UNIFORMLY QUASIREGULAR MAPS WITH TOROIDAL JULIA SETS

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ABSTRACT. The iterates of a uniformly quasiregular map acting on a Riemannian manifold are quasiregular with a uniformly bound on the dilatation. There is a Fatou-Julia type theory associated with the dynamical system obtained by iterating these mappings. We construct the first examples of uniformly quasiregular mappings that have a 2-torus as the Julia set. The spaces supporting this type of mappings include Hopf link complement and its lens space quotients.

1. INTRODUCTION

We construct uniformly quasiregular mappings on Riemannian manifolds that have a 2-torus as the Julia set. The spaces supporting these mappings include the Hopf link complement in $\mathbb{S}^3$ and its lens space quotients when equipped with a Semmes-type metric.

Semmes' idea of creating new metrics on subsets of Euclidean spaces relating the geometry of the metrics to the topological characteristics of the sets was introduced to provide counterexamples to some natural conjectures on the bilipschitz and quasisymmetric parametrizations of metric 3-spheres [11], [12]. This approach has been applied for example in [2] to study spaces that are bi-Lipschitz inequivalent to the standard $\mathbb{S}^3$ nevertheless admit maps of bounded length distortion onto $\mathbb{S}^3$, in [9] to prove a sharp non-Euclidean Picard-type theorem, and in [3] to construct geometrically nice metric n-spheres in dimension $n \geq 4$, that do not admit quasisymmetric parametrization by $\mathbb{S}^n$.

A continuous mapping $f \in W^{1,n}_{loc}$ between two oriented Riemannian n-manifolds M and N is K-*quasiregular* if it satisfies the distortion inequality

$$|Df|^n \leq KJ_f \ \ \text{a.e.in} \, M,$$

where $|Df|$ is the operator norm of the differential Df and J_f is the Jacobian determinant of Df. A non-injective mapping $f : M \to M$ is called *uniformly quasiregular*

(uqr) if there exists a constant $1 \leq K \leq \infty$ such that all the iterates f^k are K-quasiregular with distortion independent of the number of iterates.

The *Fatou set* $\mathcal{F}_f$ of the uniformly quasiregular mapping $f : M \to M$ is the set where the family of the iterates $\{f^k \mid k = 1, 2, \ldots\}$ of is normal. That is,

$$\mathcal{F}_f = \{x \in M : \quad \text{there exists an open set } U \subset M$$
$$\text{such that } x \in U \text{ and } \{f^k|_U\} \text{ is normal}\}.$$

The *Julia set* $\mathcal{J}_f$ of the uniformly quasiregular mapping f is the complement of its Fatou set. We denote by B_f the *branch set* of a quasiregular mapping f. It consists of those points in the domain of f where the mapping is not locally injective.

A theorem stated in [4, Theorem 24.4.1] summarizes the procedure in constructing a certain type of uqr maps. The proof of Iwaniec and Martin in [4] is written for the case $M = \bar{\mathbb{R}}^n$, but it holds more generally on a Riemannian manifold M without changes.

Theorem 1.1. *Let Γ be a discrete group such that $h : \mathbb{R}^n \to M$ is automorphic with respect to Γ in the strong sense. If there is a similarity $A = \lambda \mathcal{O}$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $\mathcal{O}$ an orthogonal transformation, such that*

$$A\Gamma A^{-1} \subset \Gamma,$$

then there is a unique solution $f : h(\mathbb{R}^n) \to h(\mathbb{R}^n)$ to the Schröder functional equation

$$(1.1) \qquad\qquad\qquad\qquad f \circ h = h \circ A,$$

and f is a uniformly quasiregular mapping, if h is quasiregular.

Note that following from (1.1) we have the equation $f^k \circ h = h \circ A^k$ for all k. Thus the dilatation of the uqr mapping f^k is at most the dilatation of h^2 for all $k = 1, 2 \ldots$.

We call the mappings arising from Theorem 1.1 *uniformly quasiregular mappings of Lattès type*. This type of mappings acting on the n-sphere have been studied by Volker Mayer in [7] and [8]. In fact he showed that in addition to the so-called chaotic rational maps [5] there exists analogues of planar power mappings $z \mapsto z^d$ as well as Tchebychev polynomials such as $z \mapsto z^2 - 2$. In this respect the theory of uqr maps in higher dimensional spheres is equally rich as in two dimensions. The uniformly quasiregular counterparts for power mappings have a codimension 1 sphere as a Julia set with origin and infinity as super-attracting and completely invariant fixed points. The uqr counterpart of Tchebychev polynomials have a codimension 1 closed unit disk as a Julia set and completely invariant fixed point infinity. In [1] a variety of uqr maps of Lattès type are constructed acting on closed Riemannian manifolds that include spaces that can be products of spheres, tori and odd dimensional real projective spaces. Typically these maps are chaotic, but in some cases the maps produced there have interesting Julia sets. For example, every odd dimensional projective space supports a uqr map which has a codimension one (non-orientable!) real projective

space as a Julia set. This is possible in the presence of exactly one superattractive fixed point and basin of attraction.

We do not know whether lens spaces $L(p, q)$, where $p > q > 0$ are coprime integers and $p \geq 3$, support Lattès type mappings. G. Martin and the second author has shown that the 2-torus cannot appear as a Julia set either on the standard $\mathbb{S}^3$ or on a lens space $L(p, q)$ when equipped with the quotient metric [6]. The book of Rolfsen [10] contains illustrated descriptions of these spaces.

2. UQR MAPPING WITH TOROIDAL JULIA SET

First we formulate the main theorem of this section.

Theorem 2.1. *Let H be a Hopf link. There exists a Riemannian metric g_H on $\mathbb{S}^3 \backslash H$ and a uniformly quasiregular map $f : (\mathbb{S}^3 \backslash H, g_H) \to (\mathbb{S}^3 \backslash H, g_H)$ such that the branch set $B_f = \emptyset$ and the Julia set of f is a 2-torus.*

Suppose H is a Hopf link on $\mathbb{S}^3$. P. Pankka and K. Rajala proved in [9] that the manifold $\mathbb{S}^3 \setminus H$ may be equipped with a Riemannian metric g_H so that it is quasiregularly elliptic. That is, there exists a quasiregular mapping $h : \mathbb{R}^3 \to (\mathbb{S}^3 \setminus H, g_H)$. In fact it is even possible to choose this metric in such a way that mapping h is conformal.

With the help of this mapping h and the metric g_H, we define a uniformly quasiregular map f of Lattès type on $(\mathbb{S}^3 \setminus H, g_H)$, for an arbitrary dilation $A_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, t) \mapsto (\lambda x, \lambda y, \lambda t)$, and an integer $\lambda \geq 2$.

$$(2.1) \quad \begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{A_\lambda} & \mathbb{R}^3 \\ h \downarrow & & \downarrow h \\ \mathbb{S}^3 \setminus H & \xrightarrow{f} & \mathbb{S}^3 \setminus H \end{array}$$

Consider $\mathbb{S}^3 = \{(z, w) | \ |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$ and the Hopf link $H = S_0 \cup S_1$ with $S_0 = \{(0, e^{2\pi i y} | \ y \in (0, 1]\}$ and $S_1 = \{(e^{2\pi i x}, 0) | \ x \in (0, 1]\}$. Define mapping $h : \mathbb{R}^3 \to \mathbb{S}^3 \backslash H$ by setting

$$h(x, y, t) = (\alpha(t) e^{2\pi i x}, \beta(t) e^{2\pi i y}),$$

where $\alpha, \ \beta : \mathbb{R} \to (0, 1)$ are C^∞ smooth diffeomorphisms defined by conditions

$$\alpha(t) = (\frac{1}{\pi} \arctan t + \frac{1}{2})^{\frac{1}{2}}, \ \beta(t) = (1 - \alpha(t)^2)^{\frac{1}{2}}.$$

On $\mathbb{R}^3$ we consider the Euclidean metric and on $\mathbb{S}^3 \backslash H$ we consider the push-forward metric induced by a C^∞ smooth diffeomorphism $\Phi : \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^3 \backslash H$ while $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ is equipped with the standard product metric. In the above, the unit

circles $\mathbb{S}^1 \subset \mathbb{C}$ are equipped with the ambient Euclidean metric in the plane and $\mathbb{R}$ is the standard real line, and the mapping Φ is defined by the condition

$$(\mathrm{e}^{2\pi\mathrm{i}x}, \mathrm{e}^{2\pi\mathrm{i}y}, t) \mapsto (\alpha(t)\mathrm{e}^{2\pi\mathrm{i}x}, \beta(t)\mathrm{e}^{2\pi\mathrm{i}y})$$

with functions α and β defined earlier. If g is the product metric in $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$, the smooth metric g_H on $\mathbb{S}^3 \setminus H$ that makes mapping Φ an isometry is defined by the condition

$$(g_H)_p(v, w) = g\left(((D\Phi)_{\Phi^{-1}p})^{-1}v, ((D\Phi)_{\Phi^{-1}p})^{-1}w\right),$$

for $p \in \mathbb{S}^3 \setminus H$ and $v, w \in T_p(\mathbb{S}^3 \setminus H)$.

Under the above choice of the metric, the mapping $h : \mathbb{R}^3 \to \mathbb{S}^3 \setminus H$ is a conformal covering map. Especially the branch set of h is empty. Each square

$$\{(x, y, a) \mid x \in (m, m+1], \ y \in (n, n+1]\},$$

$m, n \in \mathbb{Z}$ and $a \in \mathbb{R}$, is mapped bijectively onto a 2-torus

$$(2.2) \qquad M_a = \left\{ \left(\alpha(a)\mathrm{e}^{2\pi\mathrm{i}x}, \beta(a)\mathrm{e}^{2\pi\mathrm{i}y}\right) \mid x, y \in (0, 1] \right\}$$

under h. The restriction $h | \mathbb{R}^2 \times \{a\} : \mathbb{R}^2 \times \{a\} \to M_a$ is a conformal covering map of the 2-torus. Furthermore, every semi-infinite cylinder

$$\{(x, y, t) \mid x \in (m, m+1], \ y \in (n, n+1], \ t < a\}$$

is mapped bijectively onto the complement of the core S_0 in the solid torus

$$\left\{ \left(\alpha(t)\mathrm{e}^{2\pi\mathrm{i}x}, \beta(t)\mathrm{e}^{2\pi\mathrm{i}y}\right) \mid x, y \in (0, 1], \ t < a \right\};$$

and every semi-infinite cylinder

$$\{(x, y, t) \mid x \in (m, m+1], \ y \in (n, n+1], \ t > a\}$$

is mapped bijectively onto the complement of the core S_1 in the solid torus

$$\left\{ \left(\alpha(t)\mathrm{e}^{2\pi\mathrm{i}x}, \beta(t)\mathrm{e}^{2\pi\mathrm{i}y}\right) \mid x, y \in (0, 1], \ t > a \right\}.$$

The image of xy-plane under h is the 2-torus

$$M_0 = \left\{ \left(\frac{\sqrt{2}}{2}\mathrm{e}^{2\pi\mathrm{i}x}, \frac{\sqrt{2}}{2}\mathrm{e}^{2\pi\mathrm{i}y} \right) \mid x, y \in (0, 1] \right\}$$

which divides $\mathbb{S}^3 \setminus H$ into two parts $T_0 \setminus S_0$ and $T_1 \setminus S_1$, where T_0 and T_1 are two solid tori in $\mathbb{S}^3$ with a common boundary M_0.

To see that the mapping f is well-defined for any dilation $A_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, t) \mapsto (\lambda x, \lambda y, \lambda t)$ $x \mapsto \lambda x$, and an integer $\lambda \geq 2$, we fix a point

$$(z_0, w_0) = \left(\alpha(t_0)\mathrm{e}^{2\pi\mathrm{i}x_0}, \beta(t_0)\mathrm{e}^{2\pi\mathrm{i}y_0}\right) \in \mathbb{S}^3 \setminus H$$

with parameters $x_0, y_0 \in (0, 1]$ and $t_0 \in \mathbb{R}$. The set

$$h^{-1}(z_0, w_0) = \{(x_0 + m, y_0 + n, t_0) \mid m, n \in \mathbb{Z}\}$$

of preimages of (z_0, w_0) under h consists of infinitely many points in the plane $\mathbb{R}^2 \times \{0\}$, one in each square defined by the integer lattice. Hence the image of this set under $h \circ A_\lambda$ is a single point

$$\left(\alpha(\lambda t_0) e^{2\pi i \lambda x_0}, \beta(\lambda t_0) e^{2\pi i \lambda y_0} \right) \in \mathbb{S}^3 \backslash H$$

and the mapping f is well defined. Since $f^k \circ h = h \circ A_\lambda^{-k}$ for every integer $k \geq 1$, the induced map f is uniformly quasiregular on $\mathbb{S}^3 \backslash H$. Since the mapping h is conformal, the induced mapping f is also conformal with respect to the metric g_H.

The fact that the Julia set of the mapping f is the torus M_0 can be seen as follows. The origin is a repelling fixed point for the mapping $A_\lambda : (x, y, t) \mapsto (\lambda x, \lambda y, \lambda t)$. Consider its Γ-orbit under the group of isometries generated by translations $e_1 :$ $(x, y, t) \mapsto (x + 1, y, t)$ and $e_2 : (x, y, t) \mapsto (x, y + 1, t)$ in $\mathbb{R}^3$: $\Gamma(0) = \{\gamma(0) \mid \gamma \in \Gamma\} = \lambda \mathbb{Z}^2 \times \{0\}$. The set $E = \cup_{k \geq 0} A_\lambda^{-k}(\Gamma(0))$ is a dense subset of $\mathbb{R}^2 \times \{0\}$. Hence, $h(E)$ is a dense subset of $M_0 \subset \mathbb{S}^3 \backslash H$. We conclude that (f^k) cannot be equicontinuous in a neighbourhood of any point of M_0. On the other hand points $A_\lambda^k(x, y, t) \in \mathbb{R}^3$, $t \neq 0$ tend to infinity as $k \to \infty$. The corresponding points $f^k(h(x, y, t)) = \left(\alpha(\lambda^k t) e^{2\pi i \lambda^k x}, \beta(\lambda^k t) e^{2\pi i \lambda^k y} \right)$ tend to the omitted circle S_1 when $t > 0$, and to the other circle S_0 when $t < 0$. Hence the Julia set of f is the torus M_0 and the completely invariant Fatou set components of f are the interiors of sets $T_0 \backslash S_0$ and $T_1 \backslash S_1$. The degree of the mapping f is λ^2, and there is no branching.

Remark 2.2. *The mapping f extends to a mapping f^* acting on $\mathbb{S}^3$, when $\mathbb{S}^3 \backslash H$ is compactified by gluing the Hopf link H back and we define $f_{|H}^* = \mathrm{Id}_H$. This mapping f^* is a topological conjugate of a winding map acting on $\mathbb{S}^3$, whose branch set is the Hopf link H. These winding maps acting on solid tori are of form*

$$(r, \varphi, \theta) \mapsto (r, k_1 \varphi, k_2 \theta),$$

where integers $k_i \geq 1$, $i = 1, 2$ depend on λ and coordinate r measures distance from the core and angles φ, θ deviation along the core and meridian of the torus. These mappings are quasiregular (even of bounded length distortion) but not uniformly quasiregular.

3. UQR MAPPINGS ON LENS SPACES

Let $L(p, q)$ be a lens space of type (p, q), where $p > q > 0$ are two coprime integers $(p|q) = 1$. Denote by $\pi : \mathbb{S}^3 \to L(p, q)$ the standard p to 1 covering projection. Let $H = S_0 \cup S_1$ be the Hopf link and g_H be the metric on $\mathbb{S}^3 \setminus H$ previously defined. Metric $\tilde{g}_H$ on $L(p, q) \backslash \pi(H)$ is the one that makes the covering projection π a local isometry.

The main theorem of this section is the following.

Theorem 3.1. *There exists a uniformly quasiregular map* $f_L : (L(p,q)\backslash\pi(H), \tilde{g}_H) \to (L(p,q)\backslash\pi(H), \tilde{g}_H)$ *such that the branch set* $B_{f_L} = \emptyset$ *and the Julia set of* f_L *is a 2-torus.*

To describe the lens spaces $L(p,q)$, we again consider its covering space $\mathbb{S}^3$ as the unit sphere $\{(z,w)| \ |z|^2 + |w|^2 = 1\}$ in complex 2-space $\mathbb{C}^2$. Denote by $\tau : \mathbb{S}^3 \to \mathbb{S}^3$ the p-periodic homeomorphism $\tau(z,w) = (ze^{2\pi i/p}, we^{2\pi iq/p})$. The orbit space of this action is the lens space $L(p,q)$. Here the points u and v on $\mathbb{S}^3$ are identified if and only if $u = \tau^k(v)$ for some integer k. Then $\mathbb{S}^3$ is the universal covering space for $L(p,q)$ and τ is a generator of the cyclic group of covering translatios. We denote the p to 1 covering map by π. The mapping τ keeps circles S_0 and S_1 invariant. The p to 1 image of these circles under π consists of two circles: $\pi(S_0)$ is the edge of the lens and $\pi(S_1)$ is a core of the solid torus $\pi(T_1)$. Furthermore the image $\pi(M_a)$ of any 2-torus M_a in (2.2) is again a 2-torus. This follows, since the translation τ preserves the tori

$$\tau(M_a) = \left\{ \left(\alpha(a)e^{2\pi i(x+\frac{1}{p})}, \beta(a)e^{2\pi i(y+\frac{q}{p})} \right) \mid x, y \in (0,1] \right\}$$

that are subdivided into p tori accordingly under the covering projection.

We show that for any integer $\lambda \geq 2$ the dilation A_λ descends to a well defined mapping f_L on $L(p,q) \setminus \pi(H)$. Let f and h be the mappings defined in the previous section. Then the following diagram commutes and by Theorem 1.1 f_L is uniformly quasiregular.

$$
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{\ A_\lambda\ } & \mathbb{R}^3 \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\
\mathbb{S}^3 \setminus H & \xrightarrow{\ f\ } & \mathbb{S}^3 \setminus H \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
L(p,q) \setminus \pi(H) & \xrightarrow{\ f_L\ } & L(p,q) \setminus \pi(H)
\end{array}
$$

(3.1)

Consider a point $[(z_0, w_0)] \in L(p,q)\backslash\pi(H)$ whose representative in $\mathbb{S}^3$ is given by $(z_0, w_0) = (\alpha e^{2\pi i x_0}, \beta e^{2\pi i y_0})$ for some $\alpha \in (0,1)$, $\beta = (1-\alpha^2)^{1/2}$, $x_0, y_0 \in (0,1]$. It has p preimage points in $\mathbb{S}^3\backslash H$ under the covering map π:

$$\pi^{-1}([z_0, w_0]) = \left\{ \left(\alpha e^{2\pi i\left(x_0+\frac{k}{p}\right)}, \beta e^{2\pi i\left(y_0+\frac{qk}{p}\right)} \right) \mid k = 0, 1, \ldots, p-1 \right\}.$$

The preimage of this set under h is contained in the plane $\{(x,y,t_\alpha)| \ x, y \in \mathbb{R}\}$, where $t_\alpha = \tan\left(\pi(\alpha^2 - \frac{1}{2})\right)$. That is

$$h^{-1}\left(\pi^{-1}([z_0, w_0])\right) =$$

$$\left\{ (x_0 + \frac{k}{p} + m, y_0 + \frac{kq}{p} + n, t_\alpha) | \ k = 0, 1, \cdots, p-1, \ m, n \in \mathbb{Z} \right\}.$$

Under $h \circ A_\lambda$ the above points are further mapped to points

$$(3.2) \qquad h\left(A_\lambda\left(h^{-1}\left(\pi^{-1}([z_0, w_0])\right)\right)\right) =$$

$$\left\{ \left(\alpha\left(\lambda t_\alpha\right) \mathrm{e}^{2\pi\mathrm{i}\lambda\left(x_0 + \frac{k}{p}\right)}, \beta\left(\lambda t_\alpha\right) \mathrm{e}^{2\pi\mathrm{i}\lambda\left(y_0 + \frac{qk}{p}\right)} \right) \mid k = 0, 1, \ldots, p-1 \right\}$$

on $\mathbb{S}^3 \backslash H$. Set (3.2) constitutes at most p points. There is only one point if λ is a multiple of p. However, the projection π maps all these points into a single point in $L(p, q)\backslash\pi(H)$, for any integer $\lambda \geq 2$. This follows, because for any $k = 1, \ldots, p-1$, there exists $j_k = \lambda k \in \mathbb{Z}$

$$\lambda\left(x_0 + \frac{k}{p}\right) = \lambda x_0 + \frac{j_k}{p} \quad \text{and} \quad \lambda\left(y_0 + \frac{qk}{p}\right) = \lambda y_0 + \frac{j_k q}{p}.$$

Hence there is no restriction on multiplier λ for different values of q and p.

Since $f_L^k \circ \pi \circ h = \pi \circ h \circ A_\lambda^k$ holds for every $k \geq 1$ the mapping f_L is uniformly quasiregular. By an argument similar to that for the case $\mathbb{S}^3 \setminus H$, the Julia set of f_L is the 2-torus $\pi(M_0)$ that divides the lens space into two parts. On one side of $\pi(M_0)$ all points tend to the circle $\pi(S_0)$ under the iterates, while on the other side all points tend to the circle $\pi(S_1)$.

Remark 3.2. *Note that in case $\lambda = p$, the mapping f_L lifts to a conformal p to 1 covering map $\tilde{f}_L : L(p, q)\backslash\pi(H) \to S^3\backslash H$. Namely, for fixed parameters $\alpha \in (0, 1)$, $\beta = (1 - \alpha^2)^{1/2}$, and $x_0, y_0 \in (0, 1]$, these p^2 different points*

$$\left\{ \left(\alpha \mathrm{e}^{2\pi\mathrm{i}\left(x_0 + \frac{k}{p}\right)}, \beta \mathrm{e}^{2\pi\mathrm{i}\left(y_0 + \frac{jq}{p}\right)} \right) \mid k = 0, \ldots p-1, j = 0, \ldots p-1 \right\}$$

in $\mathbb{S}^3\backslash H$ descent to p different points

$$\left\{ \left[\left(\alpha \mathrm{e}^{2\pi\mathrm{i}\left(x_0 + \frac{k}{p}\right)}, \beta \mathrm{e}^{2\pi\mathrm{i}y_0} \right) \right] \mid k = 0, \ldots p-1 \right\}$$

in $L(p, q)\backslash\pi(H)$.

By tracking the preimages of these points under π and h as above, one gets the following points in $\mathbb{R}^3$:

$$\left\{ \left(x_0 + \frac{k}{p} + m, y_0 + \frac{jq}{p} + n, t_\alpha \right) \mid k, j = 0, \ldots, p-1, \; m, n \in \mathbb{Z} \right\},$$

where $t_\alpha = \tan\left(\pi(\alpha^2 - \frac{1}{2})\right)$. These points are further mapped under $h \circ A_p$ to a single point

$$\left(\alpha(pt_\alpha)\mathrm{e}^{2\pi\mathrm{i}px_0}, \beta(pt_\alpha)\mathrm{e}^{2\pi\mathrm{i}py_0} \right) \in \mathbb{S}^3\backslash H.$$

The covering map $\tilde{f}_L$ can be extended to act on the compactified spaces $L(p, q)$ and $\mathbb{S}^3$. The extension $\tilde{f}_L^* : L(p, q) \to \mathbb{S}^3$ is a topological conjugate of a map of bounded length distortion branching along the set $\pi(H)$, but $\tilde{f}_L^*$ is not uniformly quasiregular.

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