Dynamics of uniformly quasiregular mappings on compact manifolds

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Background

Uniformly quasiregular maps (UQR) acting on compact manifolds are higher dimensional counterparts of rational maps of one complex variable acting on the riemann sphere. First examples on higher dimensional spheres were discovered by T. Iwaniec and G. Martin in 1996 [IM1]. V. Mayer [M] found a special class of UQR maps that generalize all the classical Lattès mappings. Later it turned out [P] that many other compact riemannian manifolds support these mappings as well. Local dynamics of these mappings was developed in [HMM]. Recent research [BFP] shows that these mappings do exists also in more general sub-riemannian setting.

Governing PDEs

Analytic mappings acting on complex plane have a natural generalization called quasiregular maps (QR) to higher dimensional euclidean space:

Definition.
Let $\Omega \subset \overline{\mathbb{R}}^n$ be a domain and $f : \Omega \to \overline{\mathbb{R}}^n$ a non-constant mapping of the Sobolev class $W^{1,n}_{loc}(\Omega)$. We consider only orientation-preserving mappings, which means that the Jacobian determinant $J(x, f) \geq 0$ for a.e. $x \in \Omega$. Such a mapping is said to be *K-quasiregular*, where $1 \leq K < \infty$, if

$$\max_{|h|=1} |Df(x)h| \leq K \min_{|h|=1} |Df(x)h|$$

for a.e. $x \in \Omega$, when $Df(x)$ is the formal matrix of weak derivatives.

The theory of such mappings is deeply studied since 60's [R1]. This local definition makes sense also for maps $f : M \to N$ between arbitrary riemannian manifolds M and N of same dimension. A general classification problem on the existence of nontrivial QR maps between given riemannian manifolds is highly open. A beautiful interplay between analysis, geometry and cohomology of the manifold N was found by M. Bonk and J. Heinonen [BH] in the case $M = \mathbb{R}^n$ (QR-ellipticity).

In the UQR setting we study noninjective maps $f : M \to M$ whose all iterates f^n satisfy the above QR condition for fixed $K \geq 1$ independently of the number of iterates. Such maps distort the given metric by a bounded amount. They are always conformal with respect to some measurable conformal structure $x \mapsto G(x)$ that can be constructed from the semigroup $\Gamma = \{f^n | n \in \mathbb{N}\}$. The space of $n \times n$ positive definite real matrices G of determinant 1 can be equipped with a natural hyperbolic metric whose distance function relates the invariant conformal structure G to the distortion constant K of the semigroup elements. Conversely, for a given conformal structure G the weak solutions f of a generalized Beltrami equation

Generalized Beltrami Equation

$$D^t f(x) G(f(x)) Df(x) = J(x, f)^{2/n} G(x)$$

are UQR maps whose iterates satisfy the same equation. Monograph [IM2] deals with analysis to this direction.

The first Heisenberg group $\mathcal{H}^1$ is a basic example of a manifold equipped with a sub-riemannian structure. If (x, y, t) are the standard euclidean coordinates of $\mathbb{R}^3$, then the left invariant vector fields $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$ span the horizontal distribution. Together with vector field $T = \frac{\partial}{\partial t} = -\frac{1}{4}[X, Y]$ they constitute a basis of the Lie algebra of the Heisenberg group. The complex vector fields

$$Z = \frac{1}{2}(X - \mathrm{i}Y) = \frac{\partial}{\partial z} + \mathrm{i}\bar{z}\frac{\partial}{\partial t},$$

$$\bar{Z} = \frac{1}{2}(X + \mathrm{i}Y) = \frac{\partial}{\partial \bar{z}} - \mathrm{i}z\frac{\partial}{\partial t}$$

and T span the complexified tangent bundle. In the compactified first Heisenberg group $\hat{\mathcal{H}}^1$ the governing generalized horizontal Beltrami equation takes the complex form [BFP]

Horizontal Complex Beltrami Equation

$$(\bar{Z} f_I)(p) = \mu(p, f(p))(Z f_I)(p) - \mu(f(p), p)(\overline{Z f_I})(p),$$

where $f_I = f_1 + \mathrm{i} f_2$ is the horizontal part of any contact mapping $f = (f_1, f_2, f_3) \in \Gamma$ and $(p, q) \mapsto \mu(p, q)$ is a Haar measurable horizontal complex structure μ taking values in the hyperbolic unit disk $\mathbb{D}$. The structure μ and a real 2×2 positive definite matrix

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{12} \end{pmatrix}$$

are related to each other via an isometric bijection that gives correspondence

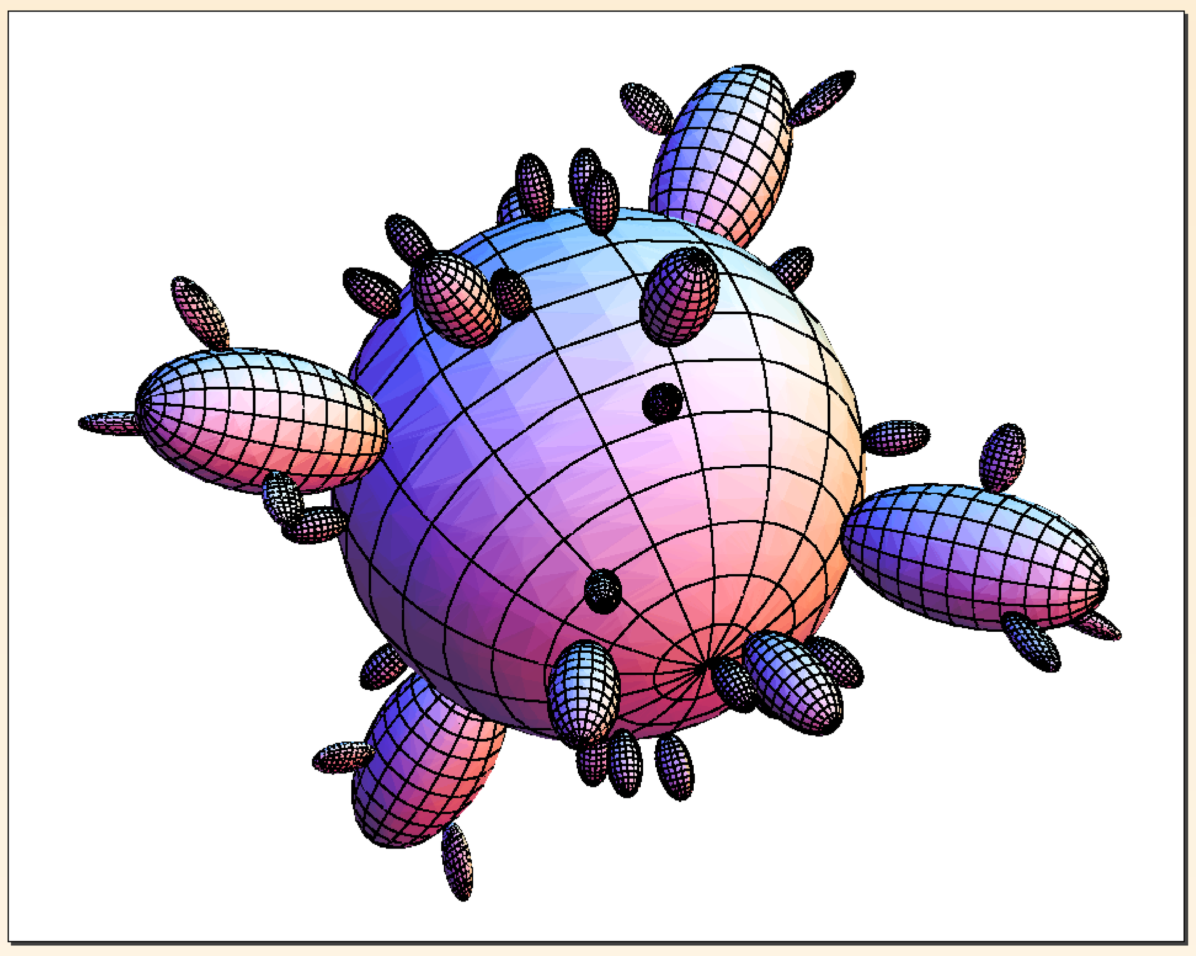
$$\mu(p, q) = \frac{g_{11}(p) - g_{22}(p) + 2\mathrm{i} g_{12}(p)}{g_{11}(p) + g_{22}(p) + g_{11}(q) + g_{22}(q)}$$

and ellipticity condition

$$|\mu(p, q)| + |\mu(q, p)| < 1.$$

Basic questions

1. Classification of compact (sub)riemannian manifolds supporting nontrivial UQR maps. A characterization of UQR maps in dimension 3 was made in [K]: A compact riemannian manifold M supports a UQR map if and only if M is QR-elliptic. In higher dimensions QR-ellipticity is a necessary condition for the existence of UQR maps. No example of a QR-elliptic manifold not supporting UQR maps is known. A simply connected 4-manifold $S^2 \times S^2 \# S^2 \times S^2$ is QR-elliptic [R2] but it is not known whether it has nontrivial UQR maps.
2. Characterize the different types of UQR maps acting on a given manifold. In addition to Lattès type examples there are constructions based on building conformal traps. The sphere and projective space are so far the only known compact manifolds that support both processes.
3. What properties of rational maps generalize to higher dimensions? In [MP1] we show that polynomials $z \mapsto z^2 + C$ can be generalized to higher dimensional spheres as well providing fractal Julia sets.



First two generations of the Julia set of the UQR counterpart of $z \mapsto z^2 - 1$ acting on three dimensional sphere (picture by Antti Rasila)

4. Classify sets that can appear as a Julia set for a UQR mapping acting on a given manifold.
5. To what extent do UQR maps take the role of analytic functions in the complex plane? In [MP2] we show that any QR map f acting on higher dimensional sphere is of form $f = g \circ h$, where g is UQR and h is quasiconformal (= QR homeomorphism). In the complex plane any QR mapping f is a composition of an analytic function g and quasiconformal mapping h.
6. Find obstructions in terms of different types of curvatures. In [MMP] we characterize the situation for constant sectional curvature spaces.

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