i) $T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$

(ii) $T \in \mathcal{B}(X_1+X_2,Y_1+Y_2)$, $T \in \mathcal{B}(X_1,Y_1)$ and $T \in \mathcal{B}(X_2,Y_2)$;

(iii) $T \in \mathcal{B}(X_1+X_2,Y_1+Y_2)$, $T[X_1] \subset Y_1$ and $T[X_2] \subset Y_2$.

Let X_0 be a dense subspace of X_1, X_2 and $X_1\cap X_2$. If $T_0 \in \mathrm{Hom}(X_0,Y_1+Y_2)$ and there is $M<\infty$ s.t.

$$\|T_0x\|_{Y_k} \leq M\|x\|_{X_k} \ \ (x \in X_0,\ k=1,2), \tag{E.7}$$

then $T_0 = T_{|X_0}$ for a unique $T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$. Moreover, T_0 has unique continuous extensions in $\mathcal{B}(X_1\cap X_2,Y_1\cap Y_2)$, $\mathcal{B}(X_1,Y_1)$, $\mathcal{B}(X_2,Y_2)$ and

$\mathcal{B}(X_1+X_2, Y_1+Y_2)$, and these are of norm $\le M$ and coincide with T on their domains.

Conversely, if T satisfies any (hence all) of (i)–(iii), then $T_0 := T|_{X_0}$ is as above.

If $X_1 = \mathrm{L}^p_\omega(J;B)$ and $X_2 = L^q_{\omega'}(J;B)$ for some $p,q \in [1,\infty)$, $\omega,\omega' \in \mathbf{R}$ (see Definition D.1.3), then we may take $X_0 := X_1 \cap X_2$, or $X_0 := C_c^\infty$ or let X_0 be the set of simple L^1 functions (or any set between these), by Theorem B.3.11.

Proof: 1° *(i)–(iii):* Trivially, (ii)⇒(i)&(iii). By Lemma A.3.6, we have (iii)⇒(ii). Obviously, $\|T\|_{\mathcal{B}(X_1+X_2,Y_1+Y_2)} \le \max_{k=1,2}\|T\|_{\mathcal{B}(X_k,Y_K)}$, so that (i) implies (ii).

2° $T_0 \leftrightarrow T$: If T satisfies (ii), then $T_0 := T|_{X_0}$ satisfies (E.7) for $M := \max_{k=1,2}\|T\|_{\mathcal{B}(X_k,Y_k)}$.

For the converse, let T_0 be as in the lemma. By Lemma A.3.10 and (E.7), T_0 has unique continuous extensions $T_0' \in \mathcal{B}(X_1\cap X_2, Y_1\cap Y_2)$ and $T_k \in \mathcal{B}(X_k,Y_k)$ ($k=1,2$), with norm $\le \|M\|$. If $\{x_n\} \subset X_0$ and $x_n \to x \in X_1\cap X_2$, then $T_0'x_n \to T_0'x$ in $X_1\cap X_2$, hence in X_1 and in X_2 too, so that $T_0'x$, T_1x and T_2x must all be equal to this limit. Thus, T_0', T_1 and T_2 coincide on $X_1\cap X_2$.

By Lemma A.3.19, X_0 is dense in X_1+X_2. One easily verifies that $\|T_0x\|_{Y_1+Y_2} \le M\|x\|_{X_1+X_2}$, so that there is a unique continuous extension $T \in \mathcal{B}(X_1+X_2, Y_1+Y_2)$ too, again with norm $\le M$, by 1°. If $X_0 \ni x_n \to x$ in X_k, then $Tx_n \to T_kx$ in Y_k and $Tx_n \to Tx$ in $Y_1\cap Y_2$, hence $T_kx = Tx$ ($x \in X_k$, $k=1,2$). Thus T coincides with T_0', T_1 and T_2. In particular, (iii) (hence (i)–(iii)) is satisfied. □

Next we give equivalent conditions for $T \in \mathcal{B}(X,Y)$:

Lemma E.1.5 ($T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ & $T[X] \subset Y \Rightarrow T \in \mathcal{B}(X,Y)$) *Let X_1, X_2, Y_1, Y_2 be normed spaces. Let X and Y be Banach spaces. Let (X_1,X_2) and (Y_1,Y_2) be sum-compatible pairs. Let $X \underset{c}{\subset} X_1+X_2$ and $Y \underset{c}{\subset} Y_1+Y_2$. Let $T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$.*

Then the following are equivalent:

(i) $T|_X \in \mathcal{B}(X,Y)$;

(ii) $T[X] \subset Y$;

(iii) there are $M < \infty$ and a dense subspace $\widetilde{X} \subset X$ s.t.

$$\|\widetilde{x}\|_X = 1 \Rightarrow \|T\widetilde{x}\|_Y \le M \quad (\widetilde{x} \in \widetilde{X}). \tag{E.8}$$

Moreover, if (iii) holds, then $\|T\|_{\mathcal{B}(X,Y)} \le M$.

Recall that $\|y\|_Y = \infty$ for $y \notin Y$.

Proof: (As the proof shows, we can allow X to be incomplete if we give up the implication (ii)⇒(i).)

Recall first that $T \in \mathcal{B}(X_1+X_2, Y_1+Y_2)$, hence $T \in \mathcal{B}(X, Y_1+Y_2)$ (because $X \underset{c}{\subset} X_1+X_2$).

1° *(i)⇒(ii):* This is obvious.

2° *(ii)⇒(i):* Now (ii) implies (i), by Lemma A.3.6, because $T \in \mathcal{B}(X, Y_1 + Y_2)$.

3° *(i)⇒(iii):* Take $\widetilde{X} = X$, $M := \|T_{|X}\|$.

4° *(iii)⇒(i):* Let $T_0 \in \mathcal{B}(X, Y)$ be the unique continuous extension of T_0. If $\widetilde{X} \ni x_n \to x$ in X, then $Tx_n \to Tx$ in $Y_1 + Y_2$ and $Tx_n = T_0 x_n \to T_0 x$ in Y, hence in $Y_1 + Y_2$ too, so that $T_0 x = Tx$. This holds for all $x \in X$, hence $T_0 = T_{|X}$. □

Now that the preparations are done, we can give four interpolation results. We start with the vector-valued forms of two celebrated theorems:

Theorem E.1.6 (Riesz–Thorin Interpolation Theorem) *Let* $p_k, q_k \in [1, \infty]$ *(*$k = 0, 1$*). Let* μ *and* μ' *be complete positive measures on sets* Q *and* Q'*, respectively. Let* B *and* B' *be complex Banach spaces.*

If $T \in \cap_{k=0,1} \mathcal{B}(\mathrm{L}^{p_k}(Q; B), \mathrm{L}^{q_k}(Q'; B'))$*, then* $T \in \mathcal{B}(\mathrm{L}^p(Q; B), \mathrm{L}^q(Q'; B'))$ *with norm*

$$M_\theta \le M_0^{1-\theta} M_1^\theta \le \max\{M_0, M_1\}, \tag{E.9}$$

provided that $0 \le \theta \le 1$, $M_k := \|T\|_{\mathcal{B}(\mathrm{L}^{p_k}(Q;B), \mathrm{L}^{q_k}(Q';B'))}$ *(*$k = 0, 1$*),*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{E.10}$$

Thus, $\log M_\theta$ is convex.

Proof: (This is an extended and rigorous version of the (scalar case) proof of Theorem 1.1.1 of [BL]. The theorem also holds for real B with $2M_0^{1-\theta} M_1^\theta$ in place of $M_0^{1-\theta} M_1^\theta$ (note that $\|T\|_{\mathcal{B}(\mathrm{L}^p(Q;B+iB), \mathrm{L}^q(Q';B'+iB'))} \le 2\|T\|_{\mathcal{B}(\mathrm{L}^p(Q;B), \mathrm{L}^q(Q';B'))}$, where $B + iB$ is the complexification of B (allow complex scalars with natural operations and, e.g., $\|x + iy\|_{B+iB} := (\|x\|_B^2 + \|y\|_B^2)^{1/2}$).)

1° *Only* $|\int_{Q'} gTf \, d\mu'| \le M$ *needs to be shown:* W.l.o.g., we assume that $0 < \theta < 1$, $1 \le p_0 < p_1 \le \infty$ and $q_0 \ne q_1$. Let $q^{-1} + q'^{-1} = 1$, $q_k^{-1} + {q'_k}^{-1} = 1$ ($k = 1, 2$). By Lemma E.1.5 and Theorem B.3.11, we only have to show that (E.8) is satisfied by the number M_θ and the set

$$\widetilde{X} := \mathrm{SMFF}(Q; B) := \{\sum_{k=1}^n x_k \chi_{E_k} \,\big|\, n \in \mathbf{N},\ \mu(E_k) < \infty,\ x_k \in B\ (k = 1, \ldots, n)\}. \tag{E.11}$$

By Theorem B.4.12, it suffices that given $f \in \mathrm{SMFF}(Q; B)$, $g \in \mathrm{SMFF}(Q; B)$ s.t. $\|f\|_p = 1 = \|g\|_{q'}$, we have $|\int_{Q'} gTf \, d\mu'| \le M$ (note that $Tf \in \mathrm{L}^{q_0}$ and $g \in \mathrm{L}^{q'_0}$, hence $gTf \in \mathrm{L}^1$, by the Hölder Inequality).

2° *We show it:* Set $1/p(z) := p_0^{-1} + z(p_1^{-1} - p_0^{-1})$, $1/q'(z) := {q'_0}^{-1} + z({q'_1}^{-1} - {q'_0}^{-1})$, as in (E.10), so that $p, q' \in \mathrm{H}(\mathbf{C})$, and $\mathrm{Re}\, p(z)^{-1} \in (p_1^{-1}, p_0^{-1})$, $\mathrm{Re}\, q'(z) \in ({q'_1}^{-1}, {q'_0}^{-1})$ when $\mathrm{Re}\, z \in (0, 1)$. Set

$$\phi(z, t) := \|f(t)\|_B^{p/p(z)} f(t) / \|f(t)\|_B, \quad \psi(z, t') := \|g(t')\|_B^{q'/q'(z)} g(t') / \|g(t')\|_B \tag{E.12}$$

for $z \in \overline{\Omega}$, $t \in Q$, $t' \in Q'$ (we set $\phi(z,t) = 0$ if $f(t) = 0$, and $\psi(z,t') = 0$ if $g(t') = 0$).

Obviously, $\phi(\cdot,q), \psi(\cdot,q) \in \mathrm{H}(\Omega;B)$ and $\operatorname{Re} p/p(z) \in (p/p_1, p/p_0) \subset (0, p/p_0)$ ($\operatorname{Re} z \in (0,1)$) (analogously for q'). It follows that $\phi, \psi, \phi_z, \psi_z$ (which are in SMFF) have a majorant in $g \in$ SMFF, independent of $z \in \overline{\Omega}$.

Therefore, $\phi \in \mathcal{C}(\overline{\Omega};\mathrm{L}^{p_0}) \cap \mathrm{H}(\Omega;\mathrm{L}^{p_0})$, by Lemma B.5.8. Analogously, $\psi \in \mathcal{C}(\overline{\Omega};\mathrm{L}^{q_0'}) \cap \mathrm{H}(\Omega;\mathrm{L}^{q_0'})$ (if $q_0' = \infty$, just note that $\psi, \psi_z \in \mathcal{C}(\overline{\Omega};\mathrm{L}^\infty)$, and use (d)&(e) of Lemma B.5.8).

By Lemma D.1.2(b1)&(b3) and the Hölder Inequality, we have $F \in \mathcal{C}(\overline{\Omega}) \cap \mathrm{H}(\Omega)$, where $F(z) := \int_Q \psi(z) T\phi(z)\, d\mu$. For any $t \in \mathbf{R}$, we have

$$\|\phi(it)\|_{p_0} = \| \, |f|^{p/p_0} \|_{p_0} = \|f\|_p^{p/p_0} = 1, \tag{E.13}$$

and, similarly, $\|\phi(1+it)\|_{p_1} = 1 = \|\psi(it)\|_{q_0'} = \|\psi(it)\|_{q_1'}$. Therefore, $|F(it)| \le M_0$ and $|F(1+it)| \le M_1$ for $t \in \mathbf{R}$, hence $|F(\theta)| \le M_0^{(1-\theta)} M_1^{\theta}$, by Lemma D.1.5. But $\phi(\theta) = f$ and $\psi(\theta) = g$, hence $F(\theta) = \int_Q gTf\, d\mu$, so we have reached our aim. □

We already know that the Fourier transform maps $\mathrm{L}^1(\mathbf{R};H) \to \mathcal{C}_0(i\mathbf{R};H) \subset \mathrm{L}^\infty(i\mathbf{R};H)$ and $\mathrm{L}^2(\mathbf{R};H) \to \mathrm{L}^2(i\mathbf{R};H)$. From the above theorem we obtain corresponding interpolation results for $\mathrm{L}^p(\mathbf{R};H)$, $1 < p < 2$:

Theorem E.1.7 (Hausdorff–Young) *Let H be a complex Hilbert space, $\omega \in \mathbf{R}$, $p \in [1,2]$ and $p^{-1} + q^{-1} = 1$. Then the Fourier transform $\mathcal{F}$ maps $\mathrm{L}^p_\omega(\mathbf{R};H)$ to $\mathrm{L}^q(\omega + i\mathbf{R};H)$ and $\mathrm{L}^p_\omega(\mathbf{R}_+;H)$ to $\mathrm{H}^q(\mathbf{C}^+_\omega;H)$ with*

$$\|\mathcal{F}f\|_q \le (2\pi)^{1/q} \|f\|_{\mathrm{L}^p_\omega} \quad (f \in \mathrm{L}^p_\omega(\mathbf{R};H)) \tag{E.14}$$

(with equality for $p = 2$).

Moreover, if $f \in \mathrm{L}^p_\omega(\mathbf{R}_+;H)$, then $\widehat{f} \in \mathrm{L}^q_\omega(\omega + i\mathbf{R};H)$ is the boundary function of $\widehat{f} \in \mathrm{H}^q(\mathbf{C}^+_\omega;H)$, and $\widehat{f * g} = \widehat{f}\widehat{g}$ on $\mathbf{C}^+_\omega$ and a.e. on $\omega + i\mathbf{R}$ for all $g \in \mathrm{L}^r_\omega(\omega + i\mathbf{R};H)$, where $r^{-1} \in [3/2 - p^{-1}, 1]$.

This means that the restriction of $\mathcal{F}$ to $\mathrm{L}^1_\omega \cap \mathrm{L}^p_\omega$ satisfies (E.14) and has hence a unique extension onto L^p. (By Lemma E.1.4, this coincides with the Plancherel Transform on $\mathrm{L}^p_\omega \cap \mathrm{L}^2_\omega$.)

Unfortunately, if H is a general Banach space, then we know this for $p = 1$ only; in particular, we cannot interpolate. E.g., $\mathcal{F}$ is not bounded $\mathcal{C}_c(\mathbf{R}^n;c_0) \to \mathrm{L}^q$ w.r.t. the L^p norm, where $p \in (1,2]$ and $c_0 := \mathcal{C}_0(\mathbf{N})$ is the space of sequences $\mathbf{N} \to \mathbf{C}$ converging to zero with the sup-norm (let $0 \ne \phi \in \mathcal{S}(i\mathbf{R})$, $\phi = \pi_{[0,1)}\phi$, $N \in \mathbf{N}$, so that $\psi := \mathcal{F}^{-1}\phi \in \mathcal{S}(\mathbf{R}) \subset \mathrm{L}^p$, and set $f_k := \mathrm{e}^{ik\cdot}\phi$ so that $\widehat{f}_k(ir) = \phi(ir - ik)$ for $k \le N$; it follows that $\|\widehat{g}\|_p^p = N\|\phi\|_p^p$).

Proof of Theorem E.1.7: (We take $\omega = 0$ w.l.o.g.)

1° $\widehat{f} \in \mathrm{L}^q$*:* We already know this for $p = 1$ and for $p = 2$. By Lemma E.1.4, $\mathcal{F}|_{\mathrm{L}^1 \cap \mathrm{L}^2}$ has continuous extensions to $\mathcal{B}(\mathrm{L}^1, \mathrm{L}^\infty)$ and $\mathcal{B}(\mathrm{L}^2, \mathrm{L}^2)$ (by uniqueness these extensions are equal to $\mathcal{F}$) and $\mathcal{F}$ extends to $\mathcal{B}(\mathrm{L}^1 + \mathrm{L}^2, \mathrm{L}^\infty + \mathrm{L}^2)$, hence $\mathcal{F} \in \mathcal{B}(\mathrm{L}^1, \mathrm{L}^\infty) \cap \mathcal{B}(\mathrm{L}^2, \mathrm{L}^2)$. Thus, we can apply Theorem E.1.6 to obtain (E.14) by a direct computation.

2° $\widehat{f} \in \mathrm{H}^q$: Let $f \in \mathrm{L}^p(\mathbf{R}_+;B)$. We have $\|f\|_{\mathrm{L}^p_\alpha} \le \|f\|_p$ for all $\alpha \ge 0$, hence $\|\widehat{f}\|_{\mathrm{H}^q} \le \|f\|_p$.

Set $h := f * g$, so that $\|h\|_v \le \|f\|_p\|g\|_r$, where $v^{-1} = p^{-1} + r^{-1} - 1$, by Lemma D.1.7. Because $v \in [p,2]$, we have $\|\widehat{h}\|_{v'} \le (2\pi)^{1/v'}\|f\|_p\|g\|_r$, where $v^{-1} + v'^{-1} = 1$.

If $f,g \in \mathcal{C}_\mathrm{c}$, then $\widehat{h} = \widehat{f}\widehat{g}$ on $\overline{\mathbf{C}^+} \cup \{\infty\}$, by Lemma D.1.11(c'). In general, if $f_n, g_n \in \mathcal{C}_\mathrm{c}$ and $f_n \to f$ in L^p, $g_n \to g$ in L^r, then $\widehat{h}_n \to \widehat{h}$ in $\mathrm{L}^{v'}$, hence a.e., and $\widehat{h}_n = \widehat{f}_n\widehat{g}_n \to \widehat{f}\widehat{g}$ a.e. (here we have replace $\{(f_n,g_n)\}$ by a suitable subsequence), so that $\widehat{h} = \widehat{f}\widehat{g}$ a.e. By Lemma D.1.11(c'), $\widehat{h} = \widehat{f}\widehat{g}$ on $\mathbf{C}^+_\alpha$ for each $\alpha > 0$ (because $f,g \in \mathrm{L}^1_\alpha$), hence on $\mathbf{C}^+$.

Let $g := \chi_{[0,1)}$, so that $h := f * g \in \mathrm{L}^p \cap \mathrm{L}^2(\mathbf{R}_+;H)$. By Theorem 3.3.1(a2) (or by the corresponding scalar result), $\widehat{h}$ is the boundary function of itself a.e. (in the sense of condition (1.) of Theorem 3.3.1(a1)), hence so is $\widehat{f}$, because $\widehat{g}(s) = (1 - \mathrm{e}^{-s})/s$ is invertible a.e. on $i\mathbf{R}$. □

By the Riesz–Thorin theorem, we can interpolate between L^p_r and L^q_r. Next we show that we can also interpolate between L^p_r and $\mathrm{L}^p_{r'}$:

Proposition E.1.8 ($\mathbf{L}^p_r$ interpolation w.r.t. r) *Let $p,q \in [1,\infty)$, $-\infty < a < b < \infty$. Let $J \subset \mathbf{R}$ be an open interval. Let $\mathcal{F}$ be the set of simple functions $J \to B$, or $\mathcal{F} = \mathrm{L}^p_a(J;B) \cap \mathrm{L}^p_b(J;B)$ or $\mathcal{F} = \mathcal{C}^\infty_\mathrm{c}(J;B)$.*

Let $\mathbb{E} : \mathcal{F} \to \mathrm{L}^q_a(J;B) \cap \mathrm{L}^q_b(J;B)$ be linear and s.t. $M_a, M_b < \infty$, where $M_r := \sup\{\|\mathbb{E}f\|_{\mathrm{L}^q_r} \mid f \in \mathcal{F},\ \|f\|_{\mathrm{L}^p_r} \le 1\}$ ($r \in [a,b]$).

Then there is a unique extension $\widetilde{\mathbb{E}}$ of $\mathbb{E}$ to $\cup_{r\in[a,b]}\mathrm{L}^p_r(J;B)$ s.t. $\widetilde{\mathbb{E}} \in \mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$ for all $r \in [a,b]$. Moreover,

$$\|\widetilde{\mathbb{E}}\|_{\mathcal{B}(\mathrm{L}^p_r,\mathrm{L}^q_r)} = M_r \le M_a^{1-\theta_r} M_b^{\theta_r} \le \max\{M_a, M_b\} \quad (r \in [a,b]), \tag{E.15}$$

where $\theta_r := (r-a)/(b-a)$.

Note that, under the above conditions, $\mathbb{E} : \mathcal{F} \to \mathrm{L}^q_a(J;B) \cap \mathrm{L}^q_b(J;B)$ has a unique continuous extension to $\mathbb{E}_r \in \mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$, because $\mathcal{C}^\infty_\mathrm{c} \subset \mathcal{F}$ is dense in L^p_r, for any $r \in [a,b]$.

Proof: (This proof is based on that of Riesz–Thorin Interpolation Theorem.)

1° *We can take $\mathcal{F} = \mathrm{L}^p_a \cap \mathrm{L}^p_b$ w.l.o.g.:* By density (see Lemma A.3.10 and Theorem B.3.11), $\mathbb{E}$ has a unique extension $\mathbb{E}_r \in \mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$ with $\|\mathbb{E}_r\| \le M_r$ for $r = a$ and $r = b$. Let $f \in \mathrm{L}^p_a \cap \mathrm{L}^p_b$. Then, by Theorem B.3.11, there are $\{f_n\} \subset \mathcal{F}$ s.t. $f_n \to f$ in both L^p_a and L^p_b. Consequently, $\mathbb{E}_a f = \lim_n \mathbb{E} f_n = \mathbb{E}_b f$ a.e., by Theorem B.3.2. Therefore, we may assume that $\mathcal{F} = \mathrm{L}^p_a \cap \mathrm{L}^p_b$.

2° *f, g, ϕ_z, ψ_z, q':* Let $f \in \mathcal{C}^\infty_\mathrm{c}(J;B)$ be s.t. $\|f\|_{\mathrm{L}^p} = 1$. Let $g \in \mathcal{C}^\infty_\mathrm{c}(J;B^*)$ be s.t. $\|g\|_{\mathrm{L}^{q'}} = 1$, where $1/q + 1/q' = 1$. Set $\phi_z := \mathrm{e}^{z\cdot} f$, $\psi_z := \mathrm{e}^{-z\cdot} g$. If follows that

$$\|\phi_z\|_{\mathrm{L}^p_{\mathrm{Re}z}} = 1 = \|\psi_z\|_{\mathrm{L}^{q'}_{-\mathrm{Re}z}} \quad (z \in \mathbf{C}). \tag{E.16}$$

3° *$\phi, \psi, F \in \mathcal{C} \cap \mathrm{H}$:* Set $\widetilde{J} := (\mathrm{supp}(f) \cup \mathrm{supp}(g))^o$. Note that the spaces $\mathrm{L}^p_t(\widetilde{J};B)$ ($t \in \mathbf{R}$) are equal with equivalent norms, hence $\pi_{\widetilde{J}}\mathbb{E}\pi_{\widetilde{J}} \in \mathcal{B}(\mathrm{L}^p, \mathrm{L}^q)$.

Using the Mean Value Theorem, one easily verifies that $(z \mapsto e^{z\cdot}) \in H(\mathbf{C}; L^v(\widetilde{J}))$ for $v = \infty$, hence for any $v \in [1,\infty]$ (because $L^\infty(\widetilde{J}) \subset L^p(\widetilde{J})$, continuously).

Consequently, $\phi \in H(\mathbf{C}; L^p(\widetilde{J}))$ and $\psi \in H(\mathbf{C}; L^{q'}(\widetilde{J}))$. Therefore,

$$F(z) := F_{\phi,\psi}(z) := \int_J \psi_z(t)(\mathbb{E}\phi_z)(t)\,dt \tag{E.17}$$

satisfies $F \in H(\mathbf{C})$, by Lemma D.1.2(b3).

4° $|F(r)| \le M_r'$: We have

$$|F(r+it)| \le \|\psi_{r+it}\|_{L^{q'}_{-r}} \|\mathbb{E}\phi_{r+it}\|_{L^q_r} \le 1 \cdot M_r \quad (r \in [a,b],\ t \in \mathbf{R}), \tag{E.18}$$

hence $|F(r+it)| \le M_r' := M_a^{1-\theta_r} M_b^{\theta_r}$ $(r \in [a,b],\ t \in \mathbf{R})$, by Lemma D.1.5.

5° *Extending* $\mathbb{E}$: For all $r, r' \in [a,b]$, the following facts hold: By Theorem B.4.12,

$$\|\mathbb{E}\phi_r\|_{L^q_r} := \|e^{-r\cdot}\mathbb{E}\phi_r\|_q = \sup_{\widetilde{g} \in C_c^\infty,\ \|\widetilde{g}\|_{q'}=1} \int \widetilde{g} e^{-r\cdot}\mathbb{E}\phi_r, \tag{E.19}$$

i.e., $\|\mathbb{E}\phi_r\|_{L^q_r}$ is the supremum of functions $|F_{\phi,\psi}(r)|$, where ψ is as above. Therefore, $\|\mathbb{E}\phi_r\|_{L^q_r} \le M_r'$.

But each $C_c^\infty(J;B)$ function is a scalar multiple of some ϕ_r of the above form, hence $\mathbb{E}$ extends to an operator $\mathbb{E}_r \in \mathcal{B}(L^p_r, L^q_r)$ with norm $\|\mathbb{E}_r\| \le M_r'$. As in 1°, we see that $\mathbb{E}_r = \mathbb{E}_{r'}$ on $L^p_r \cap L^p_{r'}$. □

Also a converse holds: if $\mathbb{E} \in \mathcal{B}(L^p_r, L^q_r)$ for all $r \in (a,b)$ with $\|\mathbb{E}\|$ having an uniform upper bound, then $\mathbb{E} \in \mathcal{B}(L^p_r, L^q_r)$ for all $r \in [a,b]$:

Lemma E.1.9 *Let $p,q,a,b,J,\mathcal{F},M_r$ be as in Proposition E.1.8 (we may also allow $\mathcal{F} = L^p_r \cup L^p_s$ for some $r,s \in [a,b]$). Let $\mathbb{E} : \mathcal{F} \to L(J;B)$ be linear, and let $D \subset [a,b]$ be dense.*

Then $M := \sup_{r\in D} M_r = \sup_{r\in[a,b]} M_r = \max\{M_a, M_b\}$; in particular, $\mathbb{E}$ has a (unique) continuous extension to $\mathcal{B}(L^p_r, L^q_r)$ for each $r \in [a,b]$ iff $M < \infty$.

Proof: (In fact, D need not be dense; it suffices that a and b are in the closure of D.)

Part "only if" and the moreover-claim follow from Proposition E.1.8. Therefore, we assume that $\mathbb{E} \in \mathcal{B}(L^p_r, L^q_r)$ for all $r \in D$, hence for all $r \in (a,b)$ (by the proposition, we can use continuous extensions for r's between elements of D).

Let ϕ be a simple measurable function with $\|\phi\|_{L^p_a} = 1$. Then $\|\phi\|_{L^p_r} \to 1$ as $r \to a+$, by Lemma D.1.10(a3), and

$$M\|\phi\|_{L^p_a} \leftarrow M\|\phi\|_{L^p_r} \ge \|\mathbb{E}\phi\|_{L^q_r} \to \|\mathbb{E}\phi\|_{L^q_a}. \tag{E.20}$$

Because ϕ was arbitrary, we have $\mathbb{E} \in \mathcal{B}(L^p_a, L^q_a)$ with norm $\le M$. The same holds with b in place of a. □

Notes

It seems that Proposition E.1.8 and Lemma E.1.9 cannot be obtained as corollaries to standard interpolation theorems (see, e.g., [BL]) unless we introduce additional constants.

The Riesz–Thorin and Hausdorff–Young theorems and their proofs are from [BL], except that we have made the concepts and the proofs more rigorous and more general. See [BL] for an extensive interpolation theory, further references and historical remarks.