Notes on glottal flow and acoustic inertial effects

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Abstract

This text is a compilation of some of the notes that the author has written during the development of the low-order model "DICO" [2, 8, 10, 11] for vowel phonation and the even more rudimentary glottal flow model [9] for processing high-speed glottal video data.

The following subject matters are covered: (i) Incompressible, laminar, lossless flow models for idealised rectangular and wedge shape vocal fold geometries. Equations of motion and the pressure distribution are computed in a closed form for each model using the unsteady Bernoulli's theorem; (ii) The assumption of incompressibility and energy loss (i.e., irrecoverable pressure drop) of the airflow in airways (including the glottis) is discussed using steady compressible Bernoulli theorem as the main tool; (iii) Inertia of an uniform waveguide is studied in terms of the low-frequency limit of the the (acoustic) impedance transfer function. It is observed that the inductive loading in the boundary condition sums up with the waveguide inertance in an expected way; (iv) It is shown that an acoustic waveguide, modelled by Webster's lossless equation with Dirichlet boundary condition at the far end, will produce the expected mass inertance of the fluid column as the low-frequency limit of the impedance transfer function.

1 Introduction

This text is a cleaned-up compilation of notes that have been written during the development of the low-order model "DICO" [2, 8, 10, 11] for vowel phonation and the even more rudimentary glottal flow model [9] for processing high-speed glottal video data. While it is not possible to include all details and derivations in journal articles, building even a modest model of phonation will require a great number of mathematical and physical considerations, idealisations, and approximations. There exists a number of fora (such as arXiv.org) where complementary material can be presented almost without any limitations, making excuses of obscurity somewhat moot nowadays. Unfortunately, much of the background material of the above mentioned publications is not included here due to shortness of time and other resources of the author. So much for the justification of this text; let's move on.[1]

Briefly, the subject of these notes is an air column inside a perfectly smooth, acoustically reflecting tubular boundary. The air column is both translating (when it is considered incompressible) as well as it is an acoustic medium (where compressibility is a prerequisite for the finite speed of sound). The tube, i.e., the flow channel consists of three parts having finite lengths: the *subglottal tract* (SGT), the *glottis*, and the *vocal tract* (VT). The much shorter glottis is positioned between SGT and VT, and it is the only part of the tube boundary that is assumed to be time-dependent. The walls of the tube at the glottis are called *vocal folds*. As is sometimes required, the SGT may be considered as having been extended from its free end by a *piston* made of incompressible material (even, perhaps, fluid) that has a much higher density than air. The mass and the dimensions of the piston add to the total (flow-mechanical part of the) mass inertia if they are taken into consideration. The acoustics is considered only in the VT part using Webster's model, hence restricting it to an immobile boundary.

An outline of these notes is as follows: In Sections 2 and 3, incompressible, laminar, lossless flow models are developed for two kinds of idealised glottis geometries: *rectangular* and *wedge* shape vocal folds. Equations of motion and the pressure distribution are computed in a closed form for each model using the unsteady Bernoulli's theorem. This is motivated by, and only feasible due to the extremely simplified nature of the glottis geometry. From these equations, the coefficient of *inertance* shows up for each of the three parts of the flow channel, and their sum – the total inertance – regulates the mass inertial effects in the fluid movement.

In Section 4, the assumption of incompressibility is examined from the point of view of the flow. Constriction areas and thermodynamic state are computed for glottal openings where the compressible *steady* flow would reach Mach 0.3 (often considered as the upper limit for air flow to be treatable as incompressible) and Mach 1.0. In Section 5, the VT inertance discovered in the flow models of Sections 2 and 3 is associated to acoustics using the uniform diameter acoustic waveguide as a model. It is, in particular, observed that the inductive acoustic termination at the waveguide end (mouth) will add to the inertance of the acoustic system. That the same holds for general acoustic waveguides with nonconstant intersectional areas is indicated in Section 6.

The lossless, incompressible models introduced in Sections 2 and 3 are not as such suitable for simulation modelling of vowels even when combined with VT and SGT acoustics models. The two main reasons are the following:

(i) For performance reasons, it is desirable to have some *pruning of terms* in the equations of motion (see Eqs. (4) and (10)) as well as in the equations of the hydrodynamic pressure components (see Eqs. (6) and (19)) resulting

[1] The author has tried his best not to leave any obvious mistakes (mathematical, or of some other kind) in the material. If an interested reader makes observations, the author is pleased to receive comments by e-mail: jarmo.malinen@aalto.fi.

in the aerodynamic force to the vocal folds (not considered in these notes). Pruning of a term is considered acceptable if its effect can be compensated by tuning of the model parameters in numerical simulations.

(ii) For accuracy reasons, some sort of (unrecoverable) pressure loss terms must be added. Such pressure loss is due to fluid viscosity or various other entrance-exit effects not accounted for by the classical Hagen–Poiseuille's law. Indeed, the pressure head lost due to friction cannot accelerate the column during the glottal open phase, and the flow at the glottal constriction is viscosity dominated right before the moment of closure. Incidentally, the glottal inertance is singular at the moment of closure as well.

To conclude, the underlying story of these notes is the inertance that makes appearance not only in the (incompressible) flow model for the fluid column acceleration but in the equations of the acoustics of the same fluid volume. All acoustic inertia in the proposed system is also flow mechanical inertia but this is not true conversely: The inertia of the piston (i.e., moving tissues during exhalation) as well as the air jet separating from the lips are parts of the flow mechanical loading, only.

The reader of these semi-informal notes should be warned that this text is not, neither will it be, a proper scientific article. Many necessary attributes of scientific articles are missing, including much of the wider scientific context and all references to works of other authors.

2 Rectangular glottis

In this section, we consider a rectangular glottis where an incompressible, laminar[2], lossless flow takes place. The length of the SGT, glottis, and VT denoted by L_{SGT}, L_G, L_{VT}, respectively. The subintervals $[-L_{SGT}, 0)$, $[0, L_G)$, and $[L_G, L_G + L_{VT}]$ denote these parts of the flow channel, and the flow intersection area $A(\cdot)$ is assumed to be time-dependent only on $[0, L_G)$.

2.1 An elementary treatment

The velocity, given the time-variant volume velocity $U = U(t)$

$$v(x,t) = \begin{cases} \frac{U(t)}{A} & \text{for } x \in [-L_{SGT}, 0), \\ \frac{U(t)}{hg(t)} & \text{for } x \in [0, L_G), \\ \frac{U(t)}{A} & \text{for } x \in [L_G, L_G + L_{VT}). \end{cases}$$

[2]...and what a convenient assumption this is!

The corresponding velocity potential, remembering its continuity, is given by

$$\psi(x,t) = \begin{cases} \frac{U(t)x}{A} & \text{for } x \in [-L_{SGT}, 0), \\ \frac{U(t)x}{hg(t)} & \text{for } x \in [0, L_G), \\ U(t)\left[\frac{x-L_G}{A} + \frac{L_G}{hg(t)}\right] & \text{for } x \in [L_G, L_G + L_{VT}). \end{cases}$$

The time derivative is given by

$$\frac{\partial \psi}{\partial t}(x,t) = \begin{cases} \frac{U'(t)x}{A} & \text{for } x \in [-L_{SGT}, 0), \\ \frac{x}{h}\frac{d}{dt}\left(\frac{U(t)}{g(t)}\right) & \text{for } x \in [0, L_G), \\ \frac{x-L_G}{A}U'(t) + \frac{L_G}{h}\frac{d}{dt}\left(\frac{U(t)}{g(t)}\right) & \text{for } x \in [L_G, L_G + L_{VT}). \end{cases}$$

Unsteady Bernoulli at points $x = -L_{SGT}$ with pressure $p_s = p_s(t)$ and $x = L_G + L_{VT}$ with ambient pressure $p_{amb} = 0$:

$$-\frac{U'(t)L_{SGT}}{A} + \frac{1}{2}\left(\frac{U(t)}{A}\right)^2 + \frac{p_s(t)}{\rho} = \frac{L_{VT} + L_G - L_G}{A}U'(t) + \frac{L_G}{h}\frac{d}{dt}\left(\frac{U(t)}{g(t)}\right) + \frac{1}{2}\left(\frac{U(t)}{A}\right)^2,$$

or, equivalently,

$$\frac{p_s(t)}{\rho} = \frac{L_{SGT} + L_{VT}}{A}U'(t) + \frac{L_G}{h}\frac{d}{dt}\left(\frac{U(t)}{g(t)}\right).$$

This is exactly the original version of the “lossless model” for the rectangular glottis, given in [9].

2.2 A more elegant version

It is desirable to carry out the computation so that the inertance of the full fluid column is treated in an unified manner. We show next that, in fact,

$$p_s(t) = \frac{d}{dt}\left(C_{iner}(t)U(t)\right) \tag{1}$$

where

$$\begin{aligned} C_{iner}(t) &= \rho\int_0^{L_{SGT}} \frac{ds}{A_{SGT}(s)} + \rho\int_0^{L_{VT}} \frac{ds}{A_{VT}(s)} + \frac{\rho L_G}{hg(t)} \\ &= C_{iner}^{(SGT)} + C_{iner}^{(VT)} + C_{iner}^{(G)}(t) \text{ with } C_{iner}^{(G)}(t) := \frac{\rho L_G}{hg(t)} \end{aligned} \tag{2}$$

is the *total inertance* of the subglottal tract, vocal tract, and the interglottal volume. Note that

$$hC'_{iner}(t)g(t) = -\frac{\rho L_G g'(t)}{g(t)^2}g(t) = -\frac{\rho L_G g'(t)}{g(t)} = -\rho L_G\frac{d\ln g(t)}{dt}. \tag{3}$$

Our assumptions are that the VT and SGT area functions $A_{VT}(s)$ and $A_{SGT}(s)$ have their glottal ends at $s = L_G$, $s = 0$, respectively, and that

$$\lim_{s\to L_{VT}} A_{VT}(s) = \lim_{s\to L_{SGT}} A_{SGT}(s) = \infty$$

leading to stagnation at both of these ends. For the incompressible flow, the velocity is now given by

$$v(x,t) = \begin{cases} \frac{U(t)}{A_{SGT}(-x)} & \text{for } x \in (-L_{SGT}, 0), \\ \frac{U(t)}{hg(t)} & \text{for } x \in [0, L_G), \\ \frac{U(t)}{A_{VT}(x-L_G)} & \text{for } x \in [L_G, L_G + L_{VT}). \end{cases}$$

The corresponding velocity potential, remembering its continuity, is given by

$$\psi(x,t) = \begin{cases} -U(t)\int_x^0 \frac{ds}{A(-s)} & \text{for } x \in [-L_{SGT}, 0), \\ \frac{U(t)x}{hg(t)} & \text{for } x \in [0, L_G), \\ U(t)\left[\frac{L_G}{hg(t)} + \int_{L_G}^x \frac{ds}{A_{VT}(s-L_G)}\right] & \text{for } x \in [L_G, L_G + L_{VT}). \end{cases}$$

For $-L_{SGT} < x_1 < 0 < L_G < x_2 < L_G + L_{VT}$, the pressure drop satisfies by the unsteady Bernoulli equation

$$\frac{\partial \psi}{\partial t}(x_1,t) + \frac{1}{2}\left(\frac{U(t)}{A_{SGT}(-x_1)}\right)^2 + \frac{p(x_1,t)}{\rho} = \frac{\partial \psi}{\partial t}(x_2,t) + \frac{1}{2}\left(\frac{U(t)}{A_{VT}(x_2)}\right)^2 + \frac{p(x_2,t)}{\rho};$$

that is,

$$p(x_1,t) - p(x_2,t) = \rho\frac{\partial}{\partial t}\left(\psi(x_2,t) - \psi(x_1,t)\right) + \frac{\rho}{2}\left(\frac{U(t)}{A_{VT}(x_2)}\right)^2 - \frac{\rho}{2}\left(\frac{U(t)}{A_{SGT}(-x_1)}\right)^2.$$

Denoting the stagnation pressures $p_s(t) = \lim_{x_1\to -L_{SGT}} p(x_1,t)$ and $p_{amb}(t) = \lim_{x_2\to L_{SGT}} p(x_2,t)$ at the infinitely wide ends of the tube, we get by taking the limits at the both ends

$$\begin{aligned} p_s(t) - p_{amb}(t) &= \rho\frac{\partial}{\partial t}\left(U(t)\left[\frac{L_G}{hg(t)} + \int_{L_G}^{L_G+L_{VT}} \frac{ds}{A_{VT}(s-L_G)}\right] + U(t)\int_{-L_{SGT}}^0 \frac{ds}{A(-s)}\right) \\ &= \frac{\partial}{\partial t}\left(U(t)C_{iner}(t)\right) \end{aligned}$$

by change of variables. Assuming that $p_{amb}(t) = 0$ is a constant reference ambient pressure level, the result Eq. (1) follows.

Remark 2.1. *For later comparison, Eq.* (1) *can we written in terms of the velocity at the glottal opening* $v(t) = U(t)/hg(t)$ *as*

$$\begin{aligned} v'(t) &= \frac{1}{C_{iner}(t)hg(t)}\left(p_s(t) - h\left(C_{iner}(t)g'(t) + C'_{iner}(t)g(t)\right)v(t)\right) \\ &= \frac{1}{C_{iner}(t)hg(t)}\left(p_s(t) - hg'(t)\left(C_{iner}(t) - \frac{\rho L_G}{hg(t)}\right)v(t)\right) \end{aligned}$$

Thus we get

$$v'(t) = \frac{1}{C_{iner}(t)hg(t)} \left(p_s(t) - hg'(t)C_{iner}^{(TOT)}v(t)\right) \tag{4}$$

where $C_{iner}^{(TOT)} := C_{iner}^{(SGT)} + C_{iner}^{(VT)}$ *is the total inertance* excluding *the time-dependent glottis.*

2.3 Solving the glottal flow and pressure

By integration of Eq. (1),

$$C_{iner}(t)U(t) - C_{iner}(0)U(0) = \int_0^t p_s(\tau)\, d\tau.$$

Assuming that $U(0) = 0$ (which is reasonable by fixing the opening point in time) we get $U(t) = \frac{1}{C_{iner}(t)} \int_0^t p_s(\tau)\, d\tau$. For the velocity in the rectangular glottal channel we get

$$v(t) = \frac{1}{C_{iner}(t)hg(t)} \int_0^t p_s(\tau)\, d\tau. \tag{5}$$

Using again the unsteady Bernoulli at $-L_{SGT} < x_1 < 0$ and $x \in [0, L_G]$, we get

$$p(x_1, t) - p(x, t) = \rho \frac{\partial}{\partial t} \left(\psi\left(x, t\right) - \psi(x_1, t)\right) + \frac{\rho}{2} v(t)^2 - \frac{\rho}{2} \left(\frac{U(t)}{A_{SGT}(x_1)}\right)^2.$$

Taking the limit $x_1 \to -L_{SGT}$ and noting the stagnation to pressure $p_s(t)$, we get

$$p_s(t) - p(x, t) = \frac{\partial}{\partial t} U(t) \left(\frac{\rho x}{hg(t)} + \rho \int_{-L_{SGT}}^{0} \frac{ds}{A(-s)}\right) + \frac{\rho}{2} v(t)^2.$$

Thus, the pressure in the glottis $x \in [0, L_G]$ is given by

$$\begin{aligned} p(x, t) &= p_s(t) - \frac{\rho}{2} v(t)^2 - \frac{\partial}{\partial t} U(t) \left(\frac{\rho x}{hg(t)} + C_{iner}^{(SGT)}\right) \\ &= p_s(t) - \frac{\rho}{2} v(t)^2 - C_{iner}^{(SGT)} U'(t) - \rho x \frac{\partial}{\partial t} \left(\frac{U(t)}{hg(t)}\right) \\ &= p_s(t) - \frac{\rho}{2} v(t)^2 - \rho x v'(t) - C_{iner}^{(SGT)} U'(t). \end{aligned}$$

Writing $U'(t) = h\left(g'(t)v(t) + g(t)v'(t)\right)$ yields the first version of the equations for the pressure:

$$\begin{aligned} p(x, t) &= p_s(t) - \frac{\rho}{2} v(t)^2 - C_{iner}^{(SGT)} U'(t) - \rho x \frac{\partial}{\partial t} \left(\frac{U(t)}{hg(t)}\right) \\ &= p_s(t) - \frac{\rho}{2} v(t)^2 - \left(\rho x + hC_{iner}^{(SGT)} g(t)\right) v'(t) - hC_{iner}^{(SGT)} g'(t) v(t). \end{aligned} \tag{6}$$

The third term on the RHS relates to the increase of pressure at the narrowing due to deceleration (i.e., when $v'(t) < 0$).

Now, it is possible to eliminate the $v'(t)$ term from Eq. (6). Inserting Eq. (4) gives for $x \in [0, L_G)$

$$
\begin{aligned}
p(x,t) &= p_s(t) - \frac{\rho}{2}v(t)^2 - hC_{iner}^{(SGT)}g'(t)v(t) \\
&\quad - \left(\rho x + hC_{iner}^{(SGT)}g(t)\right) \cdot \frac{1}{hC_{iner}(t)g(t)} \left(p_s(t) - hg'(t)C_{iner}^{(TOT)}v(t)\right) \\
&= \left[1 - \frac{\rho x + hC_{iner}^{(SGT)}g(t)}{hC_{iner}(t)g(t)}\right] p_s(t) - \frac{\rho}{2}v(t)^2 \\
&\quad + \left[\frac{\left(\rho x + hC_{iner}^{(SGT)}g(t)\right)\left(hg'(t)C_{iner}^{(TOT)}\right)}{hC_{iner}(t)g(t)} - hC_{iner}^{(SGT)}g'(t)\right] v(t) \\
&= \left[1 - \frac{C_{iner}^{(SGT)} + \frac{\rho x}{hg(t)}}{C_{iner}(t)}\right] p_s(t) - \frac{\rho}{2}v(t)^2 \\
&\qquad + \frac{\rho g'(t)}{C_{iner}(t)g(t)}\left[C_{iner}^{(TOT)}x - C_{iner}^{(SGT)}L_G\right] v(t).
\end{aligned}
\tag{7}
$$

A few concluding remarks are now in order. The first two terms on the RHS of the top row in Eq. (7) are familiar from the steady Bernoulli principle, and one should note that always $\frac{C_{iner}^{(SGT)} + \frac{\rho x}{hg(t)}}{C_{iner}(t)} < 1$ in the second to the last row. One could call this number *partial inertance proportion* for an obvious reason, and that correction remains there even in a stationary glottis. It is unclear to me what the last term in Eq. (7) stands for but it certainly vanishes for nonmoving glottis. The expression in the brackets can be written as $C_{iner}^{(SGT)}(x - L_g) + C_{iner}^{(VT)}x$.

3 Wedge-like glottis

Let us carry out similar computations as in Section 2 but this time for wedge-like vocal folds. We again assume that the length of the glottis is L_G, and the glottal part of the airways is the interval $[0, L_G]$. The smallest opening is denoted by $g = g(t) > 0$, and the opening at the wide end is denoted by g_0. The narrow end is always downstream.

The glottal area function is now given by

$$A_G(x) = h\left(\frac{g(t) - g_0}{L_G}x + g_0\right) \text{ for } x \in [0, L_G],$$

giving for the glottal inertance

$$C_{iner}^{(G)}(t) = \rho \int_0^{L_G} \frac{ds}{A_G(s)} = \frac{\rho L_G}{h(g_0 - g(t))} \ln\left(\frac{g_0}{g(t)}\right) \tag{8}$$

where we used the fact that

$$\int_0^x \frac{ds}{(g(t)-g_0)s+g_0L_G} = \frac{1}{g(t)-g_0}\ln\frac{(g(t)-g_0)x+g_0L_G}{g_0L_G}$$
$$= \frac{1}{g(t)-g_0}\ln\left(1-\frac{g_0-g(t)}{g_0L_G}x\right).$$

Note that the wedge geometry gives a logarithmic singularity in $g(t)$ for $C_{iner}^{(G)}(t)$ at $g(t)=0$. Compared to Section 2, the singularity is there stronger since $C_{iner}^{(G)}(t) = L_G/hg(t)$ for the rectangular glottis. In any case, it is a general fact in any geometry that the inertance of the glottis (and hence, the total inertance of the fluid column) becomes singular at the moment of closure.

3.1 Equation of motion for wedge-like vocal folds

Let us start, again, from the equation of motion for the fluid column. For the incompressible, lossless flow, the velocity is

$$v(x,t) = \begin{cases} \frac{U(t)}{A_{SGT}(-x)} & \text{for } x \in (-L_{SGT},0), \\ \frac{U(t)L_G}{h((g(t)-g_0)x+g_0L_G)} & \text{for } x \in [0,L_G), \\ \frac{U(t)}{A_{VT}(x-L_G)} & \text{for } x \in [L_G, L_G+L_{VT}). \end{cases}$$

The corresponding velocity potential, remembering its continuity, is given by

$$\psi(x,t) = \begin{cases} -U(t)\int_x^0 \frac{ds}{A(-s)} & \text{for } x \in [-L_{SGT},0), \\ \frac{L_G}{h}\frac{U(t)}{g(t)-g_0}\ln\left(1+\frac{g(t)-g_0}{g_0L_G}x\right) & \text{for } x \in [0,L_G), \\ U(t)\left[\rho^{-1}C_{iner}^{(G)}(t)+\int_{L_G}^x \frac{ds}{A_{VT}(s-L_G)}\right] & \text{for } x \in [L_G, L_G+L_{VT}). \end{cases}$$

For $-L_{SGT} < x_1 < 0 < L_G < x_2 < L_G + L_{VT}$, the pressure drop satisfies by the unsteady Bernoulli equation

$$\frac{\partial\psi}{\partial t}(x_1,t)+\frac{1}{2}\left(\frac{U(t)}{A_{SGT}(-x_1)}\right)^2+\frac{p(x_1,t)}{\rho} = \frac{\partial\psi}{\partial t}(x_2,t)+\frac{1}{2}\left(\frac{U(t)}{A_{VT}(x_2)}\right)^2+\frac{p(x_2,t)}{\rho};$$

that is,

$$p(x_1,t)-p(x_2,t) = \rho\frac{\partial}{\partial t}\left(\psi(x_2,t)-\psi(x_1,t)\right)+\frac{\rho}{2}\left(\frac{U(t)}{A_{VT}(x_2)}\right)^2-\frac{\rho}{2}\left(\frac{U(t)}{A_{SGT}(-x_1)}\right)^2.$$

Denoting the stagnation pressures $p_s(t) = \lim_{x_1\to -L_{SGT}} p(x_1,t)$ and $p_{amb}(t) = \lim_{x_2\to L_{SGT}} p(x_2,t)$ at the infinitely wide ends of the tube, we get by taking the limits at the both ends

$$p_s(t)-p_{amb}(t) = \rho\frac{\partial}{\partial t}\left(U(t)\left[\rho^{-1}C_{iner}^{(G)}(t)+\int_{L_G}^{L_G+L_{VT}}\frac{ds}{A_{VT}(s-L_G)}\right]+U(t)\int_{-L_{SGT}}^0\frac{ds}{A(-s)}\right)$$
$$= \frac{d}{dt}\left(U(t)C_{iner}(t)\right)$$

by change of variables. Assuming again that $p_{amb}(t) = 0$ is a constant reference ambient pressure level, the equation of motion (1) follows by defining the total inertance by $C_{iner}(t) = C_{iner}^{(SGT)} + C_{iner}^{(VT)} + C_{iner}^{(G)}(t)$. Obviously

$$U'(t) = \frac{1}{C_{iner}(t)} \left(p_s(t) - C_{iner}'(t)U(t)\right)$$

where $C_{iner}(t) = C_{iner}^{(TOT)} + C_{iner}^{(G)}(t)$ and

$$\begin{aligned} C_{iner}^{(G)}{}'(t) &= \frac{\rho L_G}{h} \left(\frac{-g'(t)}{(g_0 - g(t))^2} \ln\left(\frac{g_0}{g(t)}\right) + \frac{1}{g_0 - g(t)} \frac{-g'(t)}{g(t)} \right) \\ &= -\frac{g'(t)}{g_0 - g(t)} \left(C_{iner}^{(G)}(t) + \frac{\rho L_G}{hg(t)} \right). \end{aligned} \tag{9}$$

These two Eqs. together give the differential equation for the volume velocity that we will use for expressing the glottal pressure distribution:

$$\begin{aligned} U'(t) &= \frac{1}{C_{iner}(t)} \left(p_s(t) + \frac{g'(t)}{g_0 - g(t)} \left(C_{iner}^{(G)}(t) + \frac{\rho L_G}{hg(t)} \right) U(t) \right) \\ &= \frac{1}{C_{iner}(t)} \left(p_s(t) + \frac{\rho L_G g'(t)}{h(g_0 - g(t))^2} \left(\frac{g_0}{g(t)} + \ln\left(\frac{g_0}{g(t)}\right) - 1 \right) U(t) \right) \\ &= \frac{1}{C_{iner}(t)} \left(p_s(t) + \frac{\rho L_G g'(t)}{h(g_0 - g(t))^2} \left(\frac{g_0}{g(t)} + \ln\left(\frac{g_0}{eg(t)}\right) \right) U(t) \right). \end{aligned} \tag{10}$$

Recalling $g(t) \ll g_0$ and omitting the weaker logarithmic singularity, we get the approximation

$$\begin{aligned} U'(t) &= \frac{1}{C_{iner}(t)} \left(p_s(t) + \frac{\rho L_G g'(t)}{hg_0 g(t)} U(t) \right) \\ &= \frac{1}{C_{iner}(t)} \left(p_s(t) + \frac{\rho L_G}{hg_0} \cdot \frac{d \ln g(t)}{dt} U(t) \right). \end{aligned} \tag{11}$$

which is quite elegant.

3.2 Equation for the flow velocity

Instead of the equations (10) and (11) for the volume velocity $U(t)$, let us turn to the flow velocity at the glottal opening

$$v(t) = \frac{U(t)}{hg(t)}$$

which satisfies a much more involved differential equation. In explicit terms, the wedge glottis without losses gives

$$\begin{aligned} p_s(t) &= C_{iner}(t) \left(v'(t)hg(t) + v(t)hg'(t)\right) + C_{iner}^{(G)}{}'(t)v(t)hg(t) \\ &= C_{iner}(t)hg(t) \cdot v'(t) + \left(C_{iner}^{(G)}{}'(t)hg(t) + C_{iner}(t)hg'(t) \right) v(t) \end{aligned}$$

and solved for the acceleration at the glottis, it gives

$$v'(t) = \frac{1}{C_{iner}(t)hg(t)} \left(p_s(t) - h \left(C^{(G)}_{iner}{}'(t)g(t) + C_{iner}(t)g'(t) \right) v(t) \right).$$

This together with Eq. (9) yield the equation for motion

$$v'(t) = \frac{1}{C_{iner}(t)hg(t)} \left(p_s(t) + hg'(t) \left(\frac{g(t)}{g_0 - g(t)} C^{(G)}_{iner}(t) - C_{iner}(t) + \frac{\rho L_G}{h(g_0 - g(t))} \right) v(t) \right) \tag{12}$$

where $C^{(G)}_{iner}(t)$ is given by (8), $C_{iner}(t) = C^{(G)}_{iner}(t) + C^{(TOT)}_{iner}$ where $C^{(TOT)}_{iner} = C^{(SGT)}_{iner} + C^{(VT)}_{iner}$ is the total inertance of the nonmoving part of the vocal tract.

Note that the first two terms inside the parentheses can be joined as

$$C^{(G)}_{iner}(t) \left(\frac{g(t)}{g_0 - g(t)} - 1 \right) - C^{(TOT)}_{iner} = -\frac{g_0 - 2g(t)}{g_0 - g(t)} C^{(G)}_{iner}(t) - C^{(TOT)}_{iner}$$

which gives the final form of the unsimplified equations of the motion

$$\begin{aligned} v'(t) &= \frac{1}{C_{iner}(t)hg(t)} \cdot \\ &\cdot \left(p_s(t) - g'(t) \left[hC^{(TOT)}_{iner} + \frac{\rho L_G}{g_0 - g(t)} \left(\frac{g_0 - 2g(t)}{g_0 - g(t)} \ln \frac{g_0}{g(t)} - 1 \right) \right] v(t) \right) \\ &= \frac{1}{C_{iner}(t)hg(t)} \cdot \\ &\cdot \left(p_s(t) - hg'(t) \left[C^{(TOT)}_{iner} + \frac{g_0 - 2g(t)}{g_0 - g(t)} \left(C^{(G)}_{iner}(t) - \frac{\rho L_G}{h(g_0 - 2g(t))} \right) \right] v(t) \right). \end{aligned} \tag{13}$$

Remark 3.1. *Note that the expression $C_{iner}(t)hg(t)$ in the denominator of Eq.* (13) *has a removable singularity as $g(t) \to 0$ in the sense that the limit at the closing moment $= 0$. See Eq.* (8).

Simplifications based on the wedge geometry

If $g(t) \ll g_0$ as is usually the case in the wedge geometry, we get

$$v'(t) = \frac{1}{C_{iner}(t)hg(t)} \left(p_s(t) - hg'(t) \left[C_{iner}(t) - \frac{\rho L_G}{hg_0} \right] v(t) \right) \tag{14}$$

which is directly comparable with the corresponding formula Eq. (4) for the rectangular glottis. Not that the sum in brackets is always nonnegative, and it grows unboundedly as $g(t) \to 0$ right before the closure. So, at the closing glottis when $g'(t) < 0$, the effect of the additional term is to work in the same direction as the stagnation pressure $p_s(t)$.

Note that $\frac{\rho L_G}{hg_0}$ in Eq. (14) is the inertance of a tube of length L_G with uniform intersectional area hg_0. Since $g_0 \gg eg(t)$, that term is insignificant compared to $C^{(G)}_{iner}$ as well as to $C^{(TOT)}_{iner}$, and we get an even more simplified model

$$v'(t) = \frac{1}{C_{iner}(t)hg(t)} \left(p_s(t) - hg'(t)C_{iner}(t)v(t)\right). \tag{15}$$

The further approximation for Eqs. (15)–(15) is to replace $C_{iner}(t)$ given by Eq. (8) by the simplified form (again by $g(t) \ll g_0$)

$$C_{iner}(t) = C^{(TOT)}_{iner} + \frac{\rho L_G}{hg_0} \ln\left(\frac{g_0}{g(t)}\right) \text{ or even } C_{iner}(t) = C^{(TOT)}_{iner}.$$

One such variant produces from Eq. (14) the form

$$v'(t) = \frac{1}{C^{(TOT)}_{iner}hg(t)} \left(p_s(t) - hg'(t)\left[C^{(TOT)}_{iner} + \frac{\rho L_G}{hg_0} \ln\left(\frac{g_0}{eg(t)}\right)\right] v(t)\right) \tag{16}$$

where the logarithmic singularity of the glottal inertance is still present in one place.

Finally, removing the glottal opening velocity $g'(t)$, we get back to the lossless wedge model in DICO [2, 8]. Note that here the glottal gap $g(t)$ is in the denominator in Eqs. (13)–(16) since we did not extend the glottis downstream to a control surface of constant area, right above the glottis.

Remark 3.2. *Within the limits of the approximation $g(t) \ll g_0$, Eq.* (15) *can plainly written in the form*

$$v'(t) = \frac{1}{C_{iner}(t)hg(t)} \left(p_s(t) - C_{iner}(t)hg'(t)v(t)\right);$$

that is,

$$p_s(t) = C_{iner}(t)h\frac{dg(t)v(t)}{dt} = C_{iner}(t)U'(t).$$

This differs from the original equation of motion only by the approximation $C'_{iner}(t) = C^{(G)}_{iner}{}'(t) = 0$. What the above reasoning amounts to, is just showing in what approximative sense $C^{(G)}_{iner}(t)$ can be regarded as being constant of time. If we do not want to make that rather crude approximation, we should use Eq. (14) *instead of Eq.* (15)*. In fact, Eq.* (14) *is equivalent with*

$$p_s(t) = C_{iner}(t)U'(t) - \frac{\rho L_G g'(t)}{hg_0 g(t)} U(t); \tag{17}$$

that is, by the approximation $C^{(G)}_{iner}{}'(t) = -\frac{\rho L_G g'(t)}{hg_0 g(t)}$ where only one term is omitted from Eq. (9)*.*

3.3 Glottal pressure for the wedge-like vocal folds

As above for the rectangular glottis, we get from the unsteady Bernoulli principle

$$p_s(t) - p(x,t)$$
$$= \frac{\partial}{\partial t}U(t)\left(-\frac{\rho L_G}{h(g_0 - g(t))} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G}x\right) + \rho\int_{-L_{SGT}}^{0} \frac{ds}{A(-s)}\right) + \frac{\rho}{2}v(x,t)^2$$

where

$$v(x,t) = \frac{U(t)L_G}{h\left(g_0 L_G - (g_0 - g(t))x\right)} \text{ for } x \in [0, L_G].$$

Thus, the pressure in the glottis $x \in [0, L_G]$ is given by

$$\begin{aligned} p(x,t) &= p_s(t) - \frac{\rho}{2} v(x,t)^2 - \frac{\partial}{\partial t} U(t) \left(-\frac{\rho L_G}{h(g_0 - g(t))} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) + C_{iner}^{(SGT)} \right) \\ &= p_s(t) - \frac{\rho}{2} v(x,t)^2 - C_{iner}^{(SGT)} U'(t) + \frac{\rho L_G}{h} \cdot \frac{\partial}{\partial t} \left(\frac{U(t)}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right)\right). \end{aligned} \tag{18}$$

The time derivative term on the right must be manipulated as follows:

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{U(t)}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right)\right) = \frac{U'(t)}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) \\ &+ U(t) \left(\frac{g'(t)}{(g_0 - g(t))^2} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) + \frac{g'(t) x}{g_0 L_G (g_0 - g(t))} \left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right)^{-1} \right) \\ &= \frac{U'(t)}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) \\ &+ \frac{U(t) g'(t)}{g_0 - g(t)} \left(\frac{1}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) + \frac{x}{g_0 L_G - (g_0 - g(t)) x} \right) \\ &= \frac{U'(t)}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) \\ &+ \frac{U(t) g'(t)}{g_0 - g(t)} \left(\frac{1}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) + \frac{x}{g(t) x + g_0 (L_G - x)} \right). \end{aligned}$$

Combining this with Eq. (18) yields

$$\begin{aligned} p(x,t) = p_s(t) &- \frac{\rho L_G^2}{2h^2 \left(g(t)x + g_0(L_G - x)\right)^2} U(t)^2 \\ &+ \left(\frac{\rho L_G}{h} \frac{1}{g_0 - g(t)} \ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) - C_{iner}^{(SGT)} \right) U'(t) \\ &+ \frac{\rho L_G}{h} \frac{g'(t)}{(g_0 - g(t))^2} \left(\ln\left(1 - \frac{g_0 - g(t)}{g_0 L_G} x\right) + \frac{1}{1 - \frac{g_0 L_G}{(g_0 - g(t)) x}} \right) U(t). \end{aligned} \tag{19}$$

This equation together with Eq. (10) produces the pressure distribution in glottis, and its first to terms on the RHS of Eq. (19) are plainly the steady incompressible Bernoulli. The third term could be called *congestion term* (or, water hammer term) as it becomes large and positive upstream the narrowest part of the glottis just before the closure when the volume flow is decelerating. The last term vanishes when $g'(t) = 0$, and for that reason it could be called *displacement term*.

Since $g(t) \ll g_0$, we may given an approximation

$$\begin{aligned} p(x,t) = p_s(t) &- \frac{\rho L_G^2}{2h^2\left(g(t)x + g_0(L_G - x)\right)^2} U(t)^2 \\ &- \left(\frac{\rho L_G}{h g_0} \ln\left(\frac{L_G}{L_G - x} \right) + C_{iner}^{(SGT)} \right) U'(t) \\ &- \frac{\rho L_G g'(t)}{h g_0^2} \left(\ln\left(\frac{L_G}{L_G - x} \right) + \frac{x}{L_G - x} \right) U(t) \end{aligned} \tag{20}$$

where the congestion and displacement terms have been greatly simplified. This equation is best used in conjunction with Eq. (11). A further simplification can be carried out by discarding the logarithmic singularity in the displacement term, or neglecting both the congestion and the displacement terms entirely.

4 Compressible steady flow

In Sections 2 and 3, the convenient assumption of compressibility was made to derive the equation of motion for the fluid column in the flow channel. Using the equation of motion, the expression for the (hydrodynamic) pressure was derived. In vocal folds models, this pressure produces the aerodynamic forces leading to the movement of the aerodynamic surfaces.[3]

To have an educated opinion on this matter, one must explicitly deal with some form of compressible flow not accounted by the acoustic approximation. The challenge here is that the treatment of a general *nonsteady* (in)compressible flow is not possible using elementary mathematical tools and solutions in a closed form. Thus, we consider only the steady variant of a flow of an ideal gas column. In this case, the modifications due to isentropic thermodynamics are well-known. We give two examples on a steady flow having the physical dimensions resembling the glottal flow.

4.1 Generalities on isentropic ideal gas flow

We assume that the usual isentropic relations hold

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{and} \quad \frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{\gamma-1}}. \tag{21}$$

Thus

$$\frac{p}{\rho} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \left(\frac{T}{T_0}\right)^{\frac{-1}{\gamma-1}} \frac{p_0}{\rho_0} = \frac{T}{T_0} \cdot \frac{p_0}{\rho_0}$$

[3] Of course, the air flow cannot be fully incompressible since that would make the VT and SGT acoustics impossible. The question is whether the air flow is incompressible to the extent that it is a *reasonable approximation* for treating the total inertia and the resulting aerodynamic force to flow channel walls.

which makes the temperature distribution easier to compute than pressure or density distributions. Actually, it is just the usual equation of state for ideal gas. For the speed of sound, we get

$$c^2 = \gamma \frac{p}{\rho} = \frac{T}{T_0} \cdot \gamma \frac{p_0}{\rho_0}.$$

A compressible, steady Bernoulli flow inside insulated streamlines is described by

$$\frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0}. \tag{22}$$

The conservation of mass in a tube (i.e., nozzle) whose intersectional area is $A = A(x)$, given by

$$\rho v A = V_m$$

where V_m is the mass flow, considered to be constant of time. We, of course, make the assumption that $A(x)$ is "slowly varying" in the sense that an isentropic, compressible flow can be supported in the entire inner volume of the tube (i.e., tube walls are always streamlines at least in the subsonic part of the nozzle).

Putting these together

$$\frac{1}{2} \left(\frac{V_m}{A} \right)^2 + \frac{\gamma}{\gamma - 1} p\rho - \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \rho^2 = 0.$$

Now, from the isentropic relations we get

$$p\rho = p_0 \rho_0 \left(\frac{T}{T_0} \right)^{\frac{\gamma+1}{\gamma-1}} \text{ and } \rho^2 = \rho_0^2 \left(\frac{T}{T_0} \right)^{\frac{2}{\gamma-1}}.$$

Plugging in, we get

$$\frac{1}{2} \left(\frac{V_m}{A(x)} \right)^2 + \frac{\gamma p_0 \rho_0}{\gamma - 1} \left[\left(\frac{T(x)}{T_0} \right)^{\frac{\gamma+1}{\gamma-1}} - \left(\frac{T(x)}{T_0} \right)^{\frac{2}{\gamma-1}} \right] = 0 \tag{23}$$

over the length of the tube. Note that $\frac{\gamma+1}{\gamma-1} = 1 + \frac{2}{\gamma-1}$.

Another form for (23) is given by

$$\begin{aligned} v(x)^2 &= \left(\frac{V_m}{A(x)\rho(x)} \right)^2 \\ &= \frac{\rho_0^2}{\rho(x)^2} \cdot \frac{2}{\gamma - 1} \cdot \gamma \frac{p_0}{\rho_0} \frac{T(x)}{T_0} \cdot \left[\left(\frac{T(x)}{T_0} \right)^{\frac{2}{\gamma-1} - 1} - \left(\frac{T(x)}{T_0} \right)^{\frac{\gamma+1}{\gamma-1} - 1} \right] \\ &= \left(\frac{T(x)}{T_0} \right)^{-\frac{2}{\gamma-1}} \cdot \frac{2}{\gamma - 1} \cdot c(x)^2 \cdot \left[\left(\frac{T(x)}{T_0} \right)^{\frac{2}{\gamma-1} - 1} - \left(\frac{T(x)}{T_0} \right)^{\frac{2}{\gamma-1}} \right] \\ &= \frac{2}{\gamma - 1} \cdot c(x)^2 \cdot \left[\left(\frac{T(x)}{T_0} \right)^{-1} - 1 \right] \end{aligned}$$

which leads to the expression for the Mach number

$$M(x)^2 = \frac{v(x)^2}{c(x)^2} = \frac{2}{\gamma - 1}\left[\frac{T_0}{T(x)} - 1\right].$$

The speed of sound is reached at the temperature $T(x) = \frac{2}{\gamma+1}T_0$ when

$$p(x) = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} p_0 \text{ and } \rho(x) = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \rho_0.$$

The condition on the nozzle area function for reaching Mach 1 at x is given by

$$\begin{aligned} A(x)^2 &= \frac{V_m^2}{\gamma p_0 \rho_0} \frac{\rho_0^2}{\rho(x)^2} \frac{T_0}{T(x)} \\ &= \frac{V_m^2}{\gamma p_0 \rho_0}\left(\frac{T_0}{T(x)}\right)^{\frac{2}{\gamma-1}} \frac{T_0}{T(x)} = \frac{V_m^2}{p_0\rho_0}\cdot\frac{1}{\gamma}\left(\frac{T_0}{T(x)}\right)^{\frac{\gamma+1}{\gamma-1}} \\ &= \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}} \cdot \frac{V_m^2}{p_0\rho_0} = \frac{\gamma+1}{\gamma}\left(1 + \frac{\gamma-1}{2}\right)^{\frac{2}{\gamma-1}} \cdot \frac{V_m^2}{2p_0\rho_0} \end{aligned} \tag{24}$$

which yields the *critical area* A_{sonic}. The speed of sound at Mach 1 is given by

$$c(x)^2 = \gamma\frac{p_0}{\rho_0}\frac{T(x)}{T_0} = c_0^2 \cdot \frac{2}{\gamma+1} \text{ with } c_0^2 = \gamma\frac{p_0}{\rho_0}.$$

Remark 4.1. *Observe that the incompressible limit case is obtained in Eqs.* (21)–(22) *by letting* $\gamma \to \infty$*. In particular, Eq.* (21) *gives in the limit*

$$\frac{p}{p_0} = \frac{T}{T_0} \quad \textit{and} \quad \frac{\rho}{\rho_0} = 1.$$

Of course, the speed of sound $c \to \infty$ *as* $\gamma \to \infty$ *as well, and hence the Mach number* $M \to 0$ *for any fixed finite mass flow* V_m*.*

However, observe that

$$A_{sonic} \to \frac{V_m}{\sqrt{2p_0\rho_0}} \quad \textit{as } \gamma \to \infty$$

which is a rather peculiar observation since one would expect to have $A_{sonic} = 0$ *for an incompressible fluid. The incompressible steady Bernoulli gives* $\frac{1}{2}v^2 + \frac{p}{\rho} = \frac{p_0}{\rho_0}$ *which gives an upper limit for the velocity* v *since* $p/\rho \geq 0$*. This limit corresponds to the limit of the sonic area where the Venturi effect has reached vacuum.*

4.2 Two examples on diatomic ideal gas

For diatomic ideal gas, $\gamma = 7/5$, and hence $\gamma/(\gamma-1) = 7/2$, $2/(\gamma-1) = 5$, $2/(\gamma+1) = 5/6$, and $(\gamma+1)/(\gamma-1) = 6$. With these values, Eq. (23) becomes

$$\left(\frac{V_m}{A(x)}\right)^2 + 7p_0\rho_0\left[\left(\frac{T(x)}{T_0}\right)^6 - \left(\frac{T(x)}{T_0}\right)^5\right] = 0,$$

or, equivalently,

$$\frac{V_m^2}{7p_0\rho_0} = A(x)^2 \left(\frac{T(x)}{T_0}\right)^5 \left[1 - \frac{T(x)}{T_0}\right].$$

From this it follows that $T(x), T_0 > 0$ implies $T(x) \le T_0$. The restriction of remaining subsonic takes the form $T(x) > \frac{2}{\gamma+1}T_0 = \frac{5}{6}T_0$. We proceed to maximize the function $b \mapsto b^5(1-b)$ for $b \in [5/6, 1]$ corresponding the subsonic regime. Differentiating and setting $5b^4 - 6b^5 = b^4(5-6b) = 0$ leads to $b = 5/6$. So, the minimum area $A(x)$ consistent with the compressible Bernoulli principle and the isentropic process takes place when Mach number $M(x) = 1$ is reached. It also follows that $T(x)$ is a decreasing (increasing) function of $A(x)$ in the subsonic (supersonic) regime.

For diatomic ideal gas, the critical "sonic area" is

$$A_{sonic} \approx 1.4604 \cdot \frac{V_m}{\sqrt{p_0\rho_0}}.$$

At Mach 1, the speed of sound has been reduced by the factor of $\sqrt{5/6} \approx 0.913$. Let us now make two computations to estimate what kind of intersection areas lead to Mach 1 and Mach 0.3 flows for parameter values typical of the glottal flow.

Example 4.2. *Consider a flow of* $2\,\mathrm{dl/s}$ *of air with* $T_0 = 300\,\mathrm{K}$*,* $p_0 = 100\,\mathrm{kPa}$*, and* $\rho_0 = 1.2\,\mathrm{kg/m^3}$*. In these conditions, the speed of sound is* $342\,\mathrm{m/s}$*.*

Then the mass flow is $V_m = 2.4 \cdot 10^{-4}\,\mathrm{kg/s}$*, and* $A_{sonic} = 1.01 \cdot 10^{-6}\,\mathrm{m^2} = 1.01\,\mathrm{mm^2}$*. The speed of sound at the narrow point is* $c = \sqrt{\frac{5}{6}} \cdot 342\,\mathrm{m/s} = 312\,\mathrm{m/s}$*, temperature* $\frac{5}{6} \cdot 300\mathrm{K} = 250\,\mathrm{K}$*, and pressure* $52.8\,\mathrm{kPa}$*.*

If the same volume flow was incompressible through the same area, the speed would be $v = 2\,\mathrm{dl/s}/1.01\,\mathrm{mm^2} = 198\,\mathrm{m/s}$*.*

Example 4.3. *Consider a flow of* $2\,\mathrm{dl/s}$ *of air with* $T_0 = 300\,\mathrm{K}$*,* $p_0 = 100\,\mathrm{kPa}$*, and* $\rho_0 = 1.2\,\mathrm{kg/m^3}$*.*

Again, the mass flow is $V_m = 2.4 \cdot 10^{-4}\,\mathrm{kg/s}$*. At Mach* 0.3*, the temperature of the gas would be*

$$T(x) = \frac{T_0}{\frac{\gamma-1}{2}M(x)^2 + 1} = \frac{300\,\mathrm{K}}{\frac{1}{5}0.3^2 + 1} = 295\,\mathrm{K}.$$

The speed of sound at this temperature is $c(x) = \sqrt{295/300} \cdot 342\,\mathrm{m/s} = 339\,\mathrm{m/s}$*. Since we are going Mach 0.3, the speed of the air is* $102\,\mathrm{m/s}$*. The density is given by* $\rho(x) = \rho_0 \left(\frac{T(x)}{T_0}\right)^{\frac{1}{\gamma-1}} = 1.2\,\mathrm{kg/m^3} \cdot \left(\frac{295\,\mathrm{K}}{300\,\mathrm{K}}\right)^{\frac{5}{2}} = 1.15\,\mathrm{kg/m^3}$*. Now,*

$$A(x) = \frac{V_m}{\rho(x)v(x)} = \frac{2.4 \cdot 10^{-4}\,\mathrm{kg}/s}{1.15\,\mathrm{kg/m^3} \cdot 102\,\mathrm{m/s}} = 2.05\,\mathrm{mm^2}.$$

If the same volume flow was incompressible through the same area, the speed would be $v = 2\,\mathrm{dl/s}/2.05\,\mathrm{mm^2} = 97.6\,\mathrm{m/s}$*.*

Considering Examples 4.2 and 4.3, conclude that for glottal opening areas over $2\,\mathrm{mm}^2$ the usual "rule of thumb" value Mach 0.3 holds and the flow can be regarded as incompressible (even neglecting all viscosity and nonsteadyness effects). For human glottis, the area of $2\,\mathrm{mm}^2$ can be considered quite a small opening. Just by halving the opening area to $1\,\mathrm{mm}^2$, the hypothetical compressible flow already gets supersonic which certainly is counterfactual as far as the glottal flow is concerned. (The adiabatic cooling to 250 K would perhaps freeze the vocal folds, and a pressure drop to half of the ambient would be destructive as well.) Realistic pressure loss effects (such as the Poiseuille law for viscous laminar flow) will check the flow velocity at the narrowest position much before the flow gets supersonic.

4.3 Discussion on energy losses in compressible flows

The internal friction due to fluid viscosity leads to a pressure loss that can be modelled by Poiseuille's law for flow channels having circular intersections. Other intersectional geometries, such as rectangular and triangular shapes, can be given analytic descriptions, and also they are known as Poiseuille's law. For even more general geometries, one is compelled to use heuristic approximations of, e.g., hydrodynamic radii or numerical solutions. These variant of Poiseuille's law are derived for an incompressible, laminar, steady flow, too.

When the flow is compressible yet remains isentropic, additional mechanisms for kinetic energy and pressure loss emerge. There is temperature variation (adiabatic heating) at stagnation points. If such heat is conducted to surrounding structures in a lower temperature (in which case the thermodynamic system is not perfectly adiabatic), there is a total energy loss from the fluid at that position. In terms of the compressible, steady Bernoulli flow, the energy loss shows up as an *unrecoverable* pressure loss. There are reasons to believe that Poiseuille's law alone is not a sufficient description of unrecoverable pressure loss in glottal flow.

A nonexhaustive list of mechanisms that could result in a pressure loss in addition to the viscosity effects:

(i) We could have *adiabatic heating at a stagnation point* or *adiabatic cooling in a constriction*. In itself, these effects do not (by definition) amount to loss of heat from the fluid in a perfectly insulated flow channel.

In the first case, the area function changes shape in such a way that a streamline actually ends inside the tube. Such a phenomenon would surely take place in a rectangular glottis when the flow meets the glottal wall perpendicularly. Then, the temperature would increase to T_0 (but not higher!) at the stagnation point. If the channel walls at that point are at a lower temperature (driven there, e.g., by the adiabatic cooling due to the flow effect as described above), the heat conduction would actually lead to a loss of energy from the flow. Conversely, the temperature at a constriction may get lower than the wall temperature. Then the heat conduction from the wall would increase the total energy of the fluid. Note that the heat capacity of the tissue walls would be much higher than that of air.

Not that the adiabatic cooling at a constriction is offset by the opposite effect of heat production due to viscosity. It is, of course, strictly speaking wrong to consider *adiabatic* effects in such a viscous flow.

Based on the figures given in Example 4.3, the temperature variation due to adiabatic heating and/or cooling is about 1 % of the absolute temperature. The effects to the pressure are of the same order due to the thermodynamic equation of state. The heat exchange with the walls seems a very complicated phenomenon, and it is perhaps not an effect that needs accounting for in low-order models.

(ii) There could be a *dissipative boundary layer effect* transforming kinetic energy into heat by the viscosity of the fluid, not accounted by the Poiseuille's law.

(iii) There could be some form of *vortex formation* that would ultimately transform kinetic energy into heat via internal viscous losses in the fluid. This would not be a boundary layer effect, nor accounted for by the Poiseuille's law for laminar flow.

(iv) *Dynamic effects* that would depend on the moving walls, nor observed in model experiments with rigid walls.

It seems likely that properly tuned variants of Poisseuille's law serve well as a first approximation for modelling of the glottal pressure loss. At least, it has a strong theoretical background in the laminar flow regime if the compressibility is not an issue either. The "correction terms" can be motivated by experimental work on physical models (such as the work by van den Berg & al., Fulcher & al., etc.) or deep numerical computations involving Navier–Stokes -based flow models with thermodynamical coupling such as [12]. However, the author has no knowledge of such computational work.

5 Inertia and termination of a waveguide

In Sections 2 and 3, we derived equations of motion for an *incompressible fluid column* in a tubular domain where part of the domain boundary was allowed to change as a function of time. We observed that both of these (lossless) models lead to the equation of motion (1) where the only difference is the expression of the glottal inertance term $C_{iner}^{(G)}(t)$; see Eqs. (2) and (8). In both cases, the vocal tract inertance is given by the integral

$$C_{iner}^{(VT)} := \rho \int_0^{L_{VT}} \frac{ds}{A_{VT}(s)}.$$

Since acoustics is also about the movement of gas molecules (albeit by rather small distances around an equilibrium position), it is natural to ask if the same expression $C_{iner}^{(VT)}$ plays a role in acoustic equations in the same vocal tract volume. The purpose of this section is to answer this question.

For simplicity, let us consider a fluid column of length $\ell > 0$ with uniform intersectional area A. The fluid density and the speed of sound are denoted by σ and c. The assumption that $c < \infty$ means, of course, that the fluid is to some degree compressible.

5.1 Generalities

Consider a waveguide of constant intersectional area A and characteristic impedance $Z_0 = \rho c/A$. Its *acoustic impedance* is given by

$$Z_{ac}(s) = Z_0 \frac{Z_L(s) + Z_0 \tanh sT_0}{Z_0 + Z_L(s) \tanh sT_0} \tag{25}$$

where we denote the *transmission time* by $T_0 = \ell/c$ and the *termination impedance* by $Z_L(s)$. We assume that both $Z_L(s)$ and the admittance $Z_L(s)^{-1}$ are analytic, positive real functions on $\mathbb{C}_+$. Another form for the impedance is

$$Z_{ac}(s) = Z_0 \frac{Z_L(s) \cosh sT_0 + Z_0 \sinh sT_0}{Z_0 \cosh sT_0 + Z_L(s) \sinh sT_0}$$

which has the merit of having the numerator and the denominator analytic in $\mathbb{C}_+$. There is an immediate conclusion:

Proposition 5.1. *If the termination load has $Z_L(s)$ has a zero at $s = \frac{k\pi}{T_0}$, $k \in Z$, then so does have the acoustic impedance $Z_{ac}(s)$.*

Indeed, $0 = 2\sinh sT_0 = e^{sT_0} - e^{-sT_0}$ implies $e^{2sT_0} = 1$ which is possible if and only if $s = \frac{k\pi}{T_0}$. This observation appears to be of particular value when $k = 0$ since then it implies the additivity of inertances[4].

Let us begin by studing the purely resistive termination for observing the qualitative behaviour of waveguide resonances and antiresonances. If $Z_L(s) = R_L$, the numerator and the denominator are both entire functions. In this case, the zeroes of $Z_{ac}(s)$ are given by

$$-\frac{R_L}{Z_0} = \tanh sT_0 = \frac{e^{sT_0} - e^{-sT_0}}{e^{sT_0} + e^{-sT_0}} = \frac{e^{2sT_0} - 1}{e^{2sT_0} + 1};$$

that is,

$$e^{-2xT_0} \left(\cos 2yT_0 - i \sin 2yT_0\right) = e^{-2sT_0} = \frac{Z_0 + R_L}{Z_0 - R_L}$$

where $s = x + yi$. We conclude that $\sin 2yT_0 = 0$, i.e., $2yT_0 = n\pi$ for $n \in \mathbb{Z}$. Then $\cos 2yT_0 = (-1)^n$, and the equation becomes

$$(-1)^n \frac{Z_0 + R_L}{Z_0 - R_L} = e^{-2xT_0}.$$

[4] I think any Webster's resonator has a discrete number of frequencies where the zeroes of the load show through.

If $Z_0 > R_L$, i.e., $n = 2k$, we get for the zeroes

$$s = -\frac{1}{2T_0} \ln \left| \frac{Z_0 + R_L}{Z_0 - R_L} \right| + ik\frac{\pi}{T_0}.$$

If $Z_0 < R_L$, i.e., $n = 2k + 1$, we get

$$s = -\frac{1}{2T_0} \ln \left| \frac{Z_0 + R_L}{Z_0 - R_L} \right| + i\left(k + \frac{1}{2}\right)\frac{\pi}{T_0}.$$

A similar computation with a similar outcome can be carried out for the poles of $Z_{ac}(s)$. If $Z_0 = R_L$, there are no zeroes nor poles as expected.

Remark 5.2. *We observe that introducing resistance to termination does not change the resonant frequencies nor antinodal frequencies (as far as we have $Z_0 \neq R_L$) but it will add losses to the impedance/admittance system. For acoustic waveguides with nonuniform intersectional areas, the resonant frequencies do depend on the termination resistances.*

In all cases, the resonant frequencies and the antinodal frequencies are sensitive to inductive or capacitive loading at the termination.

There is one more detail whose statement has some system theoretical interest. Clearly, the impedance $Z_{ac}(s)$ is a positive-real transfer function of an impedance passive system for any positive-real termination impedance $Z_L(s)$. Is the system well-posed in the usual infinite-dimensional linear systems sense? For general impedance passive systems, the well-posedness is equivalent with the fact that the transfer function is uniformly bounded on some vertical line $x_0 + i\mathbb{R} = \{x_0 + yi \,:\, y \in \mathbb{R}\}$ for $x_0 > 0$ (Theorem 5.1 in Staffans (2002)). Let us proceed to check this condition for a transmission line with a constant area function.[5]

Proposition 5.3. *For any rational, positive-real analytic function $Z_L(s)$, the transfer function $Z_{ac}(s)$ given by Eq.* (25) *is bounded on a vertical line lying in the open right half plane $\mathbb{C}_+$.*

Proof. It is clearly enough to show that the function

$$G(s) = \frac{Z(s) + \tanh s}{1 + Z(s)\tanh s} \tag{26}$$

is bounded on such a vertical line where $Z(s)$ is a positive-real analytic function in $\mathbb{C}_+$. We first need an observation concerning the hyperbolic tangent, namely that

$$Re \tanh{(x + yi)} = \frac{\sinh 2x}{\cosh 2x + \cos 2y}, \quad x, y \in \mathbb{R},$$

[5] I think also the general impedance conservative waveguide with non-constant area function – described by Webster's partial differential equation – is well-posed but proving it would require a completely different and much more difficult approach.

implying the estimate

$$0 < 1 - \frac{1}{\sinh 2x + 1} < Re \tanh(x + yi) < 1 + \frac{1}{\cosh 2x - 1} \tag{27}$$

for all $x > 0$ since $\sinh 2x < \cosh 2x$ and $|\cos 2y| \leq 1$. In particular, $Re \tanh s > 0$ for $s \in \mathbb{C}_+$. A similar computation shows that

$$Re \frac{1}{\tanh(x + yi)} = \frac{\sinh 2x}{\cosh 2x - \cos 2y}, \quad x, y \in \mathbb{R},$$

leading to exactly the same upper and lower bounds for $Re \frac{1}{\tanh(x+yi)}$ as are given for $Re \tanh(x + yi)$ in Eq. (27). We conclude that both $Re \tanh s$ and $Re \frac{1}{\tanh s}$ are uniformly bounded from above and below on all vertical lines $x_0 + i\mathbb{R} \subset \mathbb{C}_+$ by a strictly nonnegative constant.

(i) **Case where $1/Z(s)$ is bounded on $x_0 + i\mathbb{R}$ for some $x_0 > 0$.**

Because

$$G(s) = \frac{1}{Z(s)} \left(1 + \left(1 + \frac{1}{Z(s)^2}\right) \frac{1}{\frac{1}{Z(s)} + \tanh s}\right),$$

it is enough to show that $(\frac{1}{Z(s)} + \tanh s)^{-1}$ is uniformly bounded from above on $x_0 + i\mathbb{R}$. Now

$$\left|\frac{1}{Z(s)} + \tanh s\right| = \left|\left(-\frac{1}{Z(s)}\right) - \tanh s\right| > Re \tanh s > 0 \quad \text{for} \quad s \in \mathbb{C}_+$$

since $-1/Z(s) \in \mathbb{C}_-$ and $\tanh s \in \mathbb{C}_+$. We conclude from Eq. (27) that

$$\frac{1}{\left|\frac{1}{Z(s)} + \tanh s\right|} < \frac{1}{Re \tanh s} < \frac{1}{1 - \frac{1}{\sinh 2x + 1}} = 1 + \frac{1}{\sinh 2x}$$

where $s = x + yi$, $x > 0$ and $y \in \mathbb{R}$ arbitrary.

(ii) **Case where $Z(s)$ is bounded on $x_0 + i\mathbb{R}$ for some $x_0 > 0$.**

We now write

$$G(s) = \left(\frac{Z(s)}{\tanh s} + 1\right) \left(\frac{1}{\tanh s} + Z(s)\right)^{-1}$$

and observe that $\frac{Z(s)}{\tanh s}$ is uniformly bounded from above on $x_0 + i\mathbb{R}$ by using the lower estimate in Eq. (27). We proceed to show that $\left(\frac{1}{\tanh s} + Z(s)\right)^{-1}$ is uniformly bounded on $x_0 + i\mathbb{R}$. This time

$$\left|\frac{1}{\tanh s} + Z(s)\right| = \left|\left(-\frac{1}{\tanh s}\right) - Z(s)\right| > Re \frac{1}{\tanh s} \quad \text{for} \quad s \in \mathbb{C}_+$$

since $-1/\tanh s \in \mathbb{C}_-$ and $Z(s) \in \mathbb{C}_+$. Again, we obtain the estimate

$$\frac{1}{\left|\frac{1}{\tanh s} + Z(s)\right|} < 1 + \frac{1}{\sinh 2x}$$

where $s \in \mathbb{C}_+$ and $x = Re\, s$.

We proceed to the case where $Z(s)$ is a rational function. Then either $Z(s)$ or $1/Z(s)$ is proper. A proper transfer function is bounded on some right half plane $x_0 + \mathbb{C}_+$, $x_0 > 0$. The claim of the proposition now follows from the previous two special cases. $\square$

5.2 Inertial limit

Assume that there is a piston at the input end of the tube moving at the velocity $v(t) = v_0 \sin kt$ and acceleration $a(t) = v'(t)$. Then the *inertial (counter) pressure* for $0 < k \ll 1$ is plainly given by the Newton's second law for an (nearly) *incompressible* fluid is

$$p(t) = \frac{F(t)}{A} = \frac{mv'(t)}{A} = \frac{\rho \ell A \cdot v_0 k \cos kt}{A} = \rho \ell \cdot v_0 k \cos kt$$

from which for the inertial impedance transfer function (from volume velocity to pressure) we get

$$Z_{iner}(s) = \frac{\rho \ell k v_0 \left(\frac{s}{s^2+k^2}\right)}{A v_0 \left(\frac{k}{s^2+k^2}\right)} = \frac{\rho \ell}{A} s = C_{iner} s \tag{28}$$

where $C_{iner} = \rho \ell / A$ is the *inertance* of the fluid column having a constant intersection area A.

For a *compressible* fluid in a column of same dimensions, terminated to an acoustic impedance $Z_L(s)$, we get

$$Z_{ac}(s) = Z_0 \frac{Z_L(s) + Z_0 \tanh sT_0}{Z_0 + Z_L(s) \tanh sT_0}$$

where the characteristic impedance is given by $Z_0 = \frac{\rho c}{A}$ and the transmission time $T_0 = \frac{\ell}{c}$. Note that

$$\lim_{s \to 0} Z_{ac}(s) = Z_L(0)$$

since $\tanh 0 = 0$.

Definition 5.4. *We say that the termination impedance transfer function $Z_L(s)$ is* inertially feasible *if*

$$\lim_{s \to 0} \frac{Z_{ac}(s)}{Z_{iner}(s)} = r_{iner} \in \mathbb{R}.$$

In this case, the number r_{iner} is called inertial factor.

Obviously, a necessary condition for inertial feasibility is that $Z_L(0) = 0$. For perfectly terminated waveguides $Z_L(s) = Z_0$ we have $Z_{ac}(s) = Z_0$ which is not an inertially feasible termination.

Let us then compute the value of r_{iner}. We have

$$\frac{Z_{ac}(s)}{Z_{iner}(s)} = \frac{A}{\rho\ell} \cdot \frac{Z_L(s) + Z_0 \tanh sT_0}{s} \cdot \frac{Z_0}{Z_0 + Z_L(s) \tanh sT_0}$$

The last term approaches to 1 as $s \to 0$. By l'Hospital's rule, we get

$$\lim_{s\to 0} \frac{Z_L(s) + Z_0 \tanh sT_0}{s} = \lim_{s\to 0} \left(Z_L'(s) + Z_0 T_0 (1 - \tanh^2 sT_0)\right) = Z_L'(0) + Z_0 T_0.$$

Thus,

$$r_{iner} = \frac{A Z_L'(0)}{\rho\ell} + \frac{A}{\rho\ell}\frac{\rho c}{A}\frac{\ell}{c} = 1 + \frac{A Z_L'(0)}{\rho\ell} = 1 + \frac{Z_L'(0)}{Z_0 T_0}. \tag{29}$$

Using the inertance, we get yet another formula

$$r_{iner} = 1 + \frac{Z_L'(0)}{C_{iner}}. \tag{30}$$

Clearly, $r_{iner} \geq 1$ and $r_{iner} = 1$ if and only if $Z_L(0) = Z_L'(0) = 0$.

Given the termination impedance $Z_L(s)$, the inertial factor tells the proportion how much a fluid column of length ℓ must be extended in order it to have the same inertia as a comparable transmission line of length ℓ when terminated to $Z_L(s)$.

Example 5.5. *Let us consider a commonly used boundary impedance model, namely a resistance and an inductance in parallel. Then* $R_L(s) = \frac{sRL}{R+sL}$ *and* $R_L'(0) = L$. *We get*

$$r_{iner} = 1 + \frac{AL}{\rho\ell} = 1 + \frac{Lc}{Z_0\ell} = 1 + \frac{L}{Z_0 T_0}.$$

Remark 5.6. *By Eq.* (30)*, the acoustic inertance of a waveguide can be tuned by external inductive loading to any value larger than* $C_{iner} = \rho\ell/A$. *In fact, the rational impedance of Example 5.5 is a sufficiently rich class of acoustic terminations for this purpose.*

However, the inductive loading not only increases the acoustic inertance. It also moves the positions of resonant frequencies of the system. This is inconvenient when the terminated waveguide acts as an acoustic load in a larger system for which both the inertance and the resonant frequencies must be controlled to some predetermined target values.

Remark 5.7. *It was already pointed out that the termination of an uniform diameter acoustic waveguide to its characteristic impedance will not result in an inertially feasible acoustic load. This is understandable since such a waveguide appears to be infinity long with infinite mass, and its inertance cannot be expected to have any finite value. In general, acoustically nonreflecting boundary termination of* any *does not seem to be inertially feasible. However, an absorbing boundary condition may well be a desirable feature in an acoustic (part of a) model.*

5.3 Inertial proportion

The proportion of the characteristic impedance and the inertive response of a wave guide is called *inertial proportion* at frequency f, given by

$$P_{iner}(f) = \frac{C_{iner}}{Z_0} \cdot 2\pi f = (\rho\ell/A) \cdot (A/\rho c) \cdot 2\pi f = \frac{2\pi\ell f}{c} = \frac{2\pi\ell}{\lambda}$$

where $\lambda = c/f$ is the wavelength. For low frequencies f, the number $2\pi C_{iner} f$ approximates the acoustic impedance $|Z_{ac}(2\pi f i)| = Z_0\,|\tanh 2\pi T_0 f i| = Z_0\,|\tan 2\pi\ell/\lambda|$ of the transmission line of length ℓ, terminated to a short circuit. Thus

$$P_{iner}(f) \approx \tan\frac{2\pi\ell}{\lambda} = \tan P_{iner}(f).$$

In the case of the VT, it is typical to use the value $\ell = 0.17\,\mathrm{m}$ and $c = 340\,\mathrm{m/s}$. The numerical values of $P_{iner}(f)$ and $\tan\frac{2\pi\ell}{\lambda}$ for speech relevant frequencies f are given in Table 1.

f in Hz	50	75	100	125	150	175	200	225	250
P_{iner}	0.1571	0.2356	0.3142	0.3927	0.4712	0.5498	0.6283	0.7069	0.7854
$\tan P_{iner}(f)$	0.1584	0.2401	0.3249	0.4142	0.5095	0.6128	0.7265	0.8541	1.0000
Their proportions	0.9918	0.9814	0.9669	0.9481	0.9249	0.8972	0.8648	0.8276	0.7854

Table 1: Values of inertial proportion of a 17 cm long waveguide with its comparison values. The quarter wavelength frequency of such tube is 500 Hz, corresponding to F_1.

We conclude that between $100 \ldots 200$ Hz, the acoustic impedance of a 17 cm long, uniform diameter tube (with Dirichlet boundary condition at the far end opening) may be approximated by the expression $Z_{ac}(2\pi f i) \approx P_{iner}(f) Z_0 i$, $Z_0 = \rho c/A$ without making error larger than 15 % in impedance values. This serves as a "rule of thumb" for accuracy whenever we consider the acoustic impedance of a waveguide more than one octave below its lowest resonance (here 500 Hz). Note that the impedance of an infinitely long transmission line is purely resistive $\rho c/A$ whereas the inertial transfer function $Z_{iner} = C_{iner} s$ is purely reactive.

6 Inertial limit from Webster's equation

We concluded above that the inertance C_{iner} of a constant diameter fluid column is given by $C_{iner} = \rho\ell/A$ where ℓ is the length and A is the intersectional area. Because mass inertia is an additive quantity, the *flow mechanical* inertance of the variable diameter waveguide is obtained from this by integrating the infinitesimal contributions $\rho\,dx/A(x)$; i.e.,

$$C_{iner} = \rho \int_0^\ell \frac{dx}{A(x)}. \tag{31}$$

We proceed to show that the same expression for the *acoustic* inertance can be concluded from the acoustic waveguide with a varying area function $A(x)$, $x \in [0, \ell]$. For a mathematical treatment of such waveguides through Webster's partial differential equation, see [3, 1, 4].

Since there is no possibility of expressing the impedance transfer function of such a waveguide in a closed form in the same manner as in Section 5.1, the argument must be carried out by an *a priori* technique – studying the partial differential equation rather than its solution. This is always much more difficult, and we only make the computations for the trivial termination transfer function $Z_L(s) = 0$.

We begin by identifying the differential equations for the impedance transfer function of the waveguide using boundary and system nodes introduced in [7, 5, 6]. Any internally well-posed boundary node $\Xi = (G, L, K)$ (such as the one coming from Webster's horn model) induces an infinite-dimensional system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. This system node gives rise to the dynamical system of type

$$\begin{aligned} \dot{z}(t) &= A_{-1}z(t) + Bu(t), \\ y(t) &= Cz(t) + Du(t) \quad \text{for } t \geq 0, \\ z(0) &= 0. \end{aligned} \tag{32}$$

If Ξ is also impedance passive, then *semigroup generator* $A := L\big|_{\ker(G)}$ of S is maximally dissipative, and the *transfer function* of S is defined by $\mathbf{G}(s) := C\&D \left[\begin{smallmatrix} (s-A_{-1})^{-1}B \\ I \end{smallmatrix}\right]$ for all $s \in \overline{\mathbb{C}^+}$. The transfer function can always be expressed in terms of the original boundary node Ξ as follows:

Proposition 6.1. *Let $\Xi = (G, L, K)$ be a boundary node associated to the operator node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ whose transfer function is $\mathbf{G}(\cdot)$.*

(i) Then $y_s = \mathbf{G}(s)u$ for $u \in \mathcal{U}$, $s \in \rho(A)$, and $y_s \in \mathcal{Y}$ if and only if there exists a (unique) $z_s \in \operatorname{dom}(\Xi)$ such that

$$\begin{bmatrix} G \\ L \\ K \end{bmatrix} z_s = \begin{bmatrix} u \\ sz_s \\ y_s \end{bmatrix}. \tag{33}$$

(ii) Then $y_s = \mathbf{G}'(s)u$ for $u \in \mathcal{U}$, $s \in \rho(A)$, and $y_s \in \mathcal{Y}$ if and only if there exists a (unique) $z_s \in \operatorname{dom}(\Xi)$ and $x_s \in \ker(G)$ such that

$$\begin{bmatrix} G \\ L \end{bmatrix} z_s = \begin{bmatrix} u \\ sz_s \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s - L \\ -K \end{bmatrix} x_s = \begin{bmatrix} z_s \\ y_s \end{bmatrix}. \tag{34}$$

In fact, $z_s \in \cap_{k \geq 1} \operatorname{dom}\left(L^k\right)$.

Proof. We use the following relations between the operators in Ξ and S: $C\&D = \begin{bmatrix} K & 0 \end{bmatrix} |\operatorname{dom}(S)$, $A = L|\ker(G)$, $A_{-1} = L - BG$ and $G(s - A_{-1})^{-1}B = I$.

Claim (i): Now $y_s = \mathbf{G}(s)u$ if and only if $y_s = C\&D \left[\begin{smallmatrix} (s-A_{-1})^{-1}B \\ I \end{smallmatrix}\right] u$ if and only if $y_s = Kz_s$ where $z_s = (s - A_{-1})^{-1}Bu$ if and only if $y_s = Kz_s$ where $(s - A_{-1})z_s = Bu_s$ if and only if $y_s = Kz_s$ where $(s - L)z_s = B(u - Gz_s)$. But always $Gz_s = G(s - A_{-1})^{-1}Bu = u$, and hence $y_s = \mathbf{G}(s)u$ is equivalent with the solvability of z_s in (33). Because Ξ is a boundary node, $\left[\begin{smallmatrix} G \\ s-L \end{smallmatrix}\right]$ is injective, and the solution z_s of (33) is unique.

Claim (ii): This time $y_s = \mathbf{G}'(s)u$ if and only if

$$y_s = -C\&D \begin{bmatrix} (s-A)^{-1}(s-A_{-1})^{-1}B \\ I \end{bmatrix} u = K(s-A)^{-1} \cdot (s-A_{-1})^{-1}Bu = Kx_s$$

where $x_s = (s - A)^{-1}z_s$ and $z_s = (s - A_{-1})^{-1}Bu$. By Claim (i), the vector z_s is the unique solution of the first equation in (34). Moreover, we have $x_s \in \ker(G) = \operatorname{dom}(A)$, and thus $(s - L)x_s = (s - A)x_s = z_2$ being equivalent with the second equation in (34).

It follows from (33) that $z_s = (s - A_{-1})^{-1}Bu$ satisfies $Lz_s = sz_s$, and hence $z_s \in \cap_{k\geq 1}\operatorname{dom}\left(L^k\right)$. $\square$

It remains to apply Proposition 6.1 to the Webster's waveguide model given by

$$\begin{cases} \psi_{tt} = \frac{c^2}{A(s)}\frac{\partial}{\partial s}\left(A(s)\frac{\partial \psi}{\partial s}\right) \text{ for } s \in (0,\ell) \text{ and } t \in \mathbb{R}^+, \\ -A(0)\psi_s(0,t) = i_0(t) \quad \text{for } t \in \mathbb{R}^+, \\ \psi_t(\ell,t) = \psi(\ell,t) = 0 \quad \text{for } t \in \mathbb{R}^+. \end{cases} \tag{35}$$

together with the observed signal

$$p_0(t) = \rho\psi_t(0,t) \text{ for } t \in \mathbb{R}^+. \tag{36}$$

Using the operator

$$W := \frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial}{\partial x}\right),$$

the first of the equations in (35) can be cast into first order form by using the rule

$$\psi_{tt} = c^2 W\psi \quad \hat{=} \quad \frac{d}{dt}\begin{bmatrix} \psi \\ \pi \end{bmatrix} = L \begin{bmatrix} \psi \\ \pi \end{bmatrix} \text{ where } L := \begin{bmatrix} 0 & \rho^{-1} \\ \rho c^2 W & 0 \end{bmatrix}. \tag{37}$$

Note that the rule implies $\pi = \rho\psi_t$. We have $L : \mathcal{Z} \to \mathcal{X}$ where the two Hilbert spaces are given by

$$\mathcal{Z} := H^2_{\{\ell\}}(0,\ell) \times H^1_{\{\ell\}}(0,\ell), \quad \mathcal{X} := H^1_{\{\ell\}}(0,\ell) \times L^2(0,\ell) \tag{38}$$

where the subindex $\{\ell\}$ denotes the Dirichlet boundary condition at ℓ. The endpoint control and observation operators $G : \mathcal{Z} \to \mathbb{C}$ and $K : \mathcal{Z} \to \mathbb{C}$ are defined by

$$G\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] := -A(0)w_1'(0) \quad \text{and} \quad K\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] := w_2(0).$$

Now the Webster's horn model (35)–(36) for the state $z(t) = \left[\begin{smallmatrix}\psi(t)\\ \pi(t)\end{smallmatrix}\right]$ takes the form

$$\begin{cases} \frac{d}{dt}\begin{bmatrix}\psi(t)\\ \pi(t)\end{bmatrix} &= L\begin{bmatrix}\psi(t)\\ \pi(t)\end{bmatrix}, \\ G\begin{bmatrix}\psi(t)\\ \pi(t)\end{bmatrix} &= i_0(t), \end{cases} \tag{39}$$

and

$$p_0(t) = K\begin{bmatrix}\psi(t)\\ \pi(t)\end{bmatrix} \tag{40}$$

for all $t \in \overline{\mathbb{R}}^+$. We have now constructed a (impedance passive, internally well-posed) boundary node $\Xi = (G, L, K)$ whose transfer function $\mathbf{G}(s) = Z_{ac}(s)$ is the impedance of the acoustic waveguide when the far end has been terminated to a Dirichlet boundary condition. We now wish to compute $Z_{ac}(0)$ and $Z'_{ac}(0)$, leading to the following result.

Theorem 6.2. *The trivial termination transfer function $Z_L(s) = 0$ is inertially feasible for the acoustic waveguide described by* (35)–(36)*, and the inertial factor satisfies $r_{iner} = 1$.*

In other words, for Dirichlet terminated general acoustic waveguide, the flow mechanical inertance and the acoustic inertance coincide. Thus, Eq. (30) holds for general acoustic waveguides in the special case $Z_L(s) = 0$.

Proof. We must show that $Z_{ac}(0) = 0$ and $Z'_{ac}(0) = C_{iner}$ where C_{iner} is given by (31). The first step in this direction is to solve $\left[\begin{smallmatrix}G\\ L\end{smallmatrix}\right] z_0 = \left[\begin{smallmatrix}1\\ 0\\ 0\end{smallmatrix}\right]$. Writing $z_0 = \left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]$, we get the differential equation

$$\begin{cases} -A(0)w_1'(0) = 1 \\ \rho^{-1}w_2 = 0 \\ \frac{1}{A(x)}\frac{\partial}{\partial s}\left(A(x)\frac{\partial w_1}{\partial x}\right) = 0. \end{cases}$$

Thus $w_2(x) = 0$ and $w_1(x) = C_1\int_0^x \frac{dr}{A(r)} + C_2$ for $x \in [0, \ell]$. The boundary condition $w_1'(0) = -1/A(0)\ C_1 = -1$, and $w_1(\ell) = 0$ implies $C_2 = \int_0^\ell \frac{dr}{A(r)}$. Thus, we have $z_0(x) = \left[\begin{smallmatrix}\int_x^\ell \frac{dr}{A(r)}\\ 0\end{smallmatrix}\right]$. Trivially $KZ_0 = 0$, and $Z_{ac}(0) = 0$ follows from Claim (i) of Proposition 6.1.

Writing now $x_0 = \left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]$, equation $\left[\begin{smallmatrix}-L\\ -K\end{smallmatrix}\right] x_0 = z_0 = \left[\begin{smallmatrix}\int_x^\ell \frac{dr}{A(r)}\\ 0\\ y\end{smallmatrix}\right]$ implies $w_2(x) = -\rho\int_x^\ell \frac{dr}{A(r)}$. By Claim (ii) of Proposition 6.1 we get $Z'_{ac}(0) = y = -Kw_2 = \rho\int_0^\ell \frac{dr}{A(r)} = C_{iner}$. This completes the proof. □

It remains an open question whether Eq. (30) can be generalised to general acoustic waveguides for any termination impedance satisfying $Z_L(0) = 0$ and $Z'_L(0) \in \mathbb{R}$. An educated guess is that this can be done using a same kind but a more complicated form of reasoning as presented in this section.

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