A *posteriori* error estimates for Webster's equation in wave propagation

Teemu Lukkari[a], Jarmo Malinen[b,*]

[a]*Dept. Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland*
[b]*Dept. Mathematics and System Analysis, P.O. Box 11100, FI-00076, Aalto University, Finland*

Abstract

We consider a generalised Webster's equation for describing wave propagation in curved tubular structures such as variable diameter acoustic wave guides. Webster's equation in generalised form has been rigorously derived in a previous article starting from the wave equation, and it approximates cross-sectional averages of the propagating wave. Here, the approximation error is estimated by an *a posteriori* technique.

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1. Introduction

We study wave propagation in a narrow but long, tubular domain $\Omega \subset \mathbb{R}^3$ of finite length whose cross-sections are circular and of varying area. In this case, the wave equation in the domain Ω, i.e., the topmost equation in (1.1) below, has a classical approximation depending on a single spatial variable in the long direction of tubular Ω. The approximation is known as *Webster's equation*, which is given in generalised form as the topmost equation in (1.4) below. The geometry of Ω is represented by the *area function* $A(\cdot)$ whose values are cross-sectional areas of Ω. The solution of Webster's equation approximates cross-sectional averages of the solution to the wave equation as shown in [15]. The purpose of this article is to estimate the approximation error by an *a posteriori* method, using the passivity and well-posedness estimates given in [2] as well as analytic tools presented in [15, Section 5].

Webster's original work [32] was published in 1919, but the model itself has a history spanning over 200 years and starting from the works of D. Bernoulli, Euler, and Lagrange. Early work concerning Webster's equation can be found in [7, 27, 28, 32], and a selection of contemporary approaches is provided by [13, 14, 21, 22, 23, 24] and, in particular, [25]. The derivation of Webster's equation in [23] (see also [20]) is based on asymptotic expansions that, however, does not give estimates for the approximation error. The resonance structure of Webster's equation is obtained from the associated eigenvalue problem

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which resembles the characterisation for the asymptotic spectra of Neumann–Laplacian on shrinking tubular domains in [11, 26]. This is an example of dimensional reduction that is also the basis of shell and plate models; see, e.g., [5] where the treatment is for the stationary problems, only. Similarly, strings have been considered in [4] where the tool for dimensional reduction is the Γ-convergence of energy functionals as opposed to starting from a partial differential equation. In our approach, the dimensional reduction is based on the wave equation, and it is carried out by averaging over those degrees of freedom that are not part of Webster's equation; see [15].

Our interest in Webster's equation stems from the fact that it provides a model for the acoustics of the human vocal tract as it appears during a vowel utterance. Webster's equation can be used as a part of a dynamical computational physics model of speech as discussed in [3, 6, 8, 10] and the theses [1, 19]. Further applications of Webster's equation include modelling of water waves in tapered channels, acoustic design of exhaust pipes and jet engines for controlling noise, vibration, and performance as well as construction of instruments such as loudspeakers and horns [9, p. 402–405].

The results of this article describe the interplay between two kinds of models for acoustic waveguides; i.e., wave equation and Webster's equation. The first of the models is suitable for high precision, and the latter is computationally more efficient but lacks, e.g., transversal wave propagation because of simplifications. The two models are related to each other by the common underlying geometry of the waveguide. The waveguide geometry is originally defined by the tubular domain $\Omega \subset \mathbb{R}^3$ that has the following properties. The centreline of the tube is a smooth planar curve γ of unit length and with vanishing torsion, parametrised by its arc length $s \in [0, 1]$. We assume that the cross-section of Ω, perpendicular to the tangent of γ at the point $\gamma(s)$, is the circular disk $\Gamma(s)$ with centre point $\gamma(s)$. The radius of $\Gamma(s)$ is denoted by $R(s)$ with area $A(s)$. The boundary $\partial\Omega$ of Ω consists of the *ends* of the tube, $\Gamma(0)$ and $\Gamma(1)$, and the *wall* $\Gamma := \cup_{s\in[0,1]}\partial\Gamma(s)$ of the tube.

With this notation, acoustic wave propagation in Ω can be modelled by the wave equation, written for the *(perturbation) velocity potential* $\phi : \overline{\mathbb{R}}^+ \times \Omega \to \mathbb{R}$

$$\begin{cases} \phi_{tt}(t,\mathbf{r}) = c^2\Delta\phi(t,\mathbf{r}) \quad \text{for } \mathbf{r}\in\Omega \text{ and } t\in\mathbb{R}^+, \\ c\frac{\partial\phi}{\partial\nu}(t,\mathbf{r}) + \phi_t(t,\mathbf{r}) = 2\sqrt{\frac{c}{\rho A(0)}}\,u(t,\mathbf{r}) \quad \text{for } \mathbf{r}\in\Gamma(0) \text{ and } t\in\mathbb{R}^+, \\ \phi(t,\mathbf{r}) = 0 \quad \text{for } \mathbf{r}\in\Gamma(1) \text{ and } t\in\mathbb{R}^+, \\ \frac{\partial\phi}{\partial\nu}(t,\mathbf{r}) + \alpha\phi_t(t,\mathbf{r}) = 0 \quad \text{for } \mathbf{r}\in\Gamma, \text{ and } t\in\mathbb{R}^+, \text{ and} \\ \phi(0,\mathbf{r}) = \phi_0(\mathbf{r}), \quad \rho\phi_t(0,\mathbf{r}) = p_0(\mathbf{r}) \quad \text{for } \mathbf{r}\in\Omega \end{cases} \tag{1.1}$$

with the observation defined by

$$c\frac{\partial\phi}{\partial\nu}(t,\mathbf{r}) - \phi_t(t,\mathbf{r}) = 2\sqrt{\frac{c}{\rho A(0)}}\,y(t,\mathbf{r}) \quad \text{for } \mathbf{r}\in\Gamma(0) \text{ and } t\in\mathbb{R}^+, \tag{1.2}$$

where $\mathbb{R}^+ = (0,\infty)$, $\overline{\mathbb{R}}^+ = [0,\infty)$, ν denotes the unit normal vector on $\partial\Omega$, c is the sound speed, ρ is the density of the medium, and $\alpha \geq 0$ is a parameter associated to boundary dissipation. The Dirichlet condition on $\Gamma(1)$ represents an open end, and the Neumann condition on Γ represents a hard reflective surface. The control (i.e., the input) $u(t,\mathbf{r})$ and the observation (i.e., the output) $y(t,\mathbf{r})$ are given in *scattering form* in (1.1) where

the physical dimension of both signals is power per unit area. In addition to [2], the wave equation system can be treated within operator theory in different ways; see also [12, 30, 33].

It was shown in [2, Theorem 5.1 and Corollary 5.2] that for $u \in C^2(\overline{\mathbb{R}}^+; L^2(\Gamma(0)))$ and the initial state $\begin{bmatrix} \phi_0 \\ p_0 \end{bmatrix}$ compatible with the input u (as detailed below in Assumption (ii) of Theorem 4.2), there exists a unique classical solution ϕ of (1.1) satisfying

$$
\begin{aligned}
&\phi \in C^1(\overline{\mathbb{R}}^+; H^1(\Omega)) \cap C^2(\overline{\mathbb{R}}^+; L^2(\Omega)), \\
&\nabla \phi \in C^1(\overline{\mathbb{R}}^+; L^2(\Omega; \mathbb{R}^3)), \text{ and } \Delta \phi \in C(\overline{\mathbb{R}}^+; L^2(\Omega)).
\end{aligned} \tag{1.3}
$$

Then the function y given by (1.2) satisfies $y \in C(\overline{\mathbb{R}}^+; L^2(\Gamma(0)))$. For the rest of this article, u, ϕ, and y always denote these functions.

Following [15], the generalised Webster's equation for the velocity potential $\psi : \overline{\mathbb{R}}^+ \times [0,1] \to \mathbb{R}$ is given by

$$
\begin{cases}
\psi_{tt} = \dfrac{c(s)^2}{A(s)} \dfrac{\partial}{\partial s} \left(A(s) \dfrac{\partial \psi}{\partial s} \right) - \dfrac{2\pi \alpha W(s) c(s)^2}{A(s)} \psi_t \\
\qquad\qquad\qquad\qquad \text{for } s \in (0,1) \text{ and } t \in \mathbb{R}^+, \\
-c\psi_s(t,0) + \psi_t(t,0) = 2\sqrt{\dfrac{c}{\rho A(0)}}\, \tilde{u}(t) \quad \text{for } t \in \mathbb{R}^+, \\
\psi(t,1) = 0 \quad \text{for } t \in \mathbb{R}^+, \quad \text{and} \\
\psi(0,s) = \psi_0(s), \quad \rho\psi_t(0,s) = \pi_0(s) \quad \text{for } s \in (0,1),
\end{cases} \tag{1.4}
$$

and the observation $\tilde{y}$ is defined by

$$
-c\psi_s(t,0) - \psi_t(t,0) = 2\sqrt{\dfrac{c}{\rho A(0)}}\, \tilde{y}(t) \quad \text{for } t \in \mathbb{R}^+. \tag{1.5}
$$

The constants c, ρ, α are same as in (1.1), and $A(s)$ is the area of the cross-section $\Gamma(s)$. Note that the dissipative boundary condition in (1.1) gives rise to a dissipation term in (1.4). The *stretching factor* is the function $W(s) := R(s)\sqrt{R'(s)^2 + (\eta(s) - 1)^2}$ where the *curvature ratio* is given by $\eta(s) := R(s)\kappa(s)$ and κ denotes the curvature of the centreline γ. Because of the curvature of Ω, we adjust the sound speed for (1.4) by defining $c(s) := c\Sigma(s)$ where $\Sigma(s) := \left(1 + \frac{1}{4}\eta(s)^2\right)^{-1/2}$ is the *sound speed correction factor* as introduced[1] in [15, Section 3].

Standing Assumption 1. *We require that*

(i) *the tubular domain Ω does not fold into itself; i.e., $\eta(s) < 1$ for all $s \in [0,1]$; and*

(ii) *the centreline $\gamma(\cdot)$ and the radius function $R(\cdot)$ are infinitely differentiable on $[0,1]$.*

It follows from the smoothness that the rest of the data satisfies

$$
A(\cdot), \eta(\cdot), W(\cdot), c(\cdot), \Sigma(\cdot) \in C^\infty([0,1]), \tag{1.6}
$$

---

[1] For generalised Webster's equation, we use the functions A, Σ, Ξ, E, and W that are introduced in terms of the tubular domain Ω in [15].

Figure (block diagram, left and right panels):

Left panel, top block (input $\bar{u}$):
$$\begin{cases} \dfrac{\partial^2 \psi}{\partial t^2} - \dfrac{c^2}{A(s)}\dfrac{\partial}{\partial s}\left(A(s)\dfrac{\partial \psi}{\partial s}\right) = 0 \\ -c\dfrac{\partial \psi}{\partial s}(0,t) + \dfrac{\partial \psi}{\partial t}(0,t) = 2\sqrt{\dfrac{c}{\rho A(0)}}\,\bar{u}(t) \end{cases}$$
output ψ, summed $\psi - \bar{\phi}$ with $-\bar{\phi}$.

Left panel, bottom block (input u):
$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - c^2\Delta\phi = 0, \quad x \in \Omega, \\ c\dfrac{\partial \phi}{\partial \nu}(x,t) + \dfrac{\partial \phi}{\partial t}(x,t) = 2\sqrt{\dfrac{c}{\rho A(0)}}\,u(x,t), \quad x \in \Gamma(0) \end{cases}$$

Left side blocks: $\dfrac{1}{A(0)}\int_{\Gamma(0)} u\,dA$ and $\dfrac{1}{A(s)}\int_{\Gamma(s)} \phi\,dA$ (output ϕ).

Right panel, top block (input $\bar{u}$):
$$\begin{cases} \dfrac{\partial^2 \bar{\phi}}{\partial t^2} - \dfrac{c^2}{A(s)}\dfrac{\partial}{\partial s}\left(A(s)\dfrac{\partial \bar{\phi}}{\partial s}\right) = F + G + H \\ -c\dfrac{\partial \bar{\phi}}{\partial s}(0,t) + \dfrac{\partial \bar{\phi}}{\partial t}(0,t) = 2\sqrt{\dfrac{c}{\rho A(0)}}\,\bar{u}(t) \end{cases}$$
output $\bar{\phi}$, summed with 0 and $-\bar{\phi}$.

Right panel, with F, G, H feeding into block "See Eq. (2.3)-(2.5)", with inputs ϕ and $\nabla\phi$.

Right panel, bottom block (input u):
$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - c^2\Delta\phi = 0, \quad x \in \Omega, \\ c\dfrac{\partial \phi}{\partial \nu}(x,t) + \dfrac{\partial \phi}{\partial t}(x,t) = 2\sqrt{\dfrac{c}{\rho A(0)}}\,u(x,t), \quad x \in \Gamma(0) \end{cases}$$

Right side blocks: $\dfrac{1}{A(0)}\int_{\Gamma(0)} u\,dA$ and $\dfrac{1}{A(s)}\int_{\Gamma(s)} \phi\,dA$.
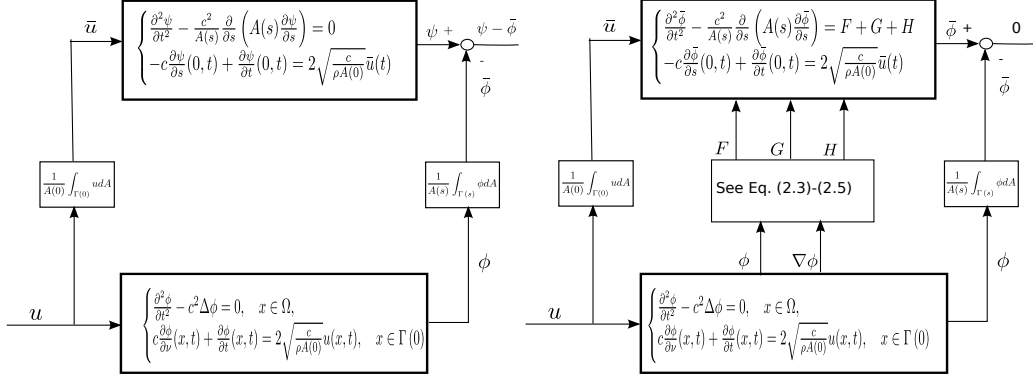
Figure 1: Left panel: Feedforward coupling describing the tracking error $e = \psi - \bar{\phi}$. Right panel: The tracking error vanishes when the additional forcing functions F, G, and H are applied. The equations in the blocks are as they appear in the lossless case $\alpha = 0$ and without curvature, i.e., $c(s) = c$.

and such a domain Ω satisfies all the assumptions listed in [2, Appendix A]. Further, since the domain doesn't fold into itself, we see that

$$0 < \min_{s\in[0,1]} c(s) \le \max_{s\in[0,1]} c(s) < \infty.$$

In addition to the regularity (1.6) of the coefficient data for the Webster's model (2.1), we make additional requirements on the geometry of Ω:

Standing Assumption 2. *We require that*

$$0 < \min_{s\in[0,1]} A(s) \le \max_{s\in[0,1]} A(s) < \infty \tag{1.7}$$

as well as $A'(0) = \kappa(0) = 0$ at the control end $\Gamma(0)$ of Ω.

The solution $\psi : [0,1] \times \overline{\mathbb{R}}^+ \to \mathbb{R}$ is *Webster's velocity potential*. It is expected to approximate the averages

$$\bar{\phi}(t,s) := \frac{1}{A(s)}\int_{\Gamma(s)} \phi\,dA \quad \text{for} \quad s \in (0,1) \quad \text{and} \quad t \in \overline{\mathbb{R}}^+ \tag{1.8}$$

of the velocity potential ϕ given by (1.1) if the inputs and initial states for both models are matched as shown in Fig. 1. We call the difference $e := \psi - \bar{\phi}$ *tracking error*, see the left panel of Fig. 1. A fundamental result on the tracking error is given in [15, Theorem 3.1], and it is presented in right panel of Fig. 1: if the generalised Webster's equation is augmented by an additional load function $f = F + G + H$, (depending on ϕ through (2.3)—(2.5) below), the tracking error will vanish. We estimate the tracking error e by a method where the exact solution ϕ of the wave equation (1.1) is assumed to be known. Hence, we call these results *a posteriori* estimates for Webster's equation even though it is a solution of another equation that needs to be known.

The article is organised as follows: we discuss the generalised Webster's equation and its weak solution in the context of [15] in Section 2 and also recall the system

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node formulation from [16]. We write the inhomogeneous Webster's equation in terms of a scattering passive system node and give the well-posedness estimate for the unique strong solution in Section 3. This is used in the next section where we show that the tracking error e satisfies the first *a posteriori* estimate, Theorem 4.2. Then, we estimate its right hand side by measuring how much ϕ differs from its planar averages, leading to the second *a posteriori* estimate, Theorem 5.1.

2. Background

2.1. Inhomogeneous Webster's equation

Let us consider the interior/boundary point control problem

$$
\begin{cases}
\psi_{tt} - \dfrac{c(s)^2}{A(s)} \dfrac{\partial}{\partial s}\left(A(s)\dfrac{\partial \psi}{\partial s}\right) + \dfrac{2\pi\alpha W(s)c(s)^2}{A(s)}\psi_t = f \\
\qquad\qquad\qquad\qquad \text{for } s \in (0,1) \text{ and } t \in \mathbb{R}^+, \\
-c\psi_s(t,0) + \psi_t(t,0) = 2\sqrt{\dfrac{c}{\rho A(0)}}\,\tilde{u}(t) \quad \text{for } t \in \mathbb{R}^+, \\
\psi(t,1) = 0 \quad \text{for } t \in \mathbb{R}^+, \quad \text{and} \\
\psi(0,s) = \psi_0(s), \quad \rho\psi_t(0,s) = \pi_0(s) \quad \text{for } s \in (0,1),
\end{cases}
\tag{2.1}
$$

with the observation $\tilde{y}$ is defined by

$$
-c\psi_s(t,0) - \psi_t(t,0) = 2\sqrt{\frac{c}{\rho A(0)}}\,\tilde{y}(t) \quad \text{for } t \in \mathbb{R}^+.
\tag{2.2}
$$

We allow for a nonvanishing load function f in (2.1). The reason for this is the fact that the spatial averages $\bar{\phi}$ of ϕ, given by (1.8), satisfy (2.1) (with properly matched initial states and boundary control) as shown in [15, Theorem 3.1] if the load term f is of particular form. We use the mapping

$$
(s, r, \theta) \mapsto \gamma(s) + r(\cos\theta\,\mathbf{n}(s) + \sin\theta\,\mathbf{b}(s))
$$

as a parameterisation of the domain Ω where $\mathbf{n}$ and $\mathbf{b}$ are the normal and binormal vectors of the centreline curve γ of Ω. Similarly, the wall Γ of the tube is parameterised by

$$
(s, \theta) \mapsto \gamma(s) + R(s)(\cos\theta\,\mathbf{n}(s) + \sin\theta\,\mathbf{b}(s));
$$

see [15, Section 2] for the details. With these parameterizations, the proper choice of the load term is $f = F + G + H \in C(\overline{\mathbb{R}}^+; L^2(0,1))$ where

$$
F(t,s) := -\frac{1}{A(s)}\frac{\partial}{\partial s}\left(A'(s)\left(\bar{\phi}(s) - \frac{1}{2\pi}\int_0^{2\pi}\phi(s,R(s),\theta)\,d\theta\right)\right);
\tag{2.3}
$$

$$
G(t,s) := -\frac{2\pi\alpha W(s)}{A(s)}\frac{\partial}{\partial t}\left(\bar{\phi}(s) - \frac{1}{2\pi}\int_0^{2\pi}\phi(s,R(s),\theta)d\theta\right); \text{ and}
\tag{2.4}
$$

$$
H(t,s) := \int_{\Gamma(s)}\frac{1}{\Xi}\nabla\left(\frac{1}{\Xi}\right)\cdot\nabla\phi\,dA - \frac{1}{A(s)}\int_{\Gamma(s)}E\Delta\phi\,dA
\tag{2.5}
$$

$$
- \frac{\alpha W(s)\eta(s)}{A(s)}\left(\int_0^{2\pi}\frac{\partial\phi}{\partial t}(s,R(s),\theta)\cos\theta d\theta\right).
$$

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Here the *curvature factor* is given by $\Xi^{-1}(s, r, \theta) := 1 - r\kappa(s)\cos\theta$, and the *error function* by

$$E(s, r, \theta) := \Xi^{-2}(s, r, \theta) - \Sigma(s)^{-2} = -2r\kappa(s)\cos\theta + \kappa(s)^2(r^2\cos^2\theta - R(s)^2/4); \quad (2.6)$$

see [15] for details. It follows from the assumed smoothness of γ and $R(\cdot)$ and from $\|\eta\|_{L^\infty([0,1])} < 1$ that $E(\cdot), \Xi(\cdot) \in C^\infty(\Omega)$.

We proceed to write (2.1) in operator form. Define $W := \frac{1}{A(s)}\frac{\partial}{\partial s}\left(A(s)\frac{\partial}{\partial s}\right)$ and $D := -\frac{2\pi W(s)}{A(s)}$. Then the first of equations in (2.1) can be cast into first order form by using the rule

$$\psi_{tt} = c(s)^2\left(W\psi + \alpha D\psi_t\right) + f \quad \hat{=} \quad \frac{d}{dt}\begin{bmatrix}\psi \\ \pi\end{bmatrix} = \begin{bmatrix}0 & \rho^{-1} \\ \rho c(s)^2 W & \alpha c(s)^2 D\end{bmatrix}\begin{bmatrix}\psi \\ \pi\end{bmatrix} + \begin{bmatrix}0 \\ \rho f\end{bmatrix}.$$

Henceforth let $L_w := \begin{bmatrix}0 & \rho^{-1} \\ \rho c(s)^2 W & \alpha c(s)^2 D\end{bmatrix} : \mathcal{Z}_w \to \mathcal{X}_w$, and

$$\mathcal{Z}_w := \left(H^1_{\{1\}}(0,1) \cap H^2(0,1)\right) \times H^1_{\{1\}}(0,1), \quad \mathcal{X}_w := H^1_{\{1\}}(0,1) \times L^2(0,1)$$

$$\text{where} \quad H^1_{\{1\}}(0,1) := \left\{f \in H^1(0,1) : f(1) = 0\right\}.$$

The Hilbert spaces $\mathcal{Z}_w$ and $\mathcal{X}_w$ are equipped with the norms

$$\|[\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}]\|^2_{\mathcal{Z}_w} := \|z_1\|^2_{H^2(0,1)} + \|z_2\|^2_{H^1(0,1)} \quad \text{and}$$

$$\|[\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}]\|^2_{H^1(0,1) \times L^2(0,1)} := \|z_1\|^2_{H^1(0,1)} + \|z_2\|^2_{L^2(0,1)},$$

respectively. For any $\rho > 0$, the *energy norm*

$$\|[\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}]\|^2_{\mathcal{X}_w} := \frac{1}{2}\left(\rho\int_0^1 |z_1'(s)|^2 A(s)\,ds + \frac{1}{\rho c^2}\int_0^1 |z_2(s)|^2 A(s)\Sigma(s)^{-2}\,ds\right) \quad (2.7)$$

is an equivalent norm for $\mathcal{X}_w$ because $\sqrt{2}\|z_1\|_{L^2(0,1)} \leq \|z_1'\|_{L^2(0,1)}$ for all $z_1 \in H^1_{\{1\}}(0,1)$.[2] We define $\mathcal{Y}_w := \mathbb{C}$ with the absolute value norm $\|u_0\|_{\mathcal{Y}_w} := |u_0|$, and the endpoint control and observation functionals $G_w : \mathcal{Z}_w \to \mathcal{Y}_w$ and $K_w : \mathcal{Z}_w \to \mathcal{Y}_w$ are defined by

$$G_w[\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}] := \frac{1}{2}\sqrt{\frac{A(0)}{\rho c(0)}}\left(-\rho c(0)z_1'(0) + z_2(0)\right) \quad \text{and}$$

$$K_w[\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}] := \frac{1}{2}\sqrt{\frac{A(0)}{\rho c(0)}}\left(-\rho c(0)z_1'(0) - z_2(0)\right).$$

Now, the generalised Webster's equation (2.1) for the state variable $x(t) = \begin{bmatrix}\psi(t) \\ \pi(t)\end{bmatrix}$ can be cast in the form

$$\begin{cases} x'(t) = L_w x(t) + \begin{bmatrix}0 \\ \rho f(t,\cdot)\end{bmatrix}, \\ \tilde{u}(t) = G_w x(t), \quad \tilde{y}(t) = K_w x(t) \quad \text{for } t \in \mathbb{R}^+, \quad \text{and} \\ x(0) = \begin{bmatrix}\psi_0 \\ \pi_0\end{bmatrix}. \end{cases} \quad (2.8)$$

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[2]We denote the (strong) derivative of a (possibly vector-valued) function of one variable by prime. In particular, f' denotes the t-derivative of load function $f = f(t,s)$ since it is regarded as the $L^2(0,1)$-valued function $t \mapsto f(t,\cdot)$. In PDE's, we denote the partial (distribution) derivatives by subindeces such as ϕ_{tt}, ϕ_{ss}, and so on.

As shown in [2, Theorem 4.1], the triple

$$\Xi^{(W)} := (G_w, L_w, K_w) \tag{2.9}$$

is a scattering passive, strong boundary node[3] on Hilbert spaces $(\mathcal{Y}_w, \mathcal{X}_w, \mathcal{Y}_w)$ which is conservative if and only if $\alpha = 0$. For $\tilde{u} \in C^2(\overline{\mathbb{R}}^+; \mathcal{Y}_w)$ and $\begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix} \in \mathcal{Z}_w$, the unique classical solution of (2.8) follows in the special case that the load function f identically vanishes (referring to the left panel in Fig. 1.).

2.2. On the weak solution of Webster's equation

Assume that ϕ is a solution of the wave equation system (1.1) satisfying the regularity properties listed in (1.3) as discussed in Section 1. It has been shown in [15, Theorem 3.1] that the averaged solution $\bar{\phi} = \bar{\phi}(t,s)$ in (1.8) satisfies

$$\bar{\phi} \in C^2(\overline{\mathbb{R}}^+; L^2(0,1)) \quad \text{and} \quad \bar{\phi}_s \in C^1(\overline{\mathbb{R}}^+; L^2(0,1)), \tag{2.10}$$

and it is a *weak* solution of the inhomogeneous Webster's equation

$$\bar{\phi}_{tt} - \frac{c(s)^2}{A(s)} \frac{\partial}{\partial s} \left(A(s) \frac{\partial \bar{\phi}}{\partial s} \right) + \frac{2\pi\alpha W(s)c(s)^2}{A(s)} \bar{\phi}_t = F + G + H \tag{2.11}$$

where the additional load term $F + G + H \in C(\overline{\mathbb{R}}^+; L^2(0,1))$ is given by (2.3)—(2.5) above. This means plainly that

$$\int_0^T \int_0^1 \left(\bar{\phi}_s \zeta_s + \frac{1}{c^2 \Sigma(s)} \bar{\phi}_{tt} \zeta \right) A(s)\, ds dt + 2\pi\alpha \int_0^T \int_0^1 W(s) \bar{\phi}_t \zeta\, ds dt$$
$$= \int_0^T \int_0^1 (F + G + H) \zeta A(s)\, ds dt \tag{2.12}$$

for all test functions $\zeta \in C_0^\infty((0,1) \times (0,T))$ and all $T > 0$.

Now, fix $t_0 \in (0,T)$ and let $\{v_\epsilon\} \subset C_0^\infty(0,T)$ for $\epsilon > 0$ be a family of non-negative functions such that $\int_0^T v_\epsilon\, dt = 1$ and $\lim_{\epsilon \to 0} v_\epsilon(t) = 0$ for all $t \in (0,T) \setminus \{t_0\}$. Let $\xi \in C_0^\infty(0,1)$ and define $\zeta(s,t) := \xi(s)v_\epsilon(t)$. By Fubini's Theorem, we get from (2.12)

$$\int_0^T \left(\int_0^1 \left(\bar{\phi}_s(t,s)\xi_s(s) + \frac{1}{c^2 \Sigma(s)} \bar{\phi}_{tt}(t,s)\xi(s) \right) A(s)\, ds \right) v_\epsilon(t)\, dt$$
$$+ 2\pi\alpha \int_0^T \left(\int_0^1 W(s)\bar{\phi}_t(t,s)\xi(s) ds \right) v_\epsilon(t)\, dt \tag{2.13}$$
$$= \int_0^T \left(\int_0^1 (F(s,t) + G(s,t) + H(s,t))\xi(s)A(s)\, ds \right) v_\epsilon(t)\, dt$$

---

[3]It is shown in [2, Theorems 4.1 and 5.1] that the wave equation model in (1.1) as well as the corresponding Webster's model in (1.4) are dynamical systems that can be represented as internally well-posed, passive boundary nodes. A short introduction of passive boundary nodes can be found in [2, Section 2].

By (2.10) and the fact that $F + G + H \in C(\overline{\mathbb{R}}^+; L^2(0,1))$, the three inner integrals in (2.13) represent continuous functions in variable t. By letting $\epsilon \to 0$, we get the identity

$$
\int_0^1 \left(\bar{\phi}_s(t_0, s) \xi_s(s) + \frac{1}{c^2 \Sigma(s)} \bar{\phi}_{tt}(t_0, s) \xi(s) \right) A(s)\, ds
$$

$$
+ 2\pi\alpha \int_0^1 W(s) \bar{\phi}_t(t_0, s) \xi(s) ds
$$

$$
= \int_0^1 (F(s, t_0) + G(s, t_0) + H(s, t_0)) \xi(s) A(s)\, ds.
$$

This means that (2.11) holds pointwise for all $t = t_0 > 0$ if the four terms in (2.11) are regarded as *distributions* for each fixed $t \in (0,1)$. By (2.10) and $F + G + H \in C(\overline{\mathbb{R}}^+; L^2(0,1))$, all other terms except the second in (2.11) are functions in $L^2(0,1)$ for any fixed $t \in (0,1)$. We conclude that the equality in (2.11) holds in $L^2(0,1)$ (understood as a subspace of distributions) for each fixed $t > 0$. Even the second term in (2.11) satisfies

$$
\frac{c(s)^2}{A(s)} \frac{\partial}{\partial s} \left(A(s) \frac{\partial \bar{\phi}}{\partial s} \right) \in C(\overline{\mathbb{R}}^+; L^2(0,1)). \tag{2.14}
$$

By continuity, Webster's equation (2.11) holds with equality in $C(\overline{\mathbb{R}}^+; L^2(0,1))$. This is the reformulation of [15, Theorem 3.1] that we use in this article.

Lemma 2.1. *Let the functions ϕ, $\bar{\phi}$, F, G, and H be defined as above. Then $x(t) = \begin{bmatrix} \bar{\phi}(t, \cdot) \\ \rho \bar{\phi}_t(t, \cdot) \end{bmatrix}$ is a solution of the first equation in (2.8) where $f = F + G + H$ and L_w is given in Section 2.1.*

Proof. We first show that $x(t) \in \mathcal{Z}_w = \text{dom}\,(L)$ for all $t \geq 0$. By the latter inclusion in (2.10) and the fact that $\bar{\phi}(t, 1) = 0$ for all $t \geq 0$, we get $\bar{\phi}(t, \cdot) \in H^1_{\{1\}}(0,1)$. Because $A(\cdot)$ is continuously differentiable, it follows from (2.14) that $\bar{\phi}_{ss}(t, \cdot) \in L^2(0,1)$; implying $\bar{\phi}(t, \cdot) \in H^2(0,1)$.

By the latter inclusion in (2.10), $\bar{\phi} \in C^1(\overline{\mathbb{R}}^+; H^1(0,1))$. Hence, $\bar{\phi}_t(t, \cdot) \in H^1_{\{1\}}(0,1)$ since $\bar{\phi}_t(t, 1) = 0$ as a consequence of $\bar{\phi}(t, 1) = 0$. We conclude that $\bar{\phi}_t(t, \cdot) \in H^1_{\{1\}}(0,1)$. We have now shown that $x(t) \in \mathcal{Z}_w$ for all t.

The claim follows from

$$
L_w \begin{bmatrix} \bar{\phi}(t, \cdot) \\ \rho \bar{\phi}_t(t, \cdot) \end{bmatrix} = \begin{bmatrix} \bar{\phi}_t(t, \cdot) \\ \rho c(s)^2 \left(W \bar{\phi}(t, \cdot) + \alpha D \bar{\phi}_t(t, \cdot) \right) \end{bmatrix} = \begin{bmatrix} \bar{\phi}_t(t, \cdot) \\ \rho \left(\bar{\phi}_{tt}(t, \cdot) - f(t, \cdot) \right) \end{bmatrix}
$$

where the last equality is by (2.11). In particular, $L_w x \in \mathcal{X}_w$. $\qquad\square$

As a consequence of (1.6), (1.7), and (2.14), the averaged solution $\bar{\phi}$ has a little more regularity that we need in Proposition 5.3:

Lemma 2.2. *The function $\bar{\phi}$ satisfies $\bar{\phi}(t, \cdot) \in H^2(0,1)$ for all $t \in \overline{\mathbb{R}}^+$.*

2.3. On system nodes

To treat the case $f \neq 0$ in (2.8), we rewrite (2.8) in terms of *system nodes* in Section 3. There exists a wide literature on system nodes, and we give a short reminder on what we need based on, e.g., [16, 29, 31]. Following [17, Definition 2.1] or [16, Definition 2.2], the system node is characterised as follows:

Definition 2.3. *An operator*

$$S := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \mathcal{X} \times \mathcal{U} \supset \mathrm{dom}\,(S) \to \mathcal{X} \times \mathcal{Y}$$

is called an system node *on the Hilbert spaces* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *if the following holds:*

(i) *A is a generator of a strongly continuous semigroup on $\mathcal{X}$.*
(ii) *$B \in \mathcal{L}(\mathcal{U}; \mathcal{X}_{-1})$ where $\mathcal{X}_{-1} = \mathrm{dom}\,(A^*)^d \supset \mathcal{X}$ is the usual extrapolation space.*
(iii) *$\mathrm{dom}\,(S) = \{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{X} \times \mathcal{U} : A_{-1}x + Bu \in \mathcal{X} \}$ where $A_{-1} \in \mathcal{L}(\mathcal{X}; \mathcal{X}_{-1})$ is the Yoshida extension of A.*
(iv) *$A\&B = \begin{bmatrix} A_{-1} & B \end{bmatrix}\big|_{\mathrm{dom}(S)}$.*
(v) *$C\&D \in \mathcal{L}(\mathrm{dom}\,(S); \mathcal{Y})$ where we use on $\mathrm{dom}\,(S)$ the graph norm of $A\&B$:*

$$\| \begin{bmatrix} x \\ u \end{bmatrix} \|^2_{\mathrm{dom}(S)} := \|x\|^2_{\mathcal{X}} + \|u\|^2_{\mathcal{U}} + \|A_{-1}x + Bu\|^2_{\mathcal{X}}.$$

Details of A_{-1} and $\mathcal{X}_{-1}$ can be found in, e.g., [16, Proposition 2.1]. We also use the Hilbert space $\mathcal{X}_1 = \mathrm{dom}\,(A)$ equipped with the graph norm of A. Whenever we refer to these spaces for the *dual node* S^d (as characterised in [16, Proposition 2.4]), we use the symbols $\mathcal{X}_1^d$ and $\mathcal{X}_{-1}^d$.

The dynamical equations for systems nodes take the form that is reminiscent of the equations in finite-dimensional linear system theory where $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$:

$$\begin{bmatrix} x'(t) \\ \tilde{y}(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ \tilde{u}(t) \end{bmatrix} \text{ for } t \in \mathbb{R}^+; \qquad x(0) = x_0. \tag{2.15}$$

Proposition 2.4. *Assume that $S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is a system node with domain $\mathrm{dom}\,(S)$. For all $x_0 \in \mathcal{X}$ and $\tilde{u} \in C^2(\overline{\mathbb{R}}^+; \mathcal{U})$ with $\begin{bmatrix} x_0 \\ \tilde{u}(0) \end{bmatrix} \in \mathrm{dom}\,(S)$ the equations (2.15) are uniquely solvable, and the solutions satisfy $x \in C^1(\overline{\mathbb{R}}^+; \mathcal{X})$, $\tilde{y} \in C(\overline{\mathbb{R}}^+; \mathcal{Y})$, and $\begin{bmatrix} x \\ u \end{bmatrix} \in C(\overline{\mathbb{R}}^+; \mathrm{dom}\,(S))$.*

This is given in [16, Proposition 2.6], and these solutions are called *classical* in the sense of mathematical systems theory. For a more complete treatment of system nodes, see [16, Section 2].

3. Inhomogeneous Webster's model

The purpose of this section is to rewrite the inhomogeneous Webster's model (2.8) as a system node with an energy inequality. As a matter of fact, we solve the following general problem in the context of mathematical systems theory: how to add an interior point control input to a passive boundary control system in the framework of system nodes.

As argued in [17, Section 2], boundary node $\Xi^{(W)} = (G_w, L_w, K_w)$ from (2.9) induces a unique system node $S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ on Hilbert spaces $(\mathcal{Y}_w, \mathcal{X}_w, \mathcal{Y}_w)$ with operators A, A_{-1}, B, and $C\&D$ as in Definition 2.3. Then, if $\tilde{u} \in C^2(\overline{\mathbb{R}}^+; \mathcal{Y}_w)$ and $x_0 = \begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix} \in \mathcal{Z}_w$, the functions x, $\tilde{y}$ in (2.8) and (2.15) are the same if $f \equiv 0$ in (2.8). We emphasize that the translation between boundary nodes like $\Xi^{(W)}$ and the corresponding system nodes S is easiest carried out using [17, Theorem 2.3]. The resulting node S is of boundary control type in the sense that $B\mathcal{Y}_w \cap \mathcal{X}_w = \{0\}$ and $\ker(B) = \{0\}$. We make use of the following relations[4] connecting S and $\Xi^{(W)}$: $\operatorname{dom}(S) = \begin{bmatrix} I \\ G_w \end{bmatrix} \mathcal{Z}_w$, $A = L_w|_{\ker(G_w)}$ with $\operatorname{dom}(A) = \ker(G_w)$, $L_w = A_{-1}|_{\mathcal{Z}_w} + BG_w$, and $C\&D = \begin{bmatrix} K_w & 0 \end{bmatrix}|_{\operatorname{dom}(S)}$; for details, see, e.g., [17, Section 2.2]. The unbounded adjoint of A is denoted by A^*, and it has been described in [2, Theorem 4.1] in the general passive case $\alpha \geq 0$. In the conservative special case $\alpha = 0$ we get $A^* := -L_w|_{\ker(K_w)}$ with $\operatorname{dom}(A^*) = \ker(K_w)$ by [17, Theorem 1.7 and Proposition 4.3]. To write (2.8) as a system node, say $S^{(W)}$, amounts to augmenting S with an additional input that accommodates the load term f.

We define the Hilbert spaces $(\mathcal{X}_w)_1 := \operatorname{dom}(A)$ and $(\mathcal{X}_w)_1^* := \operatorname{dom}(A^*)$ with the graph norms $\|z\|_{(\mathcal{X}_w)_1}^2 = \|Az\|_{\mathcal{X}_w}^2 + \|z\|_{\mathcal{X}_w}^2$ and $\|z\|_{(\mathcal{X}_w)_1^*}^2 = \|A^*z\|_{\mathcal{X}_w}^2 + \|z\|_{\mathcal{X}_w}^2$, respectively. Define $(\mathcal{X}_w)_{-1}$ to be the dual of $\operatorname{dom}(A_w^*)$ when we identify the dual of $\mathcal{X}_w$ with itself. Then $(\mathcal{X}_w)_1 \subset \mathcal{X}_w \subset (\mathcal{X}_w)_{-1}$ with continuous and dense embeddings.[5] With these definitions, $B \in \mathcal{L}(\mathcal{Y}_w; (\mathcal{X}_w)_{-1})$.

Define the control operators $B^{(e)} := \begin{bmatrix} 0 \\ \rho \end{bmatrix} : L^2(0,1) \to \mathcal{X}_w$ and $B_w := \begin{bmatrix} B & B^{(e)} \end{bmatrix} \in \mathcal{L}(\mathcal{U}_w; (\mathcal{X}_w)_{-1})$ where $\mathcal{U}_w := \mathcal{Y}_w \times L^2(0,1)$ with the norm $\|\begin{bmatrix} \tilde{u} \\ f \end{bmatrix}\|_{\mathcal{U}_w}^2 = \|\tilde{u}\|_{\mathcal{Y}_w}^2 + \|f\|_{L^2(0,1)}^2$. Define $\operatorname{dom}(S^{(W)}) := \operatorname{dom}(S) \times L^2(0,1)$ (where $\operatorname{dom}(S) = \begin{bmatrix} I \\ G_w \end{bmatrix} \mathcal{Z}_w$) with the norm

$$\left\| \begin{bmatrix} z \\ \tilde{u} \\ f \end{bmatrix} \right\|_{\operatorname{dom}(S^{(W)})}^2 = \|z\|_{\mathcal{Z}_w}^2 + \|G_w z\|_{\mathcal{Y}_w}^2 + \|f\|_{L^2(0,1)}^2$$

and the operators

$$[A\&B]_w := \begin{bmatrix} A_{-1} & B_w \end{bmatrix}\big|_{\operatorname{dom}(S^{(W)})} \quad \text{and} \quad [C\&D]_w := \begin{bmatrix} C\&D & 0 \end{bmatrix}\big|_{\operatorname{dom}(S^{(W)})}$$

yields now the system node

$$S^{(W)} := \begin{bmatrix} [A\&B]_w \\ [C\&D]_w \end{bmatrix} \tag{3.1}$$

on the Hilbert spaces $(\mathcal{U}_w, \mathcal{X}_w, \mathcal{Y}_w)$ with domain $\operatorname{dom}(S^{(W)})$. It is clear from the construction that $S^{(W)}$ has been obtained by adding a new input (using the operator $B^{(e)}$ above) to the system node S that is associated to boundary node $\Xi^{(W)}$ by [17, Theorem 2.3].

The node $S^{(W)}$ is, in particular, internally well-posed since it has the same semigroup as S. Hence, for any $\begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix} \in \mathcal{Z}_w$ and $\begin{bmatrix} \tilde{u} \\ f \end{bmatrix} \in C^2(\overline{\mathbb{R}}^+; \mathcal{U}_w)$ satisfying the compatibility

---

[4]A shorter way of writing all this is $\begin{bmatrix} L_w \\ K_w \end{bmatrix} = S \begin{bmatrix} I \\ G_w \end{bmatrix}$.

[5]Recall that $(\mathcal{X}_w)_1 \subset \mathcal{Z}_w \subset \mathcal{X}_w$ but $(\mathcal{X}_w)_1$ is not dense in $\mathcal{Z}_w$.

condition $G_w \left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix} \right] = \tilde{u}(0)$, the first and the last of the equations in

$$\begin{cases} x'(t) & = A_{-1}x(t) + B_w \left[\begin{smallmatrix} \tilde{u}(t) \\ f(t,\cdot) \end{smallmatrix} \right], \\ \tilde{y}(t) & = [C\&D]_w \left[\begin{smallmatrix} x(t) \\ \tilde{u}(t) \\ f(t,\cdot) \end{smallmatrix} \right] \quad \text{for} \quad t \in \mathbb{R}^+, \quad \text{and} \\ x(0) & = \left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix} \right] \end{cases} \qquad (3.2)$$

have a unique classical solution $x \in C^1(\overline{\mathbb{R}}^+; \mathcal{X}_w)$ with $\left[\begin{smallmatrix} x \\ \tilde{u} \\ f \end{smallmatrix} \right] \in C(\overline{\mathbb{R}}^+; \mathrm{dom}\,(S^{(W)}))$. (These equations are plainly (2.15) written for $S^{(W)}$ instead of S.) Then the output signal can be defined through the second of the equations in (3.2) since $[C\&D]_w \in \mathcal{L}(\mathrm{dom}\,(S^{(W)}); \mathcal{U}_w)$ as in Proposition 2.4. We conclude that (2.8) and (3.2) are equivalent Cauchy problems under the assumptions on $\left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix} \right]$ and $\left[\begin{smallmatrix} \tilde{u} \\ f \end{smallmatrix} \right]$ stated above.

The state $x(\cdot)$ in equations (3.2) is controlled both from the boundary points $0, 1$ (using the control function $\tilde{u}$) and also from all of the interior points of the interval $[0, 1]$ (using the control function f). We show next that that if both $\tilde{u}$ and f are twice continuously differentiable in time, the boundary and the interior point parts of the control "do not mix".

Proposition 3.1. *Let $\left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix} \right] \in \mathcal{Z}_w$, $\left[\begin{smallmatrix} \tilde{u} \\ f \end{smallmatrix} \right] \in C^2(\overline{\mathbb{R}}^+; \mathcal{U}_w)$, and $G_w \left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix} \right] = \tilde{u}(0)$. Then the classical solution x of the first and the last of equations (3.2) (associated with the system node in (3.1)) satisfies $x = z + w$ where z is the classical solution of (2.8) with $f \equiv 0$ (associated with the boundary node $\Xi^{(W)}$ in (2.9)), and $w(t) \in \ker(G_w)$ for all $t \geq 0$.*

The compatibility condition $G_w \left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix} \right] = \tilde{u}(0)$ is a peculiarity that is required by classical solutions as they are defined here. In the context of general system nodes, the role of the same compatibility condition can be understood from [16, proof of Proposition 2.6].

Proof. By linearity, the classical solution x of (3.2) can be decomposed as the sum $x = z + w$ of two classical solutions z and w for $t \in \mathbb{R}^+$ of the equations

$$z'(t) = A_{-1}z(t) + B_w \left[\begin{smallmatrix} \tilde{u}(t) \\ 0 \end{smallmatrix} \right] = A_{-1}z(t) + B\tilde{u}(t) \text{ with } z(0) = \left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix} \right]; \qquad (3.3)$$

and

$$w'(t) = A_{-1}w(t) + B_w \left[\begin{smallmatrix} 0 \\ f(t,\cdot) \end{smallmatrix} \right] = A_{-1}w(t) + B^{(e)}f(t, \cdot) \text{ with } w(0) = 0. \qquad (3.4)$$

Because the operators A_{-1} and B relate to S (as introduced in the beginning of this section) and, hence, to the boundary node $\Xi^{(W)}$ in (2.9), we have $z'(t) = L_w z(t)$ from the formulation of Cauchy problem for boundary nodes. Further, equations (3.3) give $z'(t) = L_w z(t) + B(\tilde{u}(t) - G_w z(t))$, implying $B(\tilde{u}(t) - G_w z(t)) = 0$, and hence $\tilde{u}(t) = G_w z(t)$ because $\ker(B) = \{0\}$.

Consider next the initial value problem

$$\tilde{w}'(t) = A_{-1}\tilde{w}(t) + B^{(e)}f'(t, \cdot) \quad \text{for} \quad t \in \mathbb{R}^+, \quad \tilde{w}(0) = B^{(e)}f(0), \qquad (3.5)$$

where now $f' \in C^1(\overline{\mathbb{R}}^+; L^2(0, 1))$ and $\tilde{w}(0) \in \mathcal{X}_w$. Denote by $T(\cdot)$ the strongly continuous contraction semigroup on $\mathcal{X}_w$ generated by A. Because $B^{(e)} \in \mathcal{L}(L^2(0, 1); \mathcal{X}_w)$, the variation of constants formula $\tilde{w}(t) = T(t)B^{(e)}f(0) + \int_0^t T(t - \tau)B^{(e)}f'(\tau, \cdot)\,d\tau$ gives a

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unique strong solution of (3.5) satisfying $\tilde{w} \in C(\overline{\mathbb{R}}^+; \mathcal{X}_w)$; see [29, Theorem 3.8.2(iv)]. Then w defined by $w(t) := \int_0^t \tilde{w}(\tau)\,d\tau$ satisfies $\tilde{w}(t) = A_{-1}w(t) + B^{(e)}f(t,\cdot)$ for all $t \geq 0$, as can be seen by integrating (3.5) over $[0,1]$ as a $(\mathcal{X}_w)_{-1}$-valued function. Since also $\tilde{w} = w'$ (derivative computed in the space $(\mathcal{X}_w)_{-1}$), we conclude that w equals the unique classical solution of (3.4), with $w \in C^1(\overline{\mathbb{R}}^+; \mathcal{X}_w)$.

It now follows from $A_{-1}w(t) = \tilde{w}(t) - B^{(e)}f(t,\cdot)$ that $w \in C(\overline{\mathbb{R}}^+; (\mathcal{X}_w)_1)$. Therefore $G_w w(t) = 0$ because $(X_w)_1 = \mathrm{dom}\,(A) = \ker\,(G_w)$. $\qquad\square$

In fact, the system node $S^{(W)}$ defines a well-posed linear system in the usual sense of, e.g., [16, Definition 2.7] and [29, Definition 2.2.1]:

Theorem 3.2. *The classical solution of* (3.2) *satisfies the energy inequality*

$$\frac{d}{dt}\|x(t)\|_{\mathcal{X}_w}^2 \leq |\tilde{u}(t)|^2 + 2\rho \cdot \mathrm{Re}\,\left\langle x(t), \left[\begin{smallmatrix} 0 \\ f(t,\cdot) \end{smallmatrix}\right]\right\rangle_{\mathcal{X}_w} - |\tilde{y}(t)|^2 \tag{3.6}$$

for all $t > 0$. Moreover, the well-posedness estimate

$$\|x(T)\|_{\mathcal{X}_w}^2 + \|\tilde{y}\|_{L^2((0,T);\mathcal{Y}_w)}^2 \leq K(T)\left(\|\left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix}\right]\|_{\mathcal{X}_w}^2 + \|\left[\begin{smallmatrix} \tilde{u} \\ f \end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{U}_w)}^2\right) \tag{3.7}$$

holds for all $T \geq 0$ where $K(T) := 5(\rho^2 + 1)(T + 1)$.

Proof. We first verify (3.6) for the classical solution x of (3.2) for which $\left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix}\right] \in \mathcal{Z}_w$, $\left[\begin{smallmatrix} \tilde{u} \\ f \end{smallmatrix}\right] \in C^2(\overline{\mathbb{R}}^+; \mathcal{U}_w)$, and $G_w\left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix}\right] = \tilde{u}(0)$. Proposition 3.1 gives the decomposition $x(t) = z(t) + w(t) \in \mathcal{Z}_w$ for such solutions where $z'(t) = L_w z(t)$, $w(t) \in \ker\,(G_w)$, and $w'(t) = Aw(t) + B^{(e)}f(t,\cdot)$. Hence, we get for any $t \geq 0$

$$\begin{aligned}
\frac{d}{dt}\|x(t)\|_{\mathcal{X}_w}^2 + |\tilde{y}(t)|^2 &= 2\mathrm{Re}\,\langle x(t), z'(t) + w'(t)\rangle_{\mathcal{X}_w} + |\tilde{y}(t)|^2 \\
&= 2\mathrm{Re}\,\langle x(t), L_w x(t)\rangle_{\mathcal{X}_w} + |\tilde{y}(t)|^2 + 2\mathrm{Re}\,\left\langle x(t), B^{(e)}f(t,\cdot)\right\rangle_{\mathcal{X}_w}
\end{aligned} \tag{3.8}$$

since $A = L_w|_{\ker(G_w)}$. Since $\tilde{y}(t) = K_w x(t)$ by the definition of $[C\&D]_w$, we have by the passivity of $\Xi^{(W)}$ the Green–Lagrange inequality

$$2\mathrm{Re}\,\langle x(t), L_w x(t)\rangle_{\mathcal{X}_w} + |K_w x(t)|^2 \leq |G_w x(t)|^2 = |\tilde{u}(t)|^2.$$

This, together with (3.8), gives for all $t \geq 0$ the energy estimate

$$\frac{d}{dt}\|x(t)\|_{\mathcal{X}_w}^2 + |\tilde{y}(t)|^2 \leq |\tilde{u}(t)|^2 + 2\mathrm{Re}\,\left\langle x(t), B^{(e)}f(t,\cdot)\right\rangle_{\mathcal{X}_w}. \tag{3.9}$$

Since $B^{(e)}f(t,\cdot) = \left[\begin{smallmatrix} 0 \\ \rho f(t,\cdot) \end{smallmatrix}\right]$, we conclude that (3.6) holds.

To conclude (3.7) from (3.6), we must obtain an *a priori* bound for $\|x(t)\|_{\mathcal{X}_w}$. We use again the splitting $x = z + w$ from Proposition 3.1. Because (3.3) describes the input part of the scattering passive system node S associated to $\Xi^{(W)}$ in (2.9), we get

$$\|z(t)\|_{\mathcal{X}_w}^2 \leq \|\left[\begin{smallmatrix} \psi_0 \\ \pi_0 \end{smallmatrix}\right]\|_{\mathcal{X}_w}^2 + \|\tilde{u}\|_{L^2(0,t)}^2; \tag{3.10}$$

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see [2, Theorem 4.1], based on scattering passive boundary nodes described in [18, Definition 2.3 and Theorem 2.5] that are related to passive system nodes as explained in [17, Section 2]. As in the proof of Proposition 3.1, the variation of constants formula gives $w(t) = \int_0^t T(t-\tau) \left[\begin{smallmatrix} 0 \\ \rho f(\tau,\cdot) \end{smallmatrix} \right] d\tau$ for the solution of (3.4). Because $T(\cdot)$ is a contraction semigroup, it follows from Hölder's inequality that

$$\|w(t)\|_{\mathcal{X}_w}^2 \leq t\rho^2 \|f\|_{L^2((0,t);L^2(0,1))}^2 = t\rho^2 \|f\|_{L^2((0,1)\times(0,t))}^2. \tag{3.11}$$

Combining (3.10) and (3.11) we get

$$\|x(t)\|_{\mathcal{X}_w}^2 \leq 2(\|z(t)\|_{\mathcal{X}_w}^2 + \|w(t)\|_{\mathcal{X}_w}^2) \leq 2(\|\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right]\|_{\mathcal{X}_w}^2 + (1 + t\rho^2)\|\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right]\|_{L^2((0,t);\mathcal{U}_w)}^2)$$

and thus $\|x\|_{L^2((0,T);\mathcal{X}_w)}^2 \leq 2T\|\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right]\|_{\mathcal{X}_w}^2 + (\rho^2 T^2 + 2T)\|\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{U}_w)}^2) \leq (\rho^2 T^2 + 2T)(\|\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right]\|_{\mathcal{X}_w}^2 + \|\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{U}_w)}^2)$ which implies

$$\|x\|_{L^2((0,T);\mathcal{X}_w)} \leq (\rho^2 + 1)^{1/2}(T+1)(\|\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right]\|_{\mathcal{X}_w} + \|\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{U}_w)}).$$

Now we get

$$\int_0^T \left| \langle x(t), \left[\begin{smallmatrix}0\\\rho f(t,\cdot)\end{smallmatrix}\right] \rangle_{\mathcal{X}_w} \right| dt \leq \|x\|_{L^2((0,T);\mathcal{X}_w)} \cdot \|\left[\begin{smallmatrix}0\\\rho f\end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{X}_w)}$$
$$\leq \rho(\rho^2 + 1)^{1/2}(T+1)(\|\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right]\|_{\mathcal{X}_w} + \|\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{U}_w)}) \cdot \|f\|_{L^2((0,1)\times(0,T))}$$
$$\leq (\rho^2 + 1)(T+1)(\|\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right]\|_{\mathcal{X}_w} + \|\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{U}_w)})^2$$
$$\leq 2(\rho^2 + 1)(T+1)(\|\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right]\|_{\mathcal{X}_w}^2 + \|\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right]\|_{L^2((0,T);\mathcal{U}_w)}^2).$$

This, together with integrating (3.6) over the interval $[0,T]$, produces (3.7) provided that $\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right] \in \mathcal{Z}_w$, $\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right] \in C^2(\overline{\mathbb{R}^+};\mathcal{U}_w)$, and $G_w \left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right] = \tilde{u}(0)$. $\qquad\square$

Using the well-posedness estimate of Theorem 3.2, we can move from classical solutions to more general *strong solutions* of equations (3.2).

Corollary 3.3. *The system node $S^{(W)}$ in (3.1), associated to the inhomogeneous Webster's equation described by (2.1)—(2.5), defines a well-posed linear system through equations (3.2).*

The first and the last of equations in (3.2) have a unique strong *solution x (in $\mathcal{X}_w$) for any $\left[\begin{smallmatrix}\psi_0\\\pi_0\end{smallmatrix}\right] \in \mathcal{X}_w$ and $\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right] \in L^2_{loc}(\mathbb{R}^+;\mathcal{U}_w)$ satisfying $x \in C(\overline{\mathbb{R}^+};\mathcal{X}_w) \cap W^{1,1}_{loc}(\mathbb{R}^+;(\mathcal{X}_w)_{-1})$. The output function satisfies $\tilde{y} \in L^2_{loc}(\mathbb{R}^+;\mathcal{Y}_w)$, and the well-posedness estimate (3.7) holds.*

Strong solutions are defined in [29, Definition 3.8.1] in the sense of mathematical systems theory. It is clear that classical solutions of (3.2) (as given in Proposition 2.4) are strong solutions as well. Conversely, it does not make sense to say that a strong solution would in general satisfy equations in (2.8) for, e.g, $\left[\begin{smallmatrix}\tilde{u}\\f\end{smallmatrix}\right] \notin C^2(\overline{\mathbb{R}^+};\mathcal{U}_w)$.

Proof. That $S^{(W)}$ defines a well-posed linear system follows from estimate (3.7) and [29, Lemma 4.7.8 and Theorem 4.7.15]. The existence of the strong solution follows from the definition of the well-posed linear system; see [29, Definition 2.2.1]. That the strong solution satisfies (3.7) follows by density as given in [16, Definition 2.7] and the discussion following it. $\qquad\square$

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4. Tracking error dynamics

It is now time to discuss in a rigorous way what actually is described in the right panel of Fig. 1. There, both the wave equation and Webster's equation are boundary controlled by a common external signal, apart from averaging. More precisely, the boundary control signal $u \in C^2(\overline{\mathbb{R}}^+; L^2(\Gamma(0)))$ acts as an input for the wave equation, and the scalar signal

$$\bar{u}(t) := \frac{1}{A(0)} \int_{\Gamma(0)} u \, dA \quad \text{for} \quad t \in \overline{\mathbb{R}}^+ \tag{4.1}$$

satisfying $\bar{u} \in C^2(\overline{\mathbb{R}}^+; \mathcal{Y}_w)$ is used as the input for the Webster's model. It has been shown in [15, Theorem 3.1] that the averaged solution $\bar{\phi} = \bar{\phi}(t, s)$ in (1.8), with ϕ coming from (1.1), is a *weak* solution $\psi = \bar{\phi}$ of the problem

$$\begin{cases} \psi_{tt} - \frac{c(s)^2}{A(s)} \frac{\partial}{\partial s}\left(A(s)\frac{\partial \psi}{\partial s}\right) + \frac{2\pi\alpha W(s)c(s)^2}{A(s)}\psi_t = f \\ \qquad\qquad\qquad\qquad \text{for } s \in (0,1) \text{ and } t \in \mathbb{R}^+, \\ -c\psi_s(t,0) + \psi_t(t,0) = 2\sqrt{\frac{c}{\rho A(0)}}\,\bar{u}(t) \quad \text{for } t \in \mathbb{R}^+, \\ \psi(t,1) = 0 \quad \text{for } t \in \mathbb{R}^+, \quad \text{and} \\ \psi(0,s) = \bar{\phi}(0,s), \quad \psi_t(0,s) = \bar{\phi}_t(0,s) \quad \text{for } s \in (0,1), \end{cases} \tag{4.2}$$

where the additional load term $f = F + G + H \in C(\overline{\mathbb{R}}^+; L^2(0,1))$ is given by (2.3)—(2.5) above. By [15, Theorem 3.1], the particular weak solution $\bar{\phi}$ of (4.2) has extra regularity a consequence of (1.3) as given in (2.10).

On the other hand, the system described by (4.2) and the output function $\tilde{y}$ defined by (2.2) can be reformulated in terms of the scattering passive system node as

$$\begin{bmatrix} x'(t) \\ \tilde{y}(t) \end{bmatrix} = S^{(W)} \begin{bmatrix} x(t) \\ \bar{u}(t) \\ f(t,\cdot) \end{bmatrix} \quad \text{and} \quad x(0) = \begin{bmatrix} \bar{\phi}(0,\cdot) \\ \rho\bar{\phi}_t(0,\cdot) \end{bmatrix} \tag{4.3}$$

as shown in Section 3. Equation (4.3) has a *unique strong solution* x by Corollary 3.3 which is of the form $x = \begin{bmatrix} \psi \\ \pi \end{bmatrix}$ where ψ solves (4.2), and $\pi = \rho\psi_t$. To apply the estimate (3.7) using Corollary 3.3, we need to conclude that the top component ψ of the strong solution x of (4.3) equals $\bar{\phi}$ for all $t \geq 0$.

Lemma 4.1. *Let Ω an $\Gamma(0) \subset \partial\Omega$ be defined as in Section 1, and let $u \in C^2(\overline{\mathbb{R}}^+; L^2(\Gamma(0)))$. By ϕ denote the solution of the wave equation model (1.1) satisfying the regularity conditions (1.3), and define $y \in C(\overline{\mathbb{R}}^+; L^2(\Gamma(0)))$ by (1.2). Assume that*

(i) *the function $\bar{\phi}$ is obtained from ϕ of (1.1) by the averaging operator given in (1.8);*

(ii) *the function $\bar{u} \in C^2(\overline{\mathbb{R}}^+; \mathcal{Y}_w)$ is obtained from u by (4.1); the function $\bar{y} \in C(\overline{\mathbb{R}}^+; \mathcal{Y}_w)$ is obtained similarly from y; and*

(iii) *the function $f \in C(\overline{\mathbb{R}}^+; L^2(0,1))$ is defined as $f = F + G + H$ where F, G, and H are given by (2.3)–(2.5).*

Then $x(t) = \begin{bmatrix} \bar{\phi}(t,\cdot) \\ \rho\bar{\phi}_t(t,\cdot) \end{bmatrix}$ is the (unique) strong solution of (4.3) with the output satisfying $\tilde{y} = \bar{y}$.

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We remark that the result depends essentially on the Standing Assumptions 2 as seen in the proof.

Proof. The proof is an extension of Lemma 2.1. More precisely, we need to show that (i) the functions $x(t) = \begin{bmatrix} \bar{\phi}(t,\cdot) \\ \rho\bar{\phi}_t(t,\cdot) \end{bmatrix}$ and $\begin{bmatrix} \bar{u} \\ f \end{bmatrix}$ satisfy $x(0) \in \mathcal{X}_w$, $\begin{bmatrix} \bar{u} \\ f \end{bmatrix} \in L^2_{loc}(\mathbb{R}^+; \mathcal{U}_w)$, and $x \in C(\overline{\mathbb{R}^+}; \mathcal{X}_w) \cap W^{1,1}_{loc}(\mathbb{R}^+; (\mathcal{X}_w)_{-1})$; and that (ii) the dynamical equations (4.3) are satisfied with $\tilde{y} = \bar{y}$ where the Hilbert spaces $\mathcal{U}_w$, $\mathcal{X}_w$, $(\mathcal{X}_w)_{-1}$ and the system node $S^{(W)}$ are defined in Section 3.

Now, it is immediate from assumptions that $\begin{bmatrix} \bar{u} \\ f \end{bmatrix} \in L^2_{loc}(\mathbb{R}^+; \mathcal{U}_w)$. Recalling that $\mathcal{Z}_w \subset \mathcal{X}_w$, the inclusion $x(0) \in \mathcal{X}_w$ follows because a stronger result $x(t) \in \mathcal{Z}_w$ for all $t \geq 0$ has been shown in the proof of Lemma 2.1.

We work under the regularity assumptions (1.3) on the classical solution ϕ of (1.1)–(1.2), and hence $\begin{bmatrix} \phi \\ \rho\phi_t \end{bmatrix} \in C^1(\mathbb{R}^+; H^1(\Omega) \times L^2(\Omega))$. The averaging operator $\mathcal{A}$ defined by (1.8) satisfies $\mathcal{A} \in \mathcal{L}(H^k(\Omega); H^k(0,1))$ for all $k \geq 0$ by [15, Proposition 5.3]. Thus, the averaged solution $\bar{\phi}(t,\cdot) = \mathcal{A}\phi(t,\cdot)$ satisfies $x = \begin{bmatrix} \bar{\phi} \\ \rho\bar{\phi}_t \end{bmatrix} \in C^1(\mathbb{R}^+; \mathcal{X}_w)$ since $\mathcal{X}_w = H^1_{\{1\}}(0,1) \times L^2(0,1)$ and $\bar{\phi}(t,1) = 0$ follows from the boundary condition $\phi(t,\mathbf{r}) = 0$ for all $\mathbf{r} \in \Gamma(1)$. Because $\mathcal{X}_w \subset (\mathcal{X}_w)_{-1}$ with a continuous embedding (see Definition 2.3), we have $x \in C(\overline{\mathbb{R}^+}; \mathcal{X}_w) \cap W^{1,1}_{loc}(\mathbb{R}^+; (\mathcal{X}_w)_{-1})$ as required.

Let us first check the top row of (4.3); i.e.,

$$x'(t) = [A\&B]_w \begin{bmatrix} x(t) \\ \bar{u}(t) \\ f(t,\cdot) \end{bmatrix} = A_{-1}x(t) + B\bar{u}(t) + \begin{bmatrix} 0 \\ \rho f(t,\cdot) \end{bmatrix} \tag{4.4}$$

where the operators A_{-1}, B are as in Section 3. Since $x(t) \in \mathcal{Z}_w$ (as is already stated in this proof) and $A_{-1}\big|_{\mathcal{Z}_w} = L_w - BG_w$, we conclude that $A_{-1}x(t) + Bu(t) + \begin{bmatrix} 0 \\ \rho f(t,\cdot) \end{bmatrix} = L_w x(t) + B(\bar{u}(t) - G_w x(t)) + \begin{bmatrix} 0 \\ \rho f(t,\cdot) \end{bmatrix} = x'(t) + B(\bar{u}(t) - G_w x(t))$ by Lemma 2.1. Thus, equation (4.4) holds since $\bar{u}(t) = G_w x(t)$ follows from the second equation in (1.1) as explained in [15, Eqs. (3.6) and (3.8), as shown at the end of Section 4], noting that the last two condition listed in Standing Assumptions 2 hold.

It remains to treat the bottom row of (4.3) which takes the form

$$\tilde{y}(t) = [C\&D]_w \begin{bmatrix} x(t) \\ \bar{u}(t) \\ f(t,\cdot) \end{bmatrix} = K_w x(t).$$

Similarly as above for the input equation $\bar{u}(t) = G_w x(t)$, we observe that $\bar{y}(t) = K_w x(t)$ as well. Hence $\tilde{y} = \bar{y}$ follows, and the proof is complete. $\qquad\square$

For the rest of the section, we denote by $\begin{bmatrix} \psi \\ \rho\psi_t \end{bmatrix}$ the unique solution of (4.3) with $f \equiv 0$ and output $\tilde{y}$, referring to the left panel in Fig. 1. By Lemma 4.1, the function $\begin{bmatrix} \bar{\phi} \\ \rho\bar{\phi}_t \end{bmatrix}$ is the unique solution of (4.3) with $f = F + G + H \in C(\overline{\mathbb{R}^+}; L^2(0,1))$ and output $\bar{y}$, referring to the right panel in Fig. 1. By subtracting the model equations for ψ and $\bar{\phi}$ from each other, we get the equations for the tracking error. Indeed, because both $\begin{bmatrix} \psi \\ \rho\psi_t \end{bmatrix}$ and $\begin{bmatrix} \bar{\phi} \\ \rho\bar{\phi}_t \end{bmatrix}$ are strong solutions in the sense of Corollary 3.3, the tracking error

$\tilde{e} := \begin{bmatrix} e \\ \rho e_t \end{bmatrix} = \begin{bmatrix} \psi - \bar{\phi} \\ \rho(\psi - \bar{\phi})_t \end{bmatrix}$ is the unique strong solution of the *tracking error model*

$$\begin{bmatrix} \tilde{e}'(t) \\ \tilde{y}(t) - \bar{y}(t) \end{bmatrix} = S^{(W)} \begin{bmatrix} \tilde{e}(t) \\ 0 \\ F(t,\cdot) + G(t,\cdot) + H(t,\cdot) \end{bmatrix} \quad \text{and} \quad \tilde{e}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (4.5)$$

Now, the tracking error can be estimated for $T \geq 0$ by using the well-posedness estimate (3.7) for strong solutions, given in Theorem 3.2:

$$\begin{aligned} \|\tilde{e}(T)\|^2_{\mathcal{X}_w} &+ \|\tilde{y} - \bar{y}\|^2_{L^2((0,T);\mathcal{Y}_w)} \\ &\leq 5(\rho^2 + 1)(T + 1) \cdot \|F + G + H\|^2_{L^2((0,T);L^2(0,1))}. \end{aligned} \qquad (4.6)$$

It remains to translate (4.6) to our first *a posteriori* estimate recalling the norm of $\mathcal{X}_w$ in (2.7) that was used for deriving (4.6).

Theorem 4.2. *Let the sets Ω, Γ, and $\Gamma(s)$ for $s \in [0,1]$ be defined as in Section 1, and assume that the Standing Assumptions 1 and 2 hold. Moreover, assume the following:*

(i) *Let $u \in C^2(\overline{\mathbb{R}}^+; L^2(\Gamma(0)))$, and define its spatial average $\bar{u}$ by (4.1).*

(ii) *Let $\phi_0 \in H^1(\Omega)$ with $\phi_0\big|_{\Gamma(1)} = 0$, $\Delta\phi_0 \in L^2(\Omega)$, and $\frac{\partial\phi_0}{\partial\nu}\big|_{\Gamma(0)\cup\Gamma} \in L^2(\Gamma(0) \cup \Gamma)$. Let $p_0 \in H^1(\Omega)$ with $p_0\big|_{\Gamma(1)} = 0$, and assume that the compatibility condition with the input function u holds:*

$$c\frac{\partial\phi_0}{\partial\nu}(\mathbf{r}) + \rho^{-1}p_0(\mathbf{r}) = 2\sqrt{\tfrac{c}{\rho A(0)}}\, u(0,\mathbf{r}) \quad \text{for all} \quad \mathbf{r} \in \Gamma(0).$$

(iii) *By $\phi : \overline{\mathbb{R}}^+ \times \Omega \to \mathbb{R}$ denote the solution[6] of the wave equation model (1.1) Define the output y by (1.2).*

(iv) *Define the spatially averaged version $\bar{\phi}$ of ϕ by (1.8). Similarly with $\bar{u}$, define $\bar{y}$ in terms of y.*

(v) *By $\psi : \overline{\mathbb{R}}^+ \times [0,1] \to \mathbb{R}$ denote the solution[7] of the generalised Webster's equation (1.4) with the input $\tilde{u} = \bar{u}$, and define the output $\tilde{y}$ by (1.5).*

Then the tracking error $e = \psi - \bar{\phi}$, as described by the left panel of Fig. 1, is bounded from above for all $T \geq 0$ by the inequality

$$\begin{aligned} \|(\psi - \bar{\phi})\,(T,\cdot)\|_{H^1(0,1)} &+ \|(\psi_t - \bar{\phi}_t)\,(T,\cdot)\|_{L^2(0,1)} + \|\tilde{y} - \bar{y}\|_{L^2(0,T)} \\ &\leq 4C_\Omega \rho^{-1/2}(\rho + 1)^{3/2}(T + 1)^{1/2} \cdot \|F + G + H\|_{L^2((0,T)\times(0,1))} \end{aligned}$$

where the constant C_Ω given by

$$C_\Omega^2 = \max\left(\|A(\cdot)^{-1}\|_{L^\infty(0,1)}, \|c(\cdot)^2 A(\cdot)^{-1}\|_{L^\infty(0,1)}\right) + 1, \quad c(s) = c\Sigma(s), \qquad (4.7)$$

depends only on the geometry of Ω, and the functions F, G, and H are given by (2.3) – (2.5) in terms of solution ϕ of (1.1) and the problem data.

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[6] As explained in [2, Theorem 5.1] for $\alpha > 0$ and [2, Corollary 5.2] for $\alpha = 0$.

[7] As explained in [2, Theorem 4.1].

Note that assumptions (i) and (ii) are first required for the classical solvability of equations in (1.1). After these, the next two assumption (iii) and (iv) can be made.

Proof. We observe that ϕ has the regularity required in (1.6) since it is part of [2, Theorem 5.1] for classical solutions. Hence, all that has been stated above about $\bar{\phi}$ is at our disposal. Recalling the energy norm (2.7) of $\mathcal{X}_w$, we get

$$\left\|\left(\psi_s - \bar{\phi}_s\right)(T,\cdot)\right\|^2_{L^2(0,1)} + \left\|\left(\psi_t - \bar{\phi}_t\right)(T,\cdot)\right\|^2_{L^2(0,1)}$$
$$\leq C_1 \left\|\left[\begin{smallmatrix}(\psi-\bar{\phi})(T,\cdot)\\ \rho(\psi_t-\bar{\phi}_t)(T,\cdot)\end{smallmatrix}\right]\right\|^2_{\mathcal{X}_w}$$

where

$$C_1 = \frac{2}{\rho}\max\left(\|A(\cdot)^{-1}\|_{L^\infty(0,1)}, \|c(\cdot)^2 A(\cdot)^{-1}\|_{L^\infty(0,1)}\right) \quad \text{and} \quad c(s) = c\Sigma(s).$$

Thus, using (4.6), we get

$$\left\|\left(\psi - \bar{\phi}\right)(T,\cdot)\right\|^2_{H^1(0,1)} + \left\|\left(\psi_t - \bar{\phi}_t\right)(T,\cdot)\right\|^2_{L^2(0,1)} + \|\tilde{y} - \bar{y}\|^2_{L^2(0,T)}$$
$$\leq C_2\left(\left\|\left[\begin{smallmatrix}(\psi-\bar{\phi})(T,\cdot)\\ \rho(\psi_t-\bar{\phi}_t)(T,\cdot)\end{smallmatrix}\right]\right\|^2_{\mathcal{X}_w} + \|\tilde{y} - \bar{y}\|^2_{L^2(0,T)}\right) \tag{4.8}$$
$$\leq 5C_2(\rho^2 + 1)(T+1)\cdot\|F + G + H\|^2_{L^2((0,T);L^2(0,1))}$$

where $C_2 = C_1 + 1$. Taking the square root of both sides and using $(a+b+c)^2 \leq 3\left(a^2 + b^2 + c^2\right)$ gives

$$\left\|\left(\psi - \bar{\phi}\right)(T,\cdot)\right\|_{H^1(0,1)} + \left\|\left(\psi_t - \bar{\phi}_t\right)(T,\cdot)\right\|_{L^2(0,1)} + \|\tilde{y} - \bar{y}\|_{L^2(0,T)}$$
$$\leq \sqrt{15C_2}(\rho^2+1)^{1/2}(T+1)^{1/2}\cdot\|F + G + H\|_{L^2((0,T)\times(0,1))}$$

which gives the claim after straightforward estimation, observing that $C_2 \leq \frac{\rho+1}{\rho}C_\Omega^2$. $\square$

5. A posteriori error estimate

In essence, the following Theorem 5.1 follows from Theorem 4.2 by estimating the functions F, G, and H in terms of ϕ and the problem data.

By $f\big|_\Gamma$ denote the Dirichlet trace of a function f defined on Ω. Define the Hilbert space

$$E(\Omega) = \left\{f \in H^1(\Omega) : \Delta f \in L^2(\Omega), f\big|_\Gamma \in H^1(\Gamma)\right\},$$

equipped with the norm

$$\|f\|^2_{E(\Omega)} = \|f\|^2_{H^1(\Omega)} + \|\Delta f\|^2_{L^2(\Omega)} + \|f\big|_\Gamma\|^2_{H^1(\Gamma)}.$$

Recall that $\Gamma \subset \partial\Omega$ denotes the walls of the tube Ω, excluding the ends $\Gamma(0)$ and $\Gamma(1)$.

Theorem 5.1. *Let the sets Ω, Γ, and $\Gamma(s)$ for $s \in [0,1]$ be defined as in Section 1, and assume that the Standing Assumptions 1 and 2 hold. Make the same assumptions (i)–(v) on signals u, $\bar{u}$, y, $\bar{y}$, $\tilde{y}$, and solutions ϕ, $\bar{\phi}$ as in Theorem 4.2. Denote by $\bar{\bar{\phi}}$ the extension of the averaged solution $\bar{\phi}$ to all of $\overline{\Omega}$, given by*

$$\bar{\bar{\phi}}(\cdot,\mathbf{r}) = \bar{\phi}(\cdot,s) \quad \text{for all} \quad \mathbf{r} \in \overline{\Gamma(s)}, \quad s \in [0,1]. \tag{5.1}$$

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Then the tracking error $e = \psi - \bar{\bar{\phi}}$, as described by the left panel of Fig. 1, is bounded by the inequality

$$\left\| \left(\psi - \bar{\bar{\phi}} \right)(T, \cdot) \right\|_{H^1(0,1)} + \left\| \left(\psi_t - \bar{\bar{\phi}}_t \right)(T, \cdot) \right\|_{L^2(0,1)} + \| \tilde{y} - \bar{y} \|_{L^2(0,T)}$$

$$\leq 4 C_\Omega \rho^{-1/2} (\rho + 1)^{3/2} (T+1)^{1/2}$$

$$\cdot \left(\| A^{-1} \|_{L^\infty(0,1)} \left(\| A' \|_{L^\infty(0,1)} + \| A'' \|_{L^\infty(0,1)} \right) C_\mathcal{F} \left\| \phi - \bar{\bar{\phi}} \right\|_{L^2((0,T);E(\Omega))} \right.$$

$$+ \| \max(\kappa, \kappa') \|^2_{L^\infty(0,1)} C_{\mathcal{H},1} \left\| \nabla \left(\phi - \bar{\bar{\phi}} \right) \right\|_{L^2([0,T] \times \Omega; \mathbb{C}^3)}$$

$$+ \alpha \left(C_\mathcal{G} + C_{\mathcal{H},2} \right) \left\| \left(\phi - \bar{\bar{\phi}} \right)_t \right\|_{L^2((0,T);H^1(\Omega))}$$

$$\left. + \| \kappa \|_{L^\infty(0,1)} C_{\mathcal{H},3} \left\| \Delta \phi - \overline{\overline{\Delta \phi}} \right\|_{L^2([0,T] \times \Omega)} \right)$$

for all $T \geq 0$ where

$$\left(\overline{\overline{\Delta \phi}} \right)(t, \mathbf{r}) := \mathcal{A}(\Delta \phi(t, \cdot))(s) \quad \text{for all} \quad \mathbf{r} \in \Gamma(s), \quad s \in [0,1], \quad \text{and} \quad t \in \overline{\mathbb{R}}^+,$$

the constant C_Ω is given by (4.7), the constants $C_\mathcal{F}$ and $C_\mathcal{G}$ are as given in Proposition 5.3, and the constants $C_{\mathcal{H},1}$, $C_{\mathcal{H},2}$, and $C_{\mathcal{H},3}$ are as given in Proposition 5.4.

All of the constants on the right hand side depend only on the domain Ω.

The proof of this theorem is divided into Propositions 5.2, 5.3, and 5.4. Even though the constants in Theorem 5.1 depend only on the domain Ω, their numerical values are difficult to obtain since they contain, e.g., norms of trace mappings.

For $f \in E(\Omega)$ and $g \in H^1(\Omega)$, define the linear operators

$$(\mathcal{F}f)(s) = A''(s)(\mathcal{A}f - \mathcal{B}f|_\Gamma) + A'(s)\frac{\partial}{\partial s}(\mathcal{A}f - \mathcal{B}f|_\Gamma)$$

$$(\mathcal{G}g)(s) = -\frac{2\pi W(s)}{A(s)} \left(\mathcal{A}g - \mathcal{B}g|_\Gamma \right) \tag{5.2}$$

where we use the two averaging operators that have been introduced in [15]

$$(\mathcal{A}f)(s) := \frac{1}{A(s)} \int_{\Gamma(s)} f \, dA \quad \text{and} \quad (\mathcal{B}g|_\Gamma)(s) := \frac{1}{2\pi} \int_0^{2\pi} g(s, R(s), \theta) \, d\theta$$

for all $s \in (0,1)$.

Proposition 5.2. *The operators defined in (5.2) satisfy $\mathcal{F} \in \mathcal{L}(E(\Omega); L^2(0,1))$ with the estimate*

$$\| \mathcal{F}f \|_{L^2(0,1)} \leq \left(\| A' \|_{L^\infty(0,1)} + \| A'' \|_{L^\infty(0,1)} \right) C_\mathcal{F} \| f \|_{E(\Omega)}$$

and $\mathcal{G} \in \mathcal{L}(E(\Omega); H^1(0,1)) \cap \mathcal{L}(H^1(\Omega); H^{1/2}(0,1))$.

Moreover, $\ker(\mathcal{G}) \cap E(\Omega) \subset \ker(\mathcal{F})$, $\ker(\mathcal{G})$ is closed in $H^1(\Omega)$, and $\ker(\mathcal{F})$ closed in $E(\Omega)$. If $f \in H^1(\Omega)$, and $\bar{\bar{f}}$ is defined by dilation

$$\bar{\bar{f}}(\mathbf{r}) = (\mathcal{A}f)(s) \quad \text{for all} \quad \mathbf{r} \in \Gamma(s) \quad \text{and} \quad s \in (0,1)$$

then $\bar{\bar{f}} \in \ker(\mathcal{G})$. Similarly, if $f \in E(\Omega)$ is such that $\mathcal{A}f \in H^2(0,1)$ then $\bar{\bar{f}} \in \ker(\mathcal{G}) \cap E(\Omega)$.

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Proof. As shown in [15, Propositions 5.2 and 5.3], we have

$$\mathcal{A} \in \mathcal{L}(L^2(\Omega); L^2(0,1)) \cap \mathcal{L}(H^1(\Omega); H^1(0,1)) \text{ and } \mathcal{B} \in \mathcal{L}(H^s(\Gamma); H^s(0,1))$$

for all $s \in \mathbb{R}$. Because the functions $A(\cdot)$ and $W(\cdot)$ are smooth and strictly positive, the norm estimates for $\mathcal{F}$ and $\mathcal{G}$ follow. The claims about the null spaces are evident, apart from the last one.

Since $\bar{\bar{f}}$ is constant on each $\Gamma(s)$ for $s \in (0,1)$, it would follow that the two averages $(\mathcal{A}\bar{\bar{f}})(s)$ and $(\mathcal{B}\bar{\bar{f}}|_\Gamma)(s)$ would coincide for all s. Thus, formally $\bar{\bar{f}} \in \ker(\mathcal{G})$. It remains to show that $\bar{\bar{f}} \in H^1(\Omega)$ if $f \in H^1(\Omega)$, and that $\bar{\bar{f}} \in E(\Omega)$ if $f \in E(\Omega)$ and $\mathcal{A}f \in H^2(0,1)$.

We choose a smooth curve $\mathbf{l} : [0,1] \to \Gamma$ on the tube wall such that $\Gamma(s) \cap \mathbf{l}$ consists of a single point. Then cut the tube wall open along $\mathbf{l}$, and map the surface $\Gamma \setminus \mathbf{l}$ to the unit square $[0,1] \times (0,1)$ by a smooth diffeomorphism, so that the circles $\partial\Gamma(s) \setminus \{\mathbf{l}(s)\}$ map onto $\{s\} \times (0,1)$. Now, it it clear that the extension $\tilde{f}(s,\xi) := \bar{f}(s)$ for $\xi \in (0,1)$ satisfies $\tilde{f} \in H^1((0,1)^2)$ because $f \in H^1(\Omega)$ implies $\bar{f} = \mathcal{A}f \in H^1(0,1)$ by [15, Proposition 5.2]. Thus $\bar{\bar{f}} \in H^1(\Gamma)$ by pullback. By a similar argument, we have $\bar{\bar{f}} \in H^2(\Omega) \subset E(\Omega)$ if $\bar{f} \in H^2(0,1)$ which completes the proof. $\square$

With the help of the operators $\mathcal{F}$ and $\mathcal{G}$, the forcing terms F and G given in (2.3)–(2.4) may be written as

$$F(t,s) = -A(s)^{-1}(\mathcal{F}\phi(t,\cdot))(s) \quad \text{and} \quad G(t,s) = \alpha(\mathcal{G}\phi_t(t,\cdot))(s) \tag{5.3}$$

for all $(t,s) \in \overline{\mathbb{R}}^+ \times (0,1)$.

Proposition 5.3. *Make the same assumption as in Theorem 5.1. The forcing functions F and G, given by* (2.3)–(2.4), *satisfy the estimates*

$$\|F(t,\cdot)\|_{L^2(0,1)} \le \|A^{-1}\|_{L^\infty(0,1)} \left(\|A'\|_{L^\infty(0,1)} + \|A''\|_{L^\infty(0,1)}\right) C_\mathcal{F} \left\|\left(\phi - \bar{\bar{\phi}}\right)(t,\cdot)\right\|_{E(\Omega)} \quad \text{and}$$

$$\|G(t,\cdot)\|_{L^2(0,1)} \le \alpha C_\mathcal{G} \left\|\left(\phi - \bar{\bar{\phi}}\right)_t(t,\cdot)\right\|_{H^1(\Omega)}$$

for all $t \ge 0$ where the constants $C_\mathcal{F}$ and $C_\mathcal{G}$ depend only on the geometry of Ω.

Proof. As shown in [2, Theorem 5.1 and Corollary 5.2], the unique classical solution ϕ of the wave equation (1.1) satisfies the regularity assumptions (1.3). Hence, all results of [15, Section 5] can be used: in particular, that $\phi|_\Gamma \in C(\mathbb{R}_+; H^1(\Gamma))$ by [15, Proposition 5.1] and, hence, $\phi(t,\cdot) \in E(\Omega)$ for all fixed $t \in \overline{\mathbb{R}}^+$. Further, by Lemma 2.2 we get that $\bar{\phi} \in H^2(0,1)$. Together with (5.3), we get

$$\|A(\cdot)F(t,\cdot)\|_{L^2(0,1)} \le \left\|\mathcal{F}\left(\phi(t,\cdot) - \bar{\bar{\phi}}(t,\cdot)\right)\right\|_{L^2(0,1)} + \|\mathcal{F}\bar{\bar{\phi}}(t,\cdot)\|_{L^2(0,1)}$$

$$\le C_\mathcal{F}\left(\|A'\|_{L^\infty(0,1)} + \|A''\|_{L^\infty(0,1)}\right) \cdot \left\|\left(\phi - \bar{\bar{\phi}}\right)(t,\cdot)\right\|_{E(\Omega)}$$

because $\mathcal{F}\bar{\bar{\phi}} \equiv 0$ by Proposition 5.2. The estimate involving G is proved similarly, noting that the dissipativity constant α is always nonnegative. $\square$

It remains to treat the term $H(\cdot)$ given in (2.5):

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Proposition 5.4. *Make the same assumption as in Theorem 5.1. Then the forcing function H, given by (2.5), satisfies the estimate*

$$
\begin{aligned}
\|H(t,\cdot)\|_{L^2(0,1)} &\leq \|\max\left(\kappa,\kappa'\right)\|_{L^\infty(0,1)} C_{\mathcal{H},1} \left\|\nabla\left(\phi - \bar{\bar{\phi}}\right)(t,\cdot)\right\|_{L^2(\Omega;\mathbb{C}^3)} \\
&\quad + \alpha C_{\mathcal{H},2} \left\|\left(\phi - \bar{\bar{\phi}}\right)_t (t,\cdot)\right\|_{H^1(\Omega)} + \|\kappa\|_{L^\infty(0,1)} C_{\mathcal{H},3} \left\|\left(\Delta\phi - \overline{\overline{\Delta\phi}}\right)(t,\cdot)\right\|_{L^2(\Omega)}
\end{aligned}
\tag{5.4}
$$

for all $t \geq 0$ where the constants $C_{\mathcal{H},1}$, $C_{\mathcal{H},2}$, and $C_{\mathcal{H},3}$ depend only on the geometry of Ω.

Proof. Let us begin with the first term in H in (2.5). Denoting by $\bar{\bar{\phi}}$ the extension given in (5.1), we observe by using the gradient formula in [15, Section 2] that $\nabla\bar{\bar{\phi}} = \mathbf{t}(s)\,\Xi\frac{\partial\bar{\bar{\phi}}}{\partial s}$ and $\nabla\left(\Xi^{-1}\right) = -\mathbf{t}(s)\,r\kappa'(s)\cos\theta - \mathbf{n}(s)\,\kappa(s)$. Thus, recalling that $\bar{\bar{\phi}} = \bar{\bar{\phi}}(t,s)$, we get

$$
\int_{\Gamma(s)} \nabla\left(\frac{1}{\Xi}\right)\cdot\nabla\bar{\bar{\phi}}\,\frac{dA}{\Xi} = -\kappa'(s)\frac{\partial\bar{\bar{\phi}}}{\partial s}\int_{\Gamma(s)} r\cos\theta\,dA = 0
$$

where $dA = r\,dr\,d\theta$. Hence, we get by using Hölder's inequality

$$
\begin{aligned}
\left\|\int_{\Gamma(s)} \nabla\left(\frac{1}{\Xi}\right)\cdot\nabla\phi\,\frac{dA}{\Xi}\right\|^2_{L^2(0,1)} &= \left\|\int_{\Gamma(s)} \nabla\left(\frac{1}{\Xi}\right)\cdot\nabla\left(\phi - \bar{\bar{\phi}}\right)\frac{dA}{\Xi}\right\|^2_{L^2(0,1)} \\
&\leq \int_0^1 \left(\int_{\Gamma(s)} \left|\nabla\left(\frac{1}{\Xi}\right)\right|^2\frac{dA}{\Xi}\cdot\int_{\Gamma(s)} \left|\nabla\left(\phi - \bar{\bar{\phi}}\right)\right|^2\frac{dA}{\Xi}\right)ds \\
&\leq \|\max\left(\kappa,\kappa'\right)\|^2_{L^\infty(0,1)} C^2_{\mathcal{H},1}\cdot\int_0^1 \left(\int_{\Gamma(s)} \left|\nabla\left(\phi - \bar{\bar{\phi}}\right)\right|^2\frac{dA}{\Xi}\right)ds \\
&= \|\max\left(\kappa,\kappa'\right)\|^2_{L^\infty(0,1)} C^2_{\mathcal{H},1}\cdot\left\|\nabla\left(\phi - \bar{\bar{\phi}}\right)\right\|^2_{L^2(\Omega;\mathbb{C}^3)}
\end{aligned}
$$

where $C_{\mathcal{H},1} := \sup_{s\in[0,1]} \left(\int_{\Gamma(s)} (r+1)^2\Xi^{-1}\,dA\right)^{1/2}$, since $dV = \Xi^{-1}\,dA\,ds$ and $\left|\nabla\left(\Xi^{-1}\right)\right| \leq \max\left(\kappa,\kappa'\right)(r+1)$.

Let us estimate next the last term in (2.5). Because the function $\bar{\bar{\phi}}_t \equiv \overline{\overline{\phi_t}}$ does not depend on r and θ variables at all, we have

$$
\begin{aligned}
&\left\|\int_0^{2\pi} \phi_t(t,\cdot,R(\cdot),\theta)\cos\theta d\theta\right\|_{L^2(0,1)} \\
&\leq \left\|\int_0^{2\pi} \left(\phi_t - \bar{\bar{\phi}}_t\right)(t,\cdot,R(\cdot),\theta)\cos\theta\,d\theta\right\|_{L^2(0,1)} + \left\|\bar{\bar{\phi}}_t(t,\cdot)\cdot\int_0^{2\pi}\cos\theta\,d\theta\right\|_{L^2(0,1)} \\
&= \left\|\int_0^{2\pi} \left(\phi_t - \bar{\bar{\phi}}_t\right)(t,\cdot,R(\cdot),\theta)\cos\theta\,d\theta\right\|_{L^2(0,1)}.
\end{aligned}
$$

Since the surface element on tube wall Γ is given by $dS = W(s)\,d\theta\,ds$ by [15, Section 2],

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we get for all $t \geq 0$

$$\left\| \frac{\alpha W(\cdot)\eta(\cdot)}{A(\cdot)} \int_0^{2\pi} \phi_t(t,\cdot,R(\cdot),\theta)\cos\theta d\theta \right\|_{L^2(0,1)}^2$$

$$\leq \alpha^2 C_3^2 \int_0^1 \left| \int_0^{2\pi} W(s)^{1/2} \left(\phi_t - \bar{\bar{\phi}}_t \right)(t,s,R(s),\theta)\cos\theta\, d\theta \right|^2 ds$$

$$\leq \pi\alpha^2 C_3^2 \int_0^1 \int_0^{2\pi} \left| \left(\phi_t - \bar{\bar{\phi}}_t \right)(t,s,R(s),\theta) \right|^2 W(s)\, d\theta\, ds$$

$$= \pi\alpha^2 C_3^2 \int_\Gamma \left| \left(\phi_t(t,\cdot) - \bar{\bar{\phi}}_t(t,\cdot) \right)|_\Gamma \right|^2 dS = \pi\alpha^2 C_3^2 \left\| \left(\phi_t(t,\cdot) - \bar{\bar{\phi}}_t(t,\cdot) \right)|_\Gamma \right\|_{L^2(\Gamma)}^2$$

$$\leq \pi\alpha^2 C_3^2 C_4^2 \left\| \left(\phi_t(t,\cdot) - \bar{\bar{\phi}}_t(t,\cdot) \right)|_\Gamma \right\|_{H^{1/2}(\Gamma)}^2 \leq \alpha^2 C_{\mathcal{H},2}^2 \left\| \left(\phi - \bar{\bar{\phi}} \right)_t (t,\cdot) \right\|_{H^1(\Omega)}^2$$

where $C_3 := \sup_{s\in[0,1]} \frac{\eta(s)}{A(s)}$, $C_{\mathcal{H},2} := \pi^{1/2} C_3 C_4 C_5$, and the constants C_4, C_5 are the norms of the inclusion $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ and the trace mapping from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$, respectively.

It remains to treat the second term in H in (2.5). We first observe that the error function $E = E(s,r,\theta)$ introduced in (2.6) averages to zero over each intersectional surface $\Gamma(s)$. We have

$$\int_{\Gamma(s)} E(s,r,\theta)\, dA$$

$$= -2\kappa(s) \int_{\Gamma(s)} r\cos\theta\, dA + \kappa(s)^2 \int_0^{R(s)} \int_0^{2\pi} \left(r^2\cos^2\theta - \frac{R(s)^2}{4} \right) r dr d\theta$$

$$= \kappa(s)^2 \left(\int_0^{R(s)} r^3 dr \int_0^{2\pi} \cos^2\theta\, d\theta - \frac{R(s)^2}{4} \int_0^{R(s)} r dr \int_0^{2\pi} d\theta \right)$$

$$= \kappa(s)^2 \left(\frac{1}{4} R(s)^4 \cdot \int_0^{2\pi} \cos^2\theta\, d\theta - \frac{R(s)^2}{4} \cdot \frac{1}{2} R(s)^2 \cdot 2\pi \right) = 0$$

because $\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$. Considering now the second time derivative $\bar{\bar{\phi}}_{tt} = \overline{\overline{\phi_{tt}}}$ of $\bar{\bar{\phi}}$ in (5.1), we see that also $\bar{\bar{\phi}}_{tt}$ does not depend on the variables r and θ at all. Recalling that ϕ satisfies the wave equation $\Delta\phi = c^{-2}\phi_{tt}$, we get

$$c^2 \int_{\Gamma(s)} E\, \overline{\overline{\Delta\phi}}(t,s)\, dA = \int_{\Gamma(s)} E\, \bar{\bar{\phi}}_{tt}(t,s)\, dA = \bar{\bar{\phi}}_{tt}(t,s) \int_{\Gamma(s)} E\, dA = 0$$

for all $s \in [0,1]$. Hence, we get the estimate

$$\left\| \frac{1}{A(s)} \int_{\Gamma(s)} E\Delta\phi\, dA \right\|_{L^2(0,1)} = \left\| \frac{1}{A(s)} \int_{\Gamma(s)} E\cdot\left(\Delta\phi - \overline{\overline{\Delta\phi}} \right) dA \right\|_{L^2(0,1)}$$

$$\leq \|\kappa\|_{L^\infty(0,1)} \left\| \mathcal{A}\left(\frac{E}{\kappa}\cdot\left(\Delta\phi - \overline{\overline{\Delta\phi}} \right) \right) \right\|_{L^2(0,1)} \leq \|\kappa\|_{L^\infty(0,1)} C_{\mathcal{H},3} \left\| \Delta\phi - \overline{\overline{\Delta\phi}} \right\|_{L^2(\Omega)}$$

where $C_{\mathcal{H},3} := \|\mathcal{A}\|_{\mathcal{L}(L^2(\Omega);L^2(0,1))} \|E/\kappa\|_{L^\infty(\Omega)}$; the boundedness of $\mathcal{A}$ is by [15, Propositions 5.2]. $\qquad\square$

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