On the Geometry of the Hermite – Fejér Interpolation Problem through Conservative Realizations *

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Abstract. In this paper, we give state space realizations for the classical recursive solutions of operator-valued Carathéodory, Nevanlinna – Pick and Hermite – Fejér interpolation problems. These realizations are special in the sense that they satisfy an energy balance law; hence they are called conservative. Observability, controllability and minimality (including the property known as "simplicity") of such realizations are studied, too. Finally, the main result of this paper is given, namely a geometric characterization for the McMillan degree of interpolants.

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Notation

The sets of complex and real numbers are denoted by $\mathbb{C}$ and $\mathbb{R}$, respectively. The right and left half planes are denoted by $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \Re s > 0\}$ and $\mathbb{C}_- := \{s \in \mathbb{C} \mid \Re s < 0\}$. The positive and negative real numbers are $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_- := \{x \in \mathbb{R} \mid x < 0\}$. The imaginary axis is $i\mathbb{R}$. The open unit disc and the unit circle are $\mathbb{D}$ and $\mathbb{T}$, respectively. Natural numbers, integers, nonnegative integers and negative integers are denoted by $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_- := \mathbb{Z} \setminus \mathbb{Z}_+$.

The letters U, X and Y denote separable Hilbert spaces. For any such U, its inner product is denoted by $\langle \cdot, \cdot \rangle_U$, its norm by $\| \cdot \|_U$, and its identity operator by I_U. The closure and the orthogonal complement of any set $S \subset U$ are denoted by $\overline{S}$ and $S^\perp$, respectively. Sometimes we write also $S^\perp = U \ominus S$, to emphasize that the orthogonal complement is to be taken in U.

The (external) orthogonal direct sum of Hilbert spaces X_1 and X_2 is denoted by $\begin{smallmatrix} X_1 \\ \oplus \\ X_2 \end{smallmatrix}$, and it is a Hilbert space with inner product

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle_{\begin{smallmatrix} X_1 \\ \oplus \\ X_2 \end{smallmatrix}} := \langle x_1, z_1 \rangle_{X_1} + \langle x_2, z_2 \rangle_{X_2}.$$

For any Hilbert space U and $d \in \mathbb{N}$, the d-fold (external) orthogonal sum $[U \oplus \cdots \oplus U]^T$ is denoted by U^d, for brevity.

The set of bounded linear operators between Hilbert spaces U and Y is denoted by $\mathcal{L}(U; Y)$, and $\mathcal{L}(X; X) =: \mathcal{L}(X)$. The $\mathcal{L}(U; Y)$-valued bounded analytic functions on $\mathbb{D}$ are $H^\infty(\mathbb{D}; \mathcal{L}(U; Y))$, equipped with norm

$$\|F\|_{H^\infty(\mathbb{D};\mathcal{L}(U;Y))} := \sup_{z \in \mathbb{D}} \|F(z)\|_{\mathcal{L}(U;Y)}.$$

Its unit ball, the $\mathcal{L}(U; Y)$-valued Schur class, is defined by

$$\mathcal{S}(\mathbb{D}; \mathcal{L}(U; Y)) := \{F \in H^\infty(\mathbb{D}; \mathcal{L}(U; Y)) \mid \|F\|_{H^\infty(\mathbb{D};\mathcal{L}(U;Y))} \le 1\}.$$

If $U = Y = \mathbb{C}$, then we write simply $\mathcal{L}(U) = \mathbb{C}$, $H^\infty(\mathbb{D}; \mathcal{L}(U; Y)) = H^\infty(\mathbb{D})$, and $\mathcal{S}(\mathbb{D}; \mathcal{L}(U; Y)) = \mathcal{S}(\mathbb{D})$.

1 Introduction

The classical (scalar) *Carathéodory interpolation (or extension) problem* is formulated as follows: Given the *Carathéodory data* $\underline{w} := \{w_k\}_{k=0}^d \subset \mathbb{C}$, find necessary and sufficient conditions for the existence of a Schur function $F \in \mathcal{S}(\mathbb{D})$, called the *interpolant*, whose Taylor series are of the form

$$F(z) = w_0 + w_1 z + \cdots + w_d z^d + \mathcal{O}(z^{d+1}). \tag{1}$$

Necessary and sufficient conditions for the solvability of the Carathéodory interpolation problem can be found e.g. in [12, Theorem 1.5]. Furthermore, when such solvability conditions are satisfied, the set of all interpolants $F \in \mathcal{S}(\mathbb{D})$ is to be parameterized. Classical references to various methods for the solution of the Carathéodory interpolation problem are [1, 8, 9, 22, 23, 28], and more modern treatments are [3, 20].

We shall assume henceforth from the data $\underline{w}$ that the interpolation problem has *more than one* (then, in fact, infinitely many) solutions. Necessary and sufficient conditions for this to happen can be found in [12, Theorem 1.5], too. We now describe superficially the original approach by Schur, give some names to things, and explain the techniques and purposes of this paper. Schur presented a recursive algorithm, comprising in the generic case $d+1$ *forward steps*, followed by equally many *backward steps*.

Let us first outline the recursive forward steps, parameterized by $j = 1, \ldots, d+1$. The recursion is started with the original data $\underline{v}_0 := \underline{w} = \{w_k\}_{k=0}^d$ of length $d+1$, and it is terminated when all the data has been depleted after $d+1$ steps[1]. (The precise definition of these forward steps is immaterial for now, and it will be given later.) After having computed the jth forward step, the old Carathéodory data $\underline{v}_{j-1}$ of length $d-j+2$ has been replaced by the new, updated Carathéodory data $\underline{v}_j$ of length $d-j+1$.

Indeed, the updated data $\underline{v}_j$ defines another Carathéodory interpolation problem, but this "new problem" is "easier" than any of the "old problems", as it has fewer interpolation conditions. When all $d+1$ forward steps have been taken, a trivial interpolation problem (with an empty Carathéodory data $\underline{v}_{d+1} = \emptyset$) is obtained. As no interpolation conditions are imposed, any function $g \in \mathcal{S}(\mathbb{D})$ is its solution.

In the course of carrying out the forward steps, a sequence of scalars $\underline{r} = \{r_j\}_{j=0}^d$ is extracted from the original Carathéodory data $\underline{w}$. The elements of $\underline{r}$ are called the *reflection coefficients* or *Schur parameters*. In the context of this paper, we shall always assume that these parameters satisfy $|r_j| < 1$

[1]...or the algorithm becomes impossible to continue earlier at some step, in which case infinitely many solutions do not exist. We assume that this situation does not occur.

for all $j = 0, \ldots, d$. In this case (and only in this case), the Carathéodory interpolation problem has more than one (hence, infinitely many) solutions.

The latter part – *the backward steps* – of the Schur algorithm provides us with all the interpolants solving the problem. For a Schur parameter r_{j-1}, the corresponding backward step is defined by

$$F_{j-1}(z) = \omega_{r_{j-1}}\left(zF_j(z)\right) \text{ where } \omega_r(z) := \frac{z+r}{1+\overline{r}z}, \quad z \in \mathbb{D}. \tag{2}$$

It is easy to see that

$$F_j \in \mathcal{S}(\mathbb{D}) \quad \text{if and only if} \quad F_{j-1} \in \mathcal{S}(\mathbb{D}) \text{ and } F_{j-1}(0) = r_{j-1}.$$

After all $d+1$ backward steps, we recover the interpolant from

$$F_{g,\underline{r}}(z) := F_0(z) = \omega_{r_0}\left(z\omega_{r_1}\left(\cdots, \left(\omega_{r_d}\left(zg(z)\right)\right)\cdots\right)\right), \quad z \in \mathbb{D}. \tag{3}$$

As we see, the interpolant depends on the arbitrarily chosen function $g \in \mathcal{S}(\mathbb{D})$ that is used as the initial condition $F_{d+1} = g$ for computing the backward steps (2) recursively. The function g is called the *free parameter*, defining the interpolant $F_{g,\underline{r}}$. It is well known that the full solution set of the Carathéodory interpolation problem are obtained by varying g in the set $\mathcal{S}(\mathbb{D})$.

The purpose of this paper is to present a state space realization theory (of a rather particular kind) for the solutions $F_{g,\underline{r}}$ of a number of interpolation problems, including the Carathéodory problem. Using this theory, we proceed to characterize the interpolants having a finite McMillan degree; see the main result of this paper, Theorem 5.1.

More precisely, we want to write the free parameter $g \in \mathcal{S}(\mathbb{D})$ (corresponding to the interpolant $F_{g,\underline{r}} \in \mathcal{S}(\mathbb{D})$) as a *transfer function* of a discrete time linear system (shortly: DLS) ϕ, described in the scalar case by the difference equations

$$\phi : \begin{cases} x_{j+1} &= Ax_j + bu_j \\ y_j &= \langle c, x_j \rangle_X + du_j, \quad j \geq 0. \end{cases} \tag{4}$$

Here X is a separable Hilbert space, $A \in \mathcal{L}(X)$, b, $c \in X$ and $d \in \mathbb{C}$. The sequence $\{u_j\}_{j\geq 0} \subset \mathbb{C}$ is the *input*, $\{x_j\}_{j\geq 0} \subset X$ is the *state* and $\{y_j\}_{j\geq 0} \subset \mathbb{C}$ is the *output* of the system. The operator A in (4) is called the *main operator* of ϕ. The transfer function of ϕ is defined as

$$\widehat{\mathcal{D}}(z) := d + z\left\langle c, (I - zA)^{-1}b\right\rangle_X \quad \text{for} \quad z \in \mathbb{D},$$

and the linear system ϕ is called the *state space realization* of $\widehat{\mathcal{D}}$. Moreover, the linear system ϕ is called *energy preserving* if the energy balance equations

$$\|x_{j+1}\|_X^2 - \|x_j\|_X^2 = |u_j|^2 - |y_j|^2, \quad j \geq 0$$

hold for any initial value $x_0 \in X$ and input $\{u_j\} \subset \mathbb{C}$, where x_j, u_j and y_j satisfy (4). The *dual system* ϕ^d of ϕ is described by the difference equations

$$\phi^d : \begin{cases} z_{j+1} &= A^* x_j + c v_j \\ w_j &= \langle b, x_j \rangle_X + \overline{d} v_j, \quad j \geq 0. \end{cases}$$

A system ϕ is called *conservative*, if both ϕ and ϕ^d are energy preserving. For any conservative linear system ϕ, it is well known that the structure of the main operator A is completely determined (apart from a unitary change of coordinates in state space X) by the transfer function $\widehat{\mathcal{D}}$, provided that A is completely nonunitary (c.n.u.), see e.g. [5]. Such conservative systems ϕ are called *simple*. Moreover, a complex-valued analytic function F is a transfer function of a (simple) conservative system if and only if $F \in \mathcal{S}(\mathbb{D})$, see e.g. [5].

What does all this have to do with the Schur algorithm for the Carathéodory interpolation problem? As already mentioned, the free parameter $g \in \mathcal{S}(\mathbb{D})$ (appearing in (3)) can be written as a transfer function of a (simple) conservative linear system ϕ. We shall show that each of the backward steps (as described in (2)) can be computed by using *conservative realizations* ϕ_{j-1} and ϕ_j of analytic functions F_{j-1} and F_j, instead of using these functions alone as is done in the classical approach. After each backward step, the updated realization ϕ_{j-1} for F_{j-1} is conservative, provided that the original realization ϕ_j for F_j is conservative.

Hence, starting from a conservative realization ϕ_{d+1} of the free parameter $F_{d+1} = g \in \mathcal{S}(\mathbb{D})$, we finally obtain an explicit formula for a *conservative* realization ϕ_0 of the interpolant $F_0 = F_{g,\underline{r}} \in \mathcal{S}(\mathbb{D})$. Since the theory of conservative systems is much richer than that of general linear systems, it is possible to give a number of results (like those appearing in Sections 4 and 5 of this paper) that do not hold for more general realizations of interpolants.

It is well known that the problem of Carathéodory is a special case of a more general interpolation problem, called the *Hermite – Fejér interpolation problem*, see [12, page 298]. All results of this paper will be given for this most general class of interpolations problems (including the Nevanlinna – Pick interpolation) for general operator-valued interpolants.

We finally remark that the techniques used in this paper bear a striking resemblance to the dilation theory for Hilbert space contractions, culminating in the famous commutant lifting theorem by Sarason, see e.g. in [25, 12]. Due to the generality and enormous size of the dilation theory, we shall not try to explain this connection any further here. The present system theory setting can be defended by its familiarity to the system theory community and authors, if not by any other reasons.

2 State space Carathéodory interpolation

The operator-valued Carathéodory interpolation problem is analogously defined as the scalar problem, given in Section 1. The Carathéodory data of the problem is $\underline{W} := \{W_k\}_{k=0}^d$, where each W_k is now a bounded linear operator in $\mathcal{L}(U)$. We ask for the necessary and sufficient condition for the existence of an $F \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$, such that the Taylor series of F satisfy

$$F(z) = W_0 + W_1 z + \cdots + W_d z^d + \mathcal{O}(z^{d+1}). \tag{5}$$

Moreover, when the solution set is nonempty, all the solutions F are to be parameterized. We proceed to make some definitions, following [10].

For a strict contraction $R \in \mathcal{L}(U)$, $\|R\|_{\mathcal{L}(U)} < 1$, we define the self-adjoint, boundedly invertible *defect operators* as

$$D_R := (I - R^* R)^{1/2}, \quad D_{R^*} := (I - R R^*)^{1/2}.$$

Given such R and a function $F_j \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ with $F_j(0) = R$, the *forward step* for the Schur recursion with respect to R is defined by

$$F_{j+1}(z) := \frac{1}{z} D_{R^*} \left(I - F_j(z) R^*\right)^{-1} \left(F_j(z) - R\right) D_R^{-1}, \quad z \in \mathbb{D}. \tag{6}$$

Proposition 2.1. *The following claims hold:*

(i) Assume that $F_j \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ and F_{j+1} is given by (6)*. Then $F_{j+1} \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ if and only if $F_j(0) = R$. When these equivalent conditions hold, then for all $z \in \mathbb{D}$,*

$$F_j(z) = \left(D_{R^*}^{-1} \cdot z F_{j+1}(z) + R D_R^{-1}\right) \left(D_R^{-1} + R^* D_{R^*}^{-1} \cdot z F_{j+1}(z)\right)^{-1}. \tag{7}$$

(ii) Assume that $F_{j+1} \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$, and let F_j be given by (7)*. Then $F_j \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ and $F_j(0) = R$.*

Note that as $\left(D_R^{-1} + R^* D_{R^*}^{-1} z F_{j+1}(z)\right) = D_R^{-1} \left(I + R^* z F_{j+1}(z)\right)$, the inverse in (7) exists boundedly if $F_{j+1} \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ and $\|R\|_{\mathcal{L}(U)} < 1$.

Proof. We give only an outline of the proof. To prove claim (i), define $G(z) := zF_{j+1}(z)$. We first show that G is in $\mathcal{S}(\mathbb{D}; \mathcal{L}(U))$. Because $\|R\|_{\mathcal{L}(U)} < 1$, G is analytic in $\mathbb{D}$. By some computation (or by looking it up in [10]),

$$\begin{aligned} &I - G(z)G(z)^* \\ &= D_{R^*} \left(I - F_j(z)R^*\right)^{-1} \left(I - F_j(z)F_j(z)^*\right) \left(I - RF_j(z)^*\right)^{-1} D_{R^*}, \end{aligned} \tag{8}$$

for all $z \in \mathbb{D}$. As $F_j \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$, $I - G(z)G(z)^* \geq 0$ for all $z \in \mathbb{D}$ and hence $G \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$.

Now, supposing $F_j(0) = R$, we shall show that $F_{j+1} \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$. If $F_j(0) = R$, then $F_{j+1}(z) = G(z)/z$ is analytic in $\mathbb{D}$. For all $0 < r < 1$ and $0 \leq \theta < 2\pi$, we have for $z = re^{i\theta}$,

$$\|F_{j+1}(z)\|_{\mathcal{L}(U)} \leq \sup_{z\in\mathbb{D}} \|G(z)\|_{\mathcal{L}(U)} \cdot \frac{1}{r} \leq \frac{1}{r}.$$

By the Maximum Modulus Theorem, $\sup_{z\in r\mathbb{D}} \|F_{j+1}(z)\|_{\mathcal{L}(U)} \leq 1/r$, and the claim follows by letting $r \to 1-$. The converse direction is trivial.

To prove claim (ii), note that $F_j(0) = R$ follows trivially. Denoting again $G(z) := zF_{j+1}(z)$, we have $G \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$. Rewriting and adjoining (7) gives $F_j(z)^* = D_R \left(I + G(z)^*R\right)^{-1} \left(G(z)^* + R^*\right) D_{R^*}^{-1}$, and we note that $F_j(z)^*$ depends on $G(z)^*$ in essentially the same way as $zF_{j+1}(z)$ depends on $F_j(z)$ in (6). Now it is quite easy to conclude from identity (8) (by making proper replacements) that for all $z \in \mathbb{D}$

$$\begin{aligned} &I - F_j(z)^*F_j(z) \\ &= D_R \left(I + G^*(z)R\right)^{-1} \left(I - G(z)^*G(z)\right) \left(I + R^*G(z)\right)^{-1} D_R, \end{aligned} \tag{9}$$

thus proving the claim. ∎

The mapping $F_{j+1} \mapsto F_j$, defined by (7), is called the *backward step* for the Schur recursion with respect to a strictly contractive $R \in \mathcal{L}(U)$. This mapping is denoted by $\hat{T}_R$; i.e., $F_j = \hat{T}_R(F_{j+1})$. Claim (ii) of Proposition 2.1 says that $\hat{T}_R(\mathcal{S}(\mathbb{D}; \mathcal{L}(U))) \subset \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$.

Given the Carathéodory data $\{W_k\}_{k=0}^d$, the corresponding *Schur parameters* $\underline{R} := \{R_k\}_{k=0}^d \subset \mathcal{L}(U)$ are defined recursively as follows:

- For $j = 0$, define $R_0 := W_0$ and take some $F_0 \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ of form (5)[2].

[2] We need one solution F_0 of the interpolation problem to initialize the recursion, but the computed Schur parameters will not depend on the choice of this initial value. In practical computations, one would not be able to find such F_0 before solving the interpolation problem. Nevertheless, the Schur parameters can be computed algorithmically, see [12, Chapter 1] for the scalar case.

- Assume that $0 \leq j < n$, and both $\{R_k\}_{k=0}^{j} \subset \mathcal{L}(U)$ (satisfying $\|R_k\|_{\mathcal{L}(U)} < 1$) and $\{F_k\}_{k=0}^{j} \subset \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ have already been computed. Then $F_{j+1} \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ is defined by (6) with $R = R_j$. Moreover, $R_{j+1} := F_{j+1}(0)$.

Indeed, as $R_j = F_j(0)$ and $F_j \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$, it follows from Proposition 2.1 that the updated F_{j+1} is in $\mathcal{S}(\mathbb{D}; \mathcal{L}(U))$. However, we have to *explicitly* assume at the $(j+1)$th recursive step, the new Schur parameter R_{j+1} satisfies $\|R_{j+1}\|_{\mathcal{L}(U)} < 1$. Otherwise, we might not be able to compute the following step in the recursion.

Definition 2.2. We say that the Carathéodory data $\underline{W} := \{W_k\}_{k=0}^{d}$ is *regular* if there exist operators $\underline{R} := \{R_k\}_{k=0}^{d} \subset \mathcal{L}(U)$ with $\|R_k\|_{\mathcal{L}(U)} < 1$, such that these operators appear as R_k's in the above recursion.

We shall henceforth make it a standing assumption that the Carathéodory data $\underline{W}$ is regular, and hence has a full set of Schur parameters $\underline{R}$.

As in the scalar case (see (3)), any function $F \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ of form (5) satisfies $F = F_{G,\underline{R}}$, where

$$F_{G,\underline{R}} = (\hat{T}_{R_0} \circ \hat{T}_{R_1} \circ \cdots \circ \hat{T}_{R_d})(G) \tag{10}$$

for some $G \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$. Conversely, each such $F_{G,\underline{R}}$ belongs to $\mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ and has the power series of form (5). For the matrix-valued case, see [10].

We proceed to define an extended version of the nonlinear mapping $\hat{T}_R$, appropriate for state space techniques. The idea is to write the functions F and $\tilde{F} := \hat{T}_R(F) \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ as transfer functions of conservative discrete time linear systems (DLSs) $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\tilde{\phi} := \left(\begin{smallmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{smallmatrix}\right)$; i.e. for all $z \in \mathbb{D}$ we have

$$F(z) = D + zC(I - zA)^{-1}B \quad \text{and} \quad \tilde{F}(z) = \tilde{D} + z\tilde{C}(I - z\tilde{A})^{-1}\tilde{B}.$$

Definition 2.3. Let $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS whose state space is X and both the input and output spaces equal U. Let $R \in \mathcal{L}(U)$ be a strict contraction. Then the nonlinear mapping T_R is defined by $\tilde{\phi} = T_R(\phi)$, where

$$\tilde{\phi} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} := \begin{pmatrix} \begin{bmatrix} -DR^* & C \\ -BR^* & A \end{bmatrix} & \begin{bmatrix} DD_R \\ BD_R \end{bmatrix} \\ \begin{bmatrix} D_{R^*} & 0 \end{bmatrix} & R \end{pmatrix}.$$

Note that the state space of $\tilde{\phi}$ is $\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}$ (the external orthogonal direct sum of U and X), but both the input and output spaces equal the input space U of ϕ.

Proposition 2.4. *Let ϕ and $\tilde{\phi}$ be two DLSs with transfer functions $\widehat{\mathcal{D}}_\phi$ and $\widehat{\mathcal{D}}_{\tilde{\phi}}$. Assume that $\tilde{\phi} = T_R(\phi)$ for some strict contraction $R \in \mathcal{L}(U)$. Then the following holds:*

(i) The DLS $\tilde{\phi}$ is conservative if and only if ϕ is.

(ii) Assume, in addition, that $\widehat{\mathcal{D}}_\phi \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$. Then $\widehat{\mathcal{D}}_{\tilde{\phi}} = \hat{T}_R \widehat{\mathcal{D}}_\phi$, where the mapping $\hat{T}_R$ is defined after Proposition 2.1.

Proof. Recall that $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ is conservative if and only if the block matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$ is unitary. It is easy to see that

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = P \begin{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & I_U \end{bmatrix} \begin{bmatrix} \begin{bmatrix} I_X & 0 \\ 0 & -R^* \end{bmatrix} & \begin{bmatrix} 0 \\ D_R \end{bmatrix} \\ \begin{bmatrix} 0 & D_{R^*} \end{bmatrix} & R \end{bmatrix} P', \tag{11}$$

where we use the permutations $P := \left[\begin{smallmatrix} 0 & I_U & 0 \\ I_X & 0 & 0 \\ 0 & 0 & I_U \end{smallmatrix} \right]$ and $P' := \left[\begin{smallmatrix} 0 & I_X & 0 \\ I_U & 0 & 0 \\ 0 & 0 & I_U \end{smallmatrix} \right]$. Assertion (i) follows immediately from (11), since the rotation matrix $\left[\begin{smallmatrix} -R^* & D_R \\ D_{R^*} & R \end{smallmatrix} \right]$ is unitary for any contraction R.

To prove the latter claim (ii), we show that (7) holds with $F_{j+1} = \widehat{\mathcal{D}}_\phi$ and $F_j = \widehat{\mathcal{D}}_{\tilde{\phi}}$, or equivalently for all $z \in \mathbb{D}$

$$\widehat{\mathcal{D}}_{\tilde{\phi}}(z) \left(D_R^{-1} + R^* D_{R^*}^{-1} z \widehat{\mathcal{D}}_\phi(z) \right) = \left(D_{R^*}^{-1} z \widehat{\mathcal{D}}_\phi(z) + R D_R^{-1} \right).$$

To verify this, it appears to be enough to show that for all $z \in \mathbb{D}$ and $u, y, v, w \in H^2(\mathbb{D}; U)$, the identity

$$\begin{cases} \begin{bmatrix} y(z) \\ u(z) \end{bmatrix} & = \begin{bmatrix} D_{R^*}^{-1} z & R D_R^{-1} \\ R^* D_{R^*}^{-1} z & D_R^{-1} \end{bmatrix} \begin{bmatrix} w(z) \\ v(z) \end{bmatrix}, \\ w(z) & = \widehat{\mathcal{D}}_\phi(z) v(z), \end{cases} \tag{12}$$

implies $y(z) = \widehat{\mathcal{D}}_{\tilde{\phi}}(z) u(z)$. Indeed, for any $u \in H^2(\mathbb{D}; U)$ we can find unique $y, w, v \in H^2(\mathbb{D}; U)$ such that (12) holds; but this requires some computations, using the assumptions that $\widehat{\mathcal{D}}_\phi \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ and $\|R\|_{\mathcal{L}(U)} < 1$. For example,

$$v(z) = \left(I + z R^* \widehat{\mathcal{D}}_\phi(z) \right)^{-1} D_R u(z) \quad \text{and}$$
$$y(z) = \left[I + R - \left(I + z R^* \widehat{\mathcal{D}}_\phi(z) \right)^{-1} \right] u(z).$$

By using the Z-transforms $u(z) = \sum_{k\geq 0} u_k z^k$ and $y(z) = \sum_{k\geq 0} y_k z^k$, both the identities in (12) imply the state space difference equations (solved for $k \geq 0$ with $x_0 = 0$ and $z_0 = 0$)

$$\begin{cases} \begin{bmatrix} x_{k+1} \\ y_k \\ u_k \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ D_{R^*}^{-1} & 0 & RD_R^{-1} \\ R^* D_{R^*}^{-1} & 0 & D_R^{-1} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ v_k \end{bmatrix}, \\ \begin{bmatrix} z_{k+1} \\ w_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z_k \\ v_k \end{bmatrix}, \end{cases} \tag{13}$$

by recalling that $\widehat{\mathcal{D}}_\phi$ is the transfer function of DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$.

By eliminating the variables $v_k = -R^* x_k + D_R u_k$ and $w_k = -DR^* x_k + C z_k + DD_R u_k$ from (13), we obtain

$$\begin{bmatrix} \begin{bmatrix} x_{k+1} \\ z_{k+1} \end{bmatrix} \\ y_k \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_k \\ z_k \end{bmatrix} \\ u_k \end{bmatrix},$$

where the operators $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are given by Definition 2.3. Hence $y(z) = \widehat{\mathcal{D}}_{\tilde{\phi}}(z) u(z)$ for all $z \in \mathbb{D}$, and the proof is complete. □

A mapping roughly analogous to T_R is called *diagonal transformation* in [24]. The mapping $\hat{T}_R : F_{j+1} \mapsto F_j$ (connected to $T_R(\phi)$ as in claim (ii) of the previous proposition) can be described by the feedback connection:

Indeed, the transfer function $u \mapsto y$ equals F_j, see (7).

The following theorem parameterizes the solution set of the Carathéodory interpolation problem, using a family of conservative realizations as the parameter set.

Theorem 2.5. *Assume that the Carathéodory data $\{W_k\}_{k=0}^d \subset \mathcal{L}(U)$ is regular in the sense of Definition 2.2, with Schur parameters $\underline{R} = \{R_k\}_{k=0}^d$. Then the following holds:*

(i) For any conservative DLS ϕ (with state space X and both input and output spaces equal to U), the transfer function $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{R})} \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ of

$$\tilde{\phi}(\phi,\underline{R}) := (T_{R_0} \circ T_{R_1} \circ \cdots \circ T_{R_d})(\phi) \tag{14}$$

is a solution of the Carathéodory extension problem described by (5). *Moreover, the DLS $\tilde{\phi}(\phi,\underline{R})$ is conservative, and $\widehat{\mathcal{D}}_{\phi} \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ is the free parameter associated to the interpolant $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{R})}$.*

(ii) Conversely, any solution $F \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ of the Carathéodory extension problem satisfies $F = \widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{R})}$ for $\tilde{\phi}(\phi,\underline{R})$ given by (14) *for some (simple) conservative DLS ϕ (with both input and output spaces equal to U).*

(iii) If, in addition, DLS ϕ is a simple conservative DLS, then so is $\tilde{\phi}(\phi,\underline{R})$.

Proof. Both the claim (i) and (ii) follow directly from the general discussion in the beginning of this section, together with Propositions 2.4 and A.7.

Let us prove claim (iii). Assume that ϕ is conservative, A is a c.n.u. contraction (see Appendix) and $\|R\|_{\mathcal{L}(U)} < 1$. We shall show that $\tilde{A} := \left[\begin{smallmatrix} -DR^* & C \\ -BR^* & A \end{smallmatrix}\right]$ is c.n.u., too. For contradiction, assume that there exists a nontrivial reducing subspace $V \subset \begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}$ for $\tilde{A}$, on which $\tilde{A}$ is unitary. Let $v = \left[\begin{smallmatrix} u \\ x \end{smallmatrix}\right] \in V$ such that $u \neq 0$. Then by the conservativity of ϕ and the strict contractivity of R,

$$\begin{aligned}\|\tilde{A}v\|_{\substack{U \\ \oplus \\ X}} &= \left\| \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} R^* & 0 \\ 0 & I_X \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}} \\ &= \left\| \begin{bmatrix} R^* & 0 \\ 0 & I_X \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}} < \left\| \begin{bmatrix} u \\ x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}},\end{aligned}$$

thus contradicting the fact that $\tilde{A}$ is unitary on V. Hence, $u = 0$ and $V = \begin{smallmatrix} \{0\} \\ \oplus \\ V' \end{smallmatrix}$ for some $V' \subset X$.

For the rest of this proof, we use the splitting $X = \begin{smallmatrix} V'^{\perp} \\ \oplus \\ V' \end{smallmatrix}$. Because V is $\tilde{A}$-invariant, it follows that $AV' \subset V'$; i.e. in block matrix form

$$\tilde{A} = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ \alpha & \beta & A|V' \end{bmatrix} : \begin{smallmatrix} U \\ \oplus \\ V'^{\perp} \\ \oplus \\ V' \end{smallmatrix} \to \begin{smallmatrix} U \\ \oplus \\ V'^{\perp} \\ \oplus \\ V' \end{smallmatrix}$$

for some contractions α and β, where the symbol $*$ denotes an irrelevant entry. But as $V = \left[\{0\} \oplus \{0\} \oplus V'\right]^T$ is reducing for $\tilde{A}$, we have $\alpha = 0$ and

$\beta = 0$. We conclude that V' is a reducing subspace for A, on which A operates unitarily. As A is c.n.u., we have $V' = \{0\}$ and hence V is a trivial subspace. This proves the claim. $\square$

So, by Theorem 2.5, we are able to obtain conservative realizations $\tilde{\phi}$ for interpolant $F_{G,\underline{R}}$, provided that the free parameter $G \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ is first realized by a conservative DLS. Simple conservative realizations are a very restrictive class of realizations for Schur functions, and for that reason they have much more mathematical properties than general realizations. Many practical computations turn out to have unexpectedly simple results, as various cancellations in formulas take place, see for example the results in Sections 4 and 5.

We conclude this section by showing that the Schur parameters $\underline{R} = \{R_k\}_{k=0}^{d}$ define (generalized) rotations in the state space $\begin{smallmatrix} U^d \\ \oplus \\ X \end{smallmatrix}$ of $\tilde{\phi}(\phi, \underline{R})$. This is done by giving a matrix product formula for the backward steps of the Schur algorithm.

Proposition 2.6. *Assume that the Carathéodory data $\{W_k\}_{k=0}^{d} \subset \mathcal{L}(U)$ is regular in the sense of Definition 2.2, with the Schur parameters $\underline{R} = \{R_k\}_{k=0}^{d}$. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a conservative DLS (with state space X and both input and output spaces equal to U), and define $\tilde{\phi}(\phi, \underline{R}) = \left(\begin{smallmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{smallmatrix}\right)$ by (14). Then*

$$\begin{bmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{bmatrix} = \begin{bmatrix} I_{U^{d+1}} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} M_d(R_d) M_{d-1}(R_{d-1}) \cdots M_0(R_0),$$

where

$$M_k(R_k) := \begin{bmatrix} I_{U^k} & 0 & 0 & 0 \\ 0 & R_k & D_{R_k^*} & 0 \\ 0 & D_{R_k} & -R_k^* & 0 \\ 0 & 0 & 0 & I_{\begin{smallmatrix} U^{d-k} \\ \oplus \\ X \end{smallmatrix}} \end{bmatrix}.$$

Proof. This can be verified by first rewriting (11) and then using it recursively. More precisely, note that (11) is equivalent to

$$\begin{bmatrix} D_{d-1} & C_{d-1} \\ B_{d-1} & A_{d-1} \end{bmatrix} = \begin{bmatrix} I_U & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{bmatrix} R_d & D_{R_d^*} & 0 \\ D_{R_d} & -R_d^* & 0 \\ 0 & 0 & I_X \end{bmatrix}, \tag{15}$$

where the state space of the new DLS $T_{R_d}(\phi) = \left(\begin{smallmatrix} A_{d-1} & B_{d-1} \\ C_{d-1} & D_{d-1} \end{smallmatrix}\right)$ thus obtained is $\overset{U}{\underset{X}{\oplus}}$. Augmenting the identity operator I_U to (15) gives

$$\begin{bmatrix} I_U & 0 & 0 \\ 0 & D_{d-1} & C_{d-1} \\ 0 & B_{d-1} & A_{d-1} \end{bmatrix} = \begin{bmatrix} I_{U^2} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{bmatrix} I_U & 0 & 0 & 0 \\ 0 & R_d & D_{R_d^*} & 0 \\ 0 & D_{R_d} & -R_d^* & 0 \\ 0 & 0 & 0 & I_X \end{bmatrix},$$

and applying (15) to this gives

$$\begin{bmatrix} D_{d-2} & C_{d-2} \\ B_{d-2} & A_{d-2} \end{bmatrix}$$
$$= \begin{bmatrix} I_{U^2} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{bmatrix} I_U & 0 & 0 & 0 \\ 0 & R_d & D_{R_d^*} & 0 \\ 0 & D_{R_d} & -R_d^* & 0 \\ 0 & 0 & 0 & I_X \end{bmatrix} \begin{bmatrix} R_{d-1} & D_{R_{d-1}^*} & 0 \\ D_{R_{d-1}} & -R_{d-1}^* & 0 \\ 0 & 0 & I_{\overset{U}{\underset{X}{\oplus}}} \end{bmatrix}.$$

Continuing this process all $d+1$ steps will prove the claim. $\square$

3 State space Hermite – Fejér interpolation

In this section, we shall treat a quite general interpolation problem, called *Hermite – Fejér interpolation problem.* This problem is described as follows (see also e.g. [12, p. 298]): Given the data

$$\left\{ \left(z_0, (W_0^{(0)}, \cdots, W_0^{(d_0)}) \right), \cdots, \left(z_n, \left(W_n^{(0)}, \cdots, W_n^{(d_n)} \right) \right) \right\}, \qquad (16)$$

where $z_k \in \mathbb{D}$ and $W_k^{(l)} \in \mathcal{L}(U)$ for $k = 0, 1, \ldots, n$ and $l = 0, 1, \ldots, d_k$, find necessary and sufficient conditions for the existence of an $F \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ such that, at each z_k, $k = 0, 1, \ldots, n$, the power series of F are of form

$$F(z) = W_k^{(0)} + W_k^{(1)}(z - z_k) + \cdots + W_k^{(d_k)}(z - z_k)^{d_k} + \mathcal{O}(z^{d_k+1}). \qquad (17)$$

When the solution set is nonempty, all such solutions $F \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ should be parameterized.

Remark 3.1. It is worthwhile to mention that Carathéodory and Nevanlinna – Pick interpolation problems are two special cases of Hermite – Fejér problem. Indeed, we obtain the Carathéodory interpolation problem when $n = 0$, $z_0 = 0$ and $d_0 = d$. The Nevanlinna – Pick interpolation problem occurs when $d_k = 0$ for all $k = 0, \ldots, n$.

The necessary and sufficient conditions for solvability of each of these problems are classical (see [12, 20]). It is well-known that, in the non-degenerate cases, the solution set to the general Hermite – Fejér interpolation problem can be obtained by a recursive algorithm, just as in the case of the Carathéodory problem. In this section, we reformulate (the latter part of) this recursive solution in terms of conservative realizations.

3.1 Nevanlinna – Pick interpolation

Let us outline the recursive process leading to the solution of Nevanlinna – Pick problem. We assume that the interpolation values of the problem, denoted by $\underline{W}_0 := \{W_k\}_{k=0}^n \subset \mathcal{L}(U)$, satisfy $\|W_k\|_{\mathcal{L}(U)} < 1$. We say that $F \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ solves the Nevanlinna – Pick interpolation problem, if

$$F(z_k) = W_k \quad \text{for all } k = 0, \ldots, n, \tag{18}$$

where $\underline{z}_0 := \{z_k\}_{k=0}^n \subset \mathbb{D}$ are the interpolation points. The ordered pair $(\underline{z}_0, \underline{W}_0)$ is called from now on *Nevanlinna – Pick data.*

We now proceed to describe the $n+1$ forward steps, followed by as many backward steps. In contrast to the forward step (6) for the Carathéodory problem, now the forward step consists of two operations. One of these operations is the composition operator $\hat{V}_\alpha : \mathcal{S}(\mathbb{D}; \mathcal{L}(U)) \to \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$, defined for any $\alpha \in \mathbb{D}$ by

$$(\hat{V}_\alpha F)(z) := F\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right). \tag{19}$$

It is easy to see that $\hat{V}_\alpha$, indeed, maps $\mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ onto itself, and that $\hat{V}_\alpha^{-1} = \hat{V}_{-\alpha}$.

To describe the forward steps, we shall need further sets of interpolation points and interpolation values, defined recursively as follows: Given $\underline{z}_j = \{z_{j,k}\}_{k=j}^n$ and $\underline{W}_j = \{W_{j,k}\}_{k=j}^n$ for $j < n$, we define

$$\underline{z}_{j+1} := \{z_{j+1,k}\}_{k=j+1}^n, \quad \underline{W}_{j+1} := \{W_{j+1,k}\}_{k=j+1}^n,$$

where for $k = j+1, \ldots, n$,

$$\begin{aligned} z_{j+1,k} &:= \frac{z_{j,k} - z_{j,j}}{1 - \overline{z_{j,j}} z_{j,k}}, \\ W_{j+1,k} &:= \frac{1}{z_{j+1,k}} D_{W_{j,j}^*} (I - W_{j,k} W_{j,j}^*)^{-1} (W_{j,k} - W_{j,j}) D_{W_{j,j}}^{-1}. \end{aligned} \tag{20}$$

The recursions are started with initial values $\underline{z}_0 := \{z_k\}_{k=0}^n$ and $\underline{W}_0 := \{W_k\}_{k=0}^n$ defining the original interpolation problem (18). Following Definition 2.2 for the Carathéodory case, we give now:

Definition 3.2. We say that the Nevanlinna – Pick interpolation data $(\underline{z}_0, \underline{W}_0)$ is *regular* if there exist operators

$$\{W_{j,k} \in \mathcal{L}(U) \quad | \quad 0 \le j \le n, \quad j \le k \le n\}$$

satisfying $\|W_{j,k}\|_{\mathcal{L}(U)} < 1$, such that these operators appear in recursion (20).

From now on, we shall make it a standing assumption that the Nevanlinna – Pick interpolation data $(\underline{z}_0, \underline{W}_0)$, indeed, is regular. The forward part of the recursive algorithm for the Nevanlinna – Pick problem is given next:

- For $j = 0$, take some $F_0 \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ of form (18).
- Assume that $0 \le j < n$ and $\{F_k\}_{k=0}^{j} \subset \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ have already been computed. Then $\tilde{F}_{j+1} := \hat{V}_{-z_{j,j}} F_j$ and

$$F_{j+1}(z) := \frac{1}{z} D_{W_{j,j}^*} \left(I - \tilde{F}_{j+1}(z) W_{j,j}^* \right)^{-1} \left(\tilde{F}_{j+1}(z) - W_{j,j} \right) D_{W_{j,j}}^{-1}. \quad (21)$$

A few comments are now in order. Firstly, note that the inverse mapping of $\tilde{F}_{j+1} \mapsto F_{j+1}$ in (21) is nothing but $\hat{T}_{W_{j,j}}$, as introduced immediately after the proof of Proposition 2.1. By claim (i) of Proposition 2.1 and the standing regularity assumption, we have $\tilde{F}_{j+1} \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ and $\tilde{F}_{j+1}(0) = W_{j,j}$ after j steps. So, the next step in recursion can always be computed. After all $n+1$ steps, a trivial interpolation problem is obtained, and any $F_n \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ is its solution.

As in the Carathéodory case, starting from an arbitrarily chosen $F_n := G \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ – this is the free parameter – and solving the recursion backwards give a full parameterization of interpolants $F := F_0$ of the Nevanlinna – Pick problem described by (18). More precisely, any function $F \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ satisfying (18) can be expressed as $F = F_{G,\underline{z}_0,\underline{W}_0}$, where

$$F_{G,\underline{z}_0,\underline{W}_0} = (\hat{V}_{z_{0,0}} \circ \hat{T}_{W_{0,0}} \circ \hat{V}_{z_{1,1}} \circ \hat{T}_{W_{1,1}} \circ \cdots \circ \hat{V}_{z_{n,n}} \circ \hat{T}_{W_{n,n}})(G) \quad (22)$$

for some $G \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$. Here the operator $\hat{T}_R : \mathcal{S}(\mathbb{D}; \mathcal{L}(U)) \to \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ is defined just after Proposition 2.1 and $V_\alpha : \mathcal{S}(\mathbb{D}; \mathcal{L}(U)) \to \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ by (19). Conversely, each such $F_{G,\underline{z}_0,\underline{W}_0}$ belongs to $\mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ and satisfies (18). For notational brevity, we define

$$R_j := W_{j,j}, \quad j = 0, \ldots, n, \quad \text{and} \quad \underline{R} := \{R_j\}_{j=0}^{n}. \quad (23)$$

We call the sequence $\underline{R}$ the *Schur parameters* of the Nevanlinna – Pick problem (18). By (22), the solution of this interpolation problem depends on the interpolation values $\underline{W}_0$ only via the corresponding Schur parameters $\underline{R}$.

We now proceed to translate (22) to the language of conservative realizations. As the state space variant T_R of $\hat{T}_R$ has already been treated in Definition 2.3 and Proposition 2.4, it only remains to propose a state space variant V_α for $\hat{V}_\alpha$.

Definition 3.3. Let $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS with state space X and both the input and output spaces equal to U. Then for any $\alpha \in \mathbb{D}$, the nonlinear mapping V_α (defined on all conservative DLS ϕ) is defined by $\phi_\alpha = V_\alpha(\phi)$, where

$$\phi_\alpha := \begin{pmatrix} (I + \alpha A)^{-1}(\bar{\alpha} + A) & \sqrt{1 - |\alpha|^2}(I + \alpha A)^{-1}B \\ \sqrt{1 - |\alpha|^2}C(I + \alpha A)^{-1} & D - \alpha C(I + \alpha A)^{-1}B \end{pmatrix}. \tag{24}$$

In system theory, the analogous mapping to V_α between discrete and continuous time systems is usually called *Cayley transform* or *bilinear transform*. As V_α maps DLSs to DLSs, we call it *Möbius mapping*.

The following result has a status of folklore in the theory of Hilbert space contractions, though it might often be stated in different language from ours. We include a proof only for the sake of neurotic completeness of presentation.

Lemma 3.4. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS (with state space X, and both input and output spaces equal to U), such that the main operator $A \in \mathcal{L}(X)$ is contractive. For any $\alpha \in \mathbb{D}$, denote $\phi_\alpha := V_\alpha(\phi)$ where V_α is as in Definition 3.3. Then the following holds:*

(i) The DLS ϕ is conservative if and only ϕ_α is. Moreover, ϕ is a simple conservative DLS if and only ϕ_α is.

(ii) The transfer functions of ϕ and ϕ_α are related by $\widehat{\mathcal{D}}_{\phi_\alpha} = \hat{V}_\alpha \widehat{\mathcal{D}}_\phi$, where $\hat{V}_\alpha$ is given by (19)*.*

(iii) The mapping V_α satisfies both $(V_\alpha(\phi))^d = V_{\overline{\alpha}}(\phi^d)$ and $V_\alpha^{-1} = V_{-\alpha}$. Moreover, $\operatorname{range}(\mathcal{B}_\phi)^\perp = \operatorname{range}(\mathcal{B}_{\phi_\alpha})^\perp$ *and* $\ker(\mathcal{C}_\phi) = \ker(\mathcal{C}_{\phi_\alpha})$*, where the observability and controllability maps are defined in* (42)*.*

Proof. The first part of claim (i) follows by a straightforward computation, showing that the block matrix defining ϕ_α in (24) is isometric if and only if the block matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is isometric.

For any closed $V \subset X$ and $\alpha \in \mathbb{D}$, we have $AV \subset V$ if and only if $(I + \alpha A)^{-1}(\bar{\alpha} + A)V \subset V$. Hence, V is a reducing subspace for A if and only if it is a reducing subspace for $(I + \alpha A)^{-1}(\bar{\alpha} + A)$. So, to prove the remaining part of claim (i), it is enough to show that $(I + \alpha A)^{-1}(\bar{\alpha} + A)$ is unitary if and only if A is. Note first that A is normal if and only if

$(I+\alpha A)^{-1}(\bar{\alpha}+A)$ is. Hence, the claim follows from the spectral mapping theorem for normal operators and [21, Theorem 12.26], because the mapping $z \mapsto (1+\alpha z)^{-1}(\bar{\alpha}+z)$ is a continuous bijection on $\mathbb{T}$.

We proceed to prove claim (ii). By a direct computation

$$\begin{aligned}
\left(\hat{V}_\alpha \widehat{\mathcal{D}}_\phi\right)(z) &= \widehat{\mathcal{D}}_\phi \left(\frac{z-\alpha}{1-\bar{\alpha}z}\right) \\
&= D + (z-\alpha)C\left((1-\bar{\alpha}z)I - (z-\alpha)A\right)^{-1} B \\
&= D + (z-\alpha)C\left((I+\alpha A) - (\bar{\alpha}+A)z\right)^{-1} B \\
&= D + zC\left(I - zA_\alpha\right)^{-1} B_\alpha - \alpha C\left(I - zA_\alpha\right)^{-1} B_\alpha,
\end{aligned}$$

where $A_\alpha := (I+\alpha A)^{-1}(\bar{\alpha}I + A)$ and $B_\alpha := (I+\alpha A)^{-1}B$. Noting that $(I - zA_\alpha)^{-1} = I + zA_\alpha\left(I - zA_\alpha\right)^{-1}$, we may continue the computation

$$\begin{aligned}
&= D + zC\left(I - zA_\alpha\right)^{-1} B_\alpha - \alpha C\left[I + zA_\alpha\left(I - zA_\alpha\right)^{-1}\right] B_\alpha \\
&= D - \alpha C(I+\alpha A)^{-1}B + zC\left[I - \alpha A_\alpha\left(I - zA_\alpha\right)^{-1}\right] B_\alpha \\
&= D - \alpha C(I+\alpha A)^{-1}B + zC_\alpha\left[I + \alpha A - \alpha(\bar{\alpha}+A)\left(I - zA_\alpha\right)^{-1}\right] B_\alpha \\
&= D - \alpha C(I+\alpha A)^{-1}B + (1-|\alpha|^2)zC_\alpha\left(I - zA_\alpha\right)^{-1} B_\alpha,
\end{aligned}$$

where $C_\alpha := C(I+\alpha A)^{-1}$. Hence (ii) holds.

To prove claim (iii), note first that the part concerning the duality is trivial. For the rest of this proof, we redefine the operators A_α, B_α, C_α and D_α as follows: $A_\alpha := (I+\alpha A)^{-1}(\bar{\alpha}+A)$, $B_\alpha := \sqrt{1-|\alpha|^2}(I+\alpha A)^{-1}B$, $C_\alpha := \sqrt{1-|\alpha|^2}C(I+\alpha A)^{-1}$ and $D_\alpha := D - \alpha C(I+\alpha A)^{-1}B$ for any $\alpha \in \mathbb{D}$. Clearly

$$(I - \alpha A_\alpha)^{-1} = \frac{I+\alpha A}{1-|\alpha|^2}.$$

By using this we get almost immediately $(A_\alpha)_{-\alpha} = (I-\alpha A_\alpha)^{-1}(-\bar{\alpha}+A_\alpha) = A$, $(B_\alpha)_{-\alpha} = \sqrt{1-|\alpha|^2}(I-\alpha A_\alpha)^{-1}B_\alpha = B$ and $(C_\alpha)_{-\alpha} = \sqrt{1-|\alpha|^2}C_\alpha(I - \alpha A_\alpha)^{-1} = C$. Because $\hat{V}_\alpha^{-1} = \hat{V}_{-\alpha}$, we obtain $(D_\alpha)_{-\alpha} = D$, thus proving $V_\alpha^{-1} = V_{-\alpha}$.

Let us proceed to prove the inclusion $\ker(\mathcal{C}_\phi) \subset \ker(\mathcal{C}_{\phi_\alpha})$. For any $x \in \ker(\mathcal{C}_\phi)$, we get for any $j \geq 0$

$$(I+\alpha A)^{-j-1}x = (I+\alpha A)^{-j-2}\sum_{k\geq 0}(-\alpha A)^k x \in \ker(\mathcal{C}_\phi)$$

as $A\ker(\mathcal{C}_\phi) \subset \ker(\mathcal{C}_\phi)$ and $\ker(\mathcal{C}_\phi)$ is closed. But now $(\bar{\alpha}+A)^j(I+\alpha A)^{-j-1}x \in \ker(\mathcal{C}_\phi) \subset \ker(C)$, implying that $C_\alpha A_\alpha^j x = \sqrt{1-|\alpha|^2}C(\bar{\alpha}+$

$A)^j(I+\alpha A)^{-j-1}x = 0$ for all $j \geq 0$. Hence $\ker(\mathcal{C}_\phi) \subset \ker(\mathcal{C}_{\phi_\alpha})$. Applying this to the DLS ϕ_α with parameter value $-\alpha$ gives

$$\ker(\mathcal{C}_{\phi_\alpha}) \subset \ker\left(\mathcal{C}_{V_{-\alpha}(\phi_\alpha)}\right) = \ker(\mathcal{C}_\phi),$$

as $V_{-\alpha}(\phi_\alpha) = (V_{-\alpha} \circ V_\alpha)(\phi) = \phi$ by what has already been proved. The claim involving the controllability map follows by considering the dual DLS instead. □

Now comes the Nevanlinna – Pick counterpart of Theorem 2.5.

Theorem 3.5. *Assume that the interpolation data* $(\underline{z}_0, \underline{W}_0)$ *for the Nevanlinna – Pick problem* (18) *is regular in the sense of Definition 3.2. Define the additional interpolation points* $\underline{z}_j = \{z_{j,k}\}_{k=j}^n$ *by* (20), *and the Schur parameters* $\underline{R} := \{R_j\}_{j=0}^n$ *by* (23). *Then the following holds:*

(i) For any conservative DLS ϕ *(with state space* X *and both input and output spaces equal to* U*), the transfer function* $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{z},\underline{R})} \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ *of*

$$\tilde{\phi}(\phi,\underline{z},\underline{R}) := (V_{z_{0,0}} \circ T_{R_0} \circ V_{z_{1,1}} \circ T_{R_1} \circ \cdots \circ V_{z_{n,n}} \circ T_{R_n})(\phi) \qquad (25)$$

is a solution of the Nevanlinna – Pick interpolation described by (18). *Moreover, DLS* $\tilde{\phi}(\phi,\underline{z},\underline{R})$ *is conservative, and* $\widehat{\mathcal{D}}_\phi \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ *is the free parameter associated to interpolant* $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{z},\underline{R})}$.

(ii) Conversely, any solution $F \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ *of the Nevanlinna – Pick interpolation problem* (18) *satisfies* $F = \widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{z},\underline{R})}$ *for* $\tilde{\phi}(\phi,\underline{z},\underline{R})$ *given by* (25) *for some (simple) conservative DLS* ϕ *(with both input and output spaces equal to* U*).*

(iii) If, in addition, DLS ϕ *is a simple conservative DLS, then so is* $\tilde{\phi}(\phi,\underline{z},\underline{R})$.

Proof. This theorem follows from the general discussion presented earlier in this section, together with Lemma 3.4 and Proposition A.7. □

3.2 Hermite – Fejér interpolation

It remains to give a result analogous to Theorem 3.5, but concerning the Hermite – Fejér interpolation problem. The difference to the Carathéodory and Nevanlinna – Pick problems is quite small, and we shall discuss it next.

Indeed, when comparing the backward recursion (10) of the Carathéodory problem to the backward recursion (22) of the Nevanlinna – Pick problem, we note that the only difference is as follows: the appearance of the composition operators $\hat{V}_\alpha$ in between each of the mappings $\hat{T}_R$. Recall that the mappings $\hat{V}_\alpha$ are not needed at all in the Carathéodory problem, as all the interpolation conditions are originally posed at the origin $z = 0$. This is in contrast to the Nevanlinna – Pick case, where each of the updated interpolation points $z_{j,j} \in \mathbb{D}$ (see (20)) gets mapped to the origin one by one, by repeated applications involving operators $\hat{V}_{z_{j,j}}$. After each such transformation, a single forward step (21) (of Schur recursion type, for the Carathéodory problem) is taken in order to reduce the number of remaining interpolation conditions by one.

The difference between the Nevanlinna – Pick and Hermite – Fejér interpolation problem is now clear: after the application of any $\hat{V}_{z_{j,j}}$, not only *one* but a totality of d_j (see (16)) forward steps (of Schur recursion type, for the Carathéodory problem) are required. We omit here the somewhat complicated algebraic description of these forward steps, as it is immaterial in the context of this paper. We only remark that we obtain again (in the regular case, see Definition 3.6) a family of Schur parameters, denoted henceforth by $\underline{R} := \left\{R_k^{(l)}\right\}$ where

$$R_k^{(l)} \in \mathcal{L}(U) \quad \text{for} \quad k = 0, 1, \ldots, n \quad \text{and} \quad l = 0, 1, \ldots, d_k. \tag{26}$$

Note that the parameter configuration is exactly the same as for the original interpolation values $W_k^{(l)}$ in (16). Analogously to Definitions 2.2 and 3.2, a regularity assumption must be made. This time we state it quite informally:

Definition 3.6. We say that the Hermite – Fejér interpolation data is *regular* if all the required steps so as to obtain the Schur parameters (26) produce only operators that are strictly contractive in $\mathcal{L}(U)$.

We remark that Definitions 2.2 and 3.2 are special cases of Definition 3.6. In the next theorem, we shall use the following notation for iterated compositions of mappings

$$\left(\circ_{j=0}^{n} G_j\right)(f) := (G_0 \circ G_1 \circ \cdots \circ G_n)(f).$$

The proof of the following theorem does not differ much from its special case, Theorem 3.5.

Theorem 3.7. *Assume that the interpolation data* (16) *for the Hermite – Fejér problem* (17) *is regular in the sense of Definition 3.6. Define the additional interpolation points* $\underline{z}_j = \{z_{j,k}\}_{k=j}^{n}$ *by* (20)*, and denote the Schur parameters* (26) *by* $\underline{R} := \left\{R_k^{(l)}\right\}$*. Then the following holds:*

(i) For any conservative DLS ϕ (with state space X and both input and output spaces equal to U), the transfer function $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{z},\underline{R})} \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ of

$$\tilde{\phi}(\phi,\underline{z},\underline{R}) := \left(\circ_{j=0}^{n} V_{z_{j,j}} \left(\circ_{l=0}^{d_j} T_{R_j^{(l)}}\right)\right)(\phi) \tag{27}$$

is a solution of the Hermite – Fejér interpolation described by (17). Moreover, DLS $\tilde{\phi}(\phi,\underline{z},\underline{R})$ is conservative, and $\widehat{\mathcal{D}}_{\phi} \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ is the free parameter associated to interpolant $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{z},\underline{R})}$.

(ii) Conversely, any solution $F \in \mathcal{S}(\mathbb{D};\mathcal{L}(U))$ of the Hermite – Fejér interpolation problem (17) satisfies $F = \widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{z},\underline{R})}$ for $\tilde{\phi}(\phi,\underline{z},\underline{R})$ given by (27) for some (simple) conservative DLS ϕ (with both input and output spaces equal to U).

(iii) If, in addition, DLS ϕ is a simple conservative DLS, then so is $\tilde{\phi}(\phi,\underline{z},\underline{R})$.

4 Observable and controllable subspaces

In this section, we compute the unobservable and uncontrollable subspaces of realizations $\tilde{\phi}(\phi,\underline{R})$ appearing in Theorem 2.5. Such results will be used in the main results of this paper, namely Theorem 5.1. We shall consider first a single backward step, and then proceed recursively to conclude the final results.

Given $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $R \in \mathcal{L}(U)$, we denote

$$\tilde{A}_R := \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I_X \end{bmatrix} \in \mathcal{L}\left(\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}\right). \tag{28}$$

The operator $\tilde{A}_{-R^*}$ (given by (28) with $-R^*$ in place of R) is exactly the main operator of the DLS $\tilde{\phi} := T_R(\phi)$ of Definition 2.3. The whole point of this is that both the (un)observable and (un)controllable subspaces of a conservative DLS are determined by the residual cost operators of the main operator alone, by Proposition A.3.

Proposition 4.1. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a conservative DLS (with state space X and both input and output spaces equal to U). Let $R \in \mathcal{L}(U)$ be a strict contraction, and define $\tilde{A}_R$ by (28). Define the residual cost operators $L_A \in \mathcal{L}(X)$ and $L_{\tilde{A}_R} \in \mathcal{L}\left(\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}\right)$ as in Definition A.2. Then*

$$\ker\left(I - L_{\tilde{A}_R}\right) = \begin{smallmatrix} \{0\} \\ \oplus \\ \ker(I - L_A) \end{smallmatrix} \subset \begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}.$$

Proof. Let $\left[\begin{smallmatrix} u \\ x \end{smallmatrix}\right] \in \ker\left(I - L_{\tilde{A}_R}\right)$ be arbitrary. Then by Corollary A.4 and unitarity of $\left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right]$, we obtain

$$\|u\|_U^2 + \|x\|_X^2 = \left\| \begin{bmatrix} u \\ x \end{bmatrix} \right\|^2_{\substack{U \\ \oplus \\ X}} = \left\| \tilde{A}_R \begin{bmatrix} u \\ x \end{bmatrix} \right\|^2_{\substack{U \\ \oplus \\ X}}$$

$$= \left\| \begin{bmatrix} R & 0 \\ 0 & I_X \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} \right\|^2_{\substack{U \\ \oplus \\ X}} = \|Ru\|_U^2 + \|x\|_X^2.$$

This implies $\|Ru\|_U^2 = \|u\|_U^2$ and by strong contractivity of R we get $u = 0$. Hence $\ker\left(I - L_{\tilde{A}_R}\right) \subset \substack{\{0\} \\ \oplus \\ X}$.

Now, for any $\left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] \in \ker\left(I - L_{\tilde{A}_R}\right)$ we have $\tilde{A}_R \left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] = \left[\begin{smallmatrix} Cx \\ Ax \end{smallmatrix}\right]$. Because $\tilde{A}_R \ker\left(I - L_{\tilde{A}_R}\right) \subset \ker\left(I - L_{\tilde{A}_R}\right)$ by Corollary A.4, we conclude (by what already has been proved) that $x \in \ker(C)$ and hence, by iterating,

$$\tilde{A}_R^j \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ A^j x \end{bmatrix} \text{ for all } j \geq 1.$$

Using this, together with Corollary A.4, we obtain for all $j \geq 1$

$$\|x\|_X = \left\| \begin{bmatrix} 0 \\ x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}} = \left\| \tilde{A}_R^j \begin{bmatrix} 0 \\ x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}} = \left\| \begin{bmatrix} 0 \\ A^j x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}} = \|A^j x\|_X.$$

It follows that $x \in \ker(I - L_A)$, and hence $\ker\left(I - L_{\tilde{A}_R}\right) \subset \substack{\{0\} \\ \oplus \\ \ker(I - L_A)}$.

For the converse inclusion, let $x \in \ker(I - L_A)$ be arbitrary. Because $\ker(I - L_A) = \ker(\mathcal{C}_\phi) = \cap_{j \geq 0} \ker(CA^j)$ by Proposition A.3, we have $CA^j x = 0$ for all $j \geq 0$. In the first step, we obtain $\tilde{A}_R \left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] = \left[\begin{smallmatrix} Cx \\ Ax \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ Ax \end{smallmatrix}\right]$ and by iteration

$$\tilde{A}_R^j \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} CA^{j-1} x \\ A^j x \end{bmatrix} = \begin{bmatrix} 0 \\ A^j x \end{bmatrix} \text{ for all } j \geq 1.$$

The proof is completed by an application of Corollary A.4, as for all $j \geq 1$,

$$\left\| \tilde{A}_R^j \begin{bmatrix} 0 \\ x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}} = \|A^j x\|_X = \|x\|_X = \left\| \begin{bmatrix} 0 \\ x \end{bmatrix} \right\|_{\substack{U \\ \oplus \\ X}}.$$

□

The unobservable subspace for the realization $\tilde{\phi}(\phi, \underline{R})$ for the Carathéodory interpolant in (14) is now

$$\ker\left(\mathcal{C}_{\tilde{\phi}(\phi, \underline{R})}\right) = \substack{\{0\}^{d+1} \\ \oplus \\ \ker\left(\mathcal{C}_\phi\right)}, \tag{29}$$

where $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is the realization for the free parameter, and $d+1$ is the number of the Schur parameters $\underline{R} = \{R_k\}_{k=0}^{d}$. Indeed, this follows from Proposition 2.4; the characterization of the unobservable subspaces for conservative DLSs, Proposition A.3; and a recursive application of Proposition 4.1. Note that $\ker\left(\mathcal{C}_{\tilde{\phi}(\phi,\underline{R})}\right)$ does not depend on the Schur parameters $\underline{R}$ but only on the free parameter DLS ϕ.

The case for the dual operator $\tilde{A}_R^*$ is unfortunately somewhat more complicated.

Proposition 4.2. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a conservative DLS (with state space X and both input and output spaces equal to U). Let $R \in \mathcal{L}(U)$ be a strict contraction, and define $\tilde{A}_R$ by (28). Define the residual cost operators $L_{A^*} \in \mathcal{L}(X)$ and $L_{\tilde{A}_R^*} \in \mathcal{L}\left(\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}\right)$ as in Definition A.2. Then*

$$\ker\left(I - L_{\tilde{A}_R^*}\right) = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{smallmatrix} \{0\} \\ \oplus \\ \ker(I - L_{A^*}) \end{smallmatrix}. \tag{30}$$

Proof. Assume that $\left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right]\left[\begin{smallmatrix} u \\ x \end{smallmatrix}\right] \in \ker\left(I - L_{\tilde{A}_R^*}\right)$ is arbitrary. Then by Corollary A.4 and unitarity of $\left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right]$, we obtain

$$\left\|\begin{bmatrix} u \\ x \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\|\begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\|\tilde{A}_R^* \begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\|\begin{bmatrix} R^* & 0 \\ 0 & I_X \end{bmatrix}\begin{bmatrix} u \\ x \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}},$$

whence (by the strict contractivity of R^*) we get $u = 0$. It follows that $\ker\left(I - L_{\tilde{A}_R^*}\right) \subset \left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right] \begin{smallmatrix} \{0\} \\ \oplus \\ X \end{smallmatrix}$.

Now, for any $\left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] \in \ker\left(I - L_{\tilde{A}_R^*}\right)$, we have by a direct computation $\left[\begin{smallmatrix} R^* & 0 \\ 0 & I_X \end{smallmatrix}\right]\left[\begin{smallmatrix} D^* & B^* \\ C^* & A^* \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] = \tilde{A}_R^{*2}\left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right]$. Using the norm equality

$$\left\|\tilde{A}_R^{*2} \begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\|\begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\|\begin{bmatrix} 0 \\ x \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} \tag{31}$$

from Corollary A.4, we obtain by the unitarity of $\left[\begin{smallmatrix} D^* & B^* \\ C^* & A^* \end{smallmatrix}\right]$

$$\left\|\begin{bmatrix} R^* & 0 \\ 0 & I_X \end{bmatrix}\begin{bmatrix} D^* & B^* \\ C^* & A^* \end{bmatrix}\begin{bmatrix} 0 \\ x \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\|\begin{bmatrix} 0 \\ x \end{bmatrix}\right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}}.$$

Again, by the strict contractivity of R^*, we obtain $\left[\begin{smallmatrix} D^* & B^* \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] = 0$, and hence $B^*x = 0$ and $\tilde{A}_R^{*2}\left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ A^*x \end{smallmatrix}\right]$. This together with (31) implies $\|A^*x\|_X = \|x\|_X$. Using this argument recursively gives for all $j \geq 1$

$$\tilde{A}_R^{*j} \begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} 0 \\ A^{*(j-1)}x \end{bmatrix} \quad \text{and} \quad \|A^{*(j-1)}x\|_X = \|x\|_X.$$

In other words, $x \in L_{A^*}$ and hence $\ker\left(I - L_{\tilde{A}_R^*}\right) \subset \left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right] \begin{smallmatrix} \{0\} \\ \oplus \\ \ker(I - L_{A^*}) \end{smallmatrix}$ by Corollary A.4.

It remains to prove the converse inclusion. Let $x \in \ker(I - L_{A^*})$ and define $\left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right] := \left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right]$. Now

$$\tilde{A}_R^* \begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} R^* & 0 \\ 0 & I_X \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}, \tag{32}$$

and using the norm identity $\left\| \left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] \right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\| \left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right] \right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}}$ gives $\left\| \tilde{A}_R^* \left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right] \right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \left\| \left[\begin{smallmatrix} \tilde{u} \\ \tilde{x} \end{smallmatrix}\right] \right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}}$. By Proposition A.3, $x \in \ker\left(\mathcal{B}_\phi^*\right)$, i.e. $B^* A^{*j} x = 0$ for all $j \geq 0$. This together with (32) gives for all $j \geq 2$

$$\left\| \tilde{A}_R^{*j} \begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix} \right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}} = \|A^{*(j-1)} x\|_X = \|x\|_X = \left\| \begin{bmatrix} \tilde{u} \\ \tilde{x} \end{bmatrix} \right\|_{\begin{smallmatrix} U \\ \oplus \\ X \end{smallmatrix}},$$

thus proving the claim, by Corollary A.4. □

By augmenting with identity operators, equation (30) can be written in the equivalent form for all $k \geq 1$

$$\begin{matrix} \{0\}^{k-1} \\ \oplus \\ \ker\left(I - L_{\tilde{A}_R^*}\right) \end{matrix} = \begin{bmatrix} I_{U^{k-1}} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{matrix} \{0\}^k \\ \oplus \\ \ker(I - L_{A^*}) \end{matrix}. \tag{33}$$

Applying this recursively (together with Propositions 2.4 and A.3) gives for the uncontrollable subspace of DLS $\tilde{\phi}(\phi, \underline{R})$ in (14)

$$\operatorname{range}\left(\mathcal{B}_{\tilde{\phi}(\phi,\underline{R})}\right)^\perp = \begin{bmatrix} D_1 & C_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} I_U & 0 & 0 \\ 0 & D_2 & C_2 \\ 0 & B_2 & A_2 \end{bmatrix} \cdots \tag{34}$$
$$\cdots \begin{bmatrix} I_{U^{d-1}} & 0 & 0 \\ 0 & D_d & C_d \\ 0 & B_d & A_d \end{bmatrix} \begin{bmatrix} I_{U^d} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{matrix} \{0\}^{d+1} \\ \oplus \\ \operatorname{range}\left(\mathcal{B}_\phi\right)^\perp \end{matrix}.$$

Here $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is again a conservative realization for the free parameter,

$$\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} := \left(T_{R_j} \circ T_{R_{j+1}} \circ \cdots \circ T_{R_d}\right)(\phi), \tag{35}$$

and $d+1$ is the number of the Schur parameters $\underline{R} = \{R_k\}_{k=0}^d$. Now we have proved the following result concerning the state space representation of Carathéodory interpolation:

Theorem 4.3. *Make the same assumptions and use the same notation as in Theorem 2.5. Then the unobservable subspaces of* ϕ *and* $\tilde{\phi}(\phi, \underline{R})$ *are connected by* (29)*. The uncontrollable subspaces of* ϕ *and* $\tilde{\phi}(\phi, \underline{R})$ *are connected by the unitary equivalence* (34)*.*

In particular ϕ *is approximately observable (approximately controllable, minimal) if and only if* $\tilde{\phi}(\phi, \underline{R})$ *is a DLS of the same kind.*

As a corollary, we show that the stability of the main operator A of ϕ is inherited by the realization $\tilde{\phi}(\phi, \underline{R})$ for the interpolant.

Corollary 4.4. *Make the same assumptions and use the same notation as in Theorem 2.5. Assume, in addition, that* $\dim U < \infty$ *and that the main operator* A *of* ϕ *is strongly stable:*

$$A^j x \to 0 \quad \textit{for all } x \in X.$$

Then the main operator $\tilde{A}$ *of* $\tilde{\phi}(\phi, \underline{R})$ *is strongly stable, together with its adjoint. Furthermore,* $\tilde{\phi}(\phi, \underline{R})$ *is exactly observable and exactly controllable, and the interpolant* $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{R})}$ *is inner from both sides.*

Proof. We shall use Proposition A.6 to prove this corollary. To this end, we shall show in order that $\tilde{\phi}(\phi, \underline{R})$ is minimal, and that the range of its Hankel operator is closed.

As A is strongly stable, $\widehat{\mathcal{D}}_\phi$ is inner from the left by Proposition A.5. As $\dim U < \infty$, it follows (by looking at the nontangential boundary traces) that $\widehat{\mathcal{D}}_\phi$ is inner from the right, too. Hence

$$A^* x \to 0 \quad \text{for all} \quad x \in \ker (\mathcal{C}_\phi)^\perp , \tag{36}$$

see Proposition A.5. Again, by the strong stability of A, we have $L_A = 0$ and hence ϕ is approximately observable (i.e. $\ker (\mathcal{C})^\perp = X$), by equation (45) in Proposition A.3. It follows now from (36) that also the adjoint A^* is strongly stable, implying $L_{A^*} = 0$. Hence ϕ is approximately controllable, by claim (46) of Proposition A.3. By Theorem 4.3, also the DLS $\tilde{\phi}(\phi, \underline{R})$ is both approximately observable and approximately controllable; in other words: minimal.

We conclude from (9) that the operator $T_R : \mathcal{S}(\mathbb{D}; \mathcal{L}(U)) \to \mathcal{S}(\mathbb{D}; \mathcal{L}(U))$ maps inner from the left functions to inner from the left functions. Recalling the definition of $\tilde{\phi}(\phi, \underline{R})$ (see (14)) and that $\widehat{\mathcal{D}}_{T_R(\phi)} = \hat{T}_R \widehat{\mathcal{D}}_\phi$ (see claim (ii) of Proposition 2.4), we conclude that the Carathéodory interpolant $\widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{R})}$ is inner from the left; hence it is inner from the both sides as $\dim U < \infty$. By a well-known fact that is quite easy to check, the causal Hankel operator

$\bar{\pi}_+\mathcal{D}_{\tilde{\phi}(\phi,\underline{R})}\pi_-$ (as a partial isometry) has closed range, see Proposition A.6 and the discussion preceding it.

Now, $\tilde{\phi}(\phi,\underline{R})$ is conservative, because ϕ is, by Theorem 2.5. Hence the DLS $\tilde{\phi}(\phi,\underline{R})$ satisfies all the conditions of Proposition A.6, and the proof is now complete. □

Let us proceed to consider the backward steps of the Nevanlinna – Pick problem. By Theorem 3.5, they differ slightly form the Carathéodory case: every other step in formula (25) is the operator V_α, $\alpha \in \mathbb{D}$, of the composition mapping type. Fortunately, this extra complication is not at all of serious kind:

Theorem 4.5. *Make the same assumptions and use the same notation as in Theorem 3.5. Then the unobservable subspaces of ϕ and $\tilde{\phi}(\phi,\underline{z},\underline{R})$ satisfy*

$$\ker\left(\mathcal{C}_{\tilde{\phi}(\phi,\underline{z},\underline{R})}\right) = \begin{smallmatrix}\{0\}^{n+1}\\ \oplus \\ \ker\left(\mathcal{C}_\phi\right)\end{smallmatrix}. \tag{37}$$

The uncontrollable subspaces of ϕ and $\tilde{\phi}(\phi,\underline{z},\underline{R})$ are connected by the unitary equivalence

$$\operatorname{range}\left(\mathcal{B}_{\tilde{\phi}(\phi,\underline{z},\underline{R})}\right)^\perp = \begin{bmatrix} D_1 & C_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} I_U & 0 & 0 \\ 0 & D_2 & C_2 \\ 0 & B_2 & A_2 \end{bmatrix} \cdots \tag{38}$$
$$\cdots \begin{bmatrix} I_{U^{n-2}} & 0 & 0 \\ 0 & D_n & C_n \\ 0 & B_n & A_n \end{bmatrix} \begin{bmatrix} I_{U^{n-1}} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{smallmatrix}\{0\}^{n+1}\\ \oplus \\ \operatorname{range}\left(\mathcal{B}_\phi\right)^\perp\end{smallmatrix},$$

where

$$\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} := (V_{z_{j,j}} \circ T_{R_j} \circ V_{z_{j+1,j+1}} \circ T_{R_{j+1}} \circ \cdots \circ V_{z_{n,n}} \circ T_{R_n})(\phi).$$

In particular ϕ is approximately observable (approximately controllable, minimal) if and only if $\tilde{\phi}(\phi,\underline{z},\underline{R})$ is of same kind.

Proof. Only the part involving equation (38) deserves a comment. So, let us check what happens in the very first backward step, described by the composite operation $V_{z_{n,n}} \circ T_{R_n}$ when applied on the free parameter DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

The uncontrollable subspace of DLS $T_{R_n}(\phi)$ is given by (33) with $k = n$ and $R = -R_n^*$; namely

$$\begin{smallmatrix}\{0\}^{n-1}\\ \oplus \\ \operatorname{range}\left(\mathcal{B}_{T_{R_n}(\phi)}\right)^\perp\end{smallmatrix} = \begin{bmatrix} I_{U^{n-1}} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{smallmatrix}\{0\}^{n}\\ \oplus \\ \operatorname{range}\left(\mathcal{B}_\phi\right)^\perp\end{smallmatrix}. \tag{39}$$

Indeed, just note that

$$\operatorname{range}\left(\mathcal{B}_{T_{R_n}(\phi)}\right)^{\perp} = \ker\left(I - L_{\tilde{A}^*_{-R_n^*}}\right) \text{ and } \operatorname{range}\left(\mathcal{B}_{\phi}\right)^{\perp} = \ker\left(I - L_{\tilde{A}^*}\right)$$

by claim (46) of Proposition A.3, and the conservativity of both $T_{R_n}(\phi)$ and ϕ, see Proposition 2.4. By claim (iii) of Lemma 3.4, equation (39) remains true if the DLS $T_{R_n}(\phi)$ is replaced by $\left(V_{z_{n,n}} \circ T_{R_n}\right)(\phi)$ on the left hand side. This takes care of the full first backward step, and continuing in the similar manner proves finally (38). $\square$

Even though the operators $V_{z_{j,j}}$ do not change the uncontrollable subspace in any of the steps (see claim (iii) of Lemma 3.4), the uncontrollable space range $\left(\mathcal{B}_{\tilde{\phi}(\phi,\underline{z},\underline{R})}\right)^{\perp}$ nevertheless depends on all the interpolation points $\{z_k\}_{k=0}^{n}$ (through sequence $\underline{z} = \{z_{j,j}\}_{j=0}^{n}$ given by (20)). Indeed, the DLSs $\left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right)$ depend on the sequence $\underline{z}$.

The result analogous to Theorems 4.3 and 4.5 concerning the Hermite – Fejér interpolation problem is left to the interested reader.

5 McMillan degree of rational interpolants

Both Carathéodory and Nevanlinna – Pick interpolation are used extensively in various engineering applications; see [11], circuit theory [29], system identification and signal processing [6, 13], and robust control [26, 27, 16]. In many of such applications, $\dim U < \infty$ and the interpolant is required to be a rational function of low McMillan degree. To fulfill this requirement for robust control or signal processing, the degree of interpolant is kept low appropriately in [19, 6], based on optimization of an entropy functional in [7]. However, finding (or merely characterizing, e.g. in terms of the free parameter) the *minimal* degree interpolants (preferably in an algorithmically effective way) is a long standing problem, see [29, 14, 13].

As an instructive special case, let us recall the scalar–valued Carathéodory problem discussed in Section 1. If the free parameter g in (3) is a rational function, then the corresponding interpolant $F_{g,\underline{r}}$ is rational, too. However, due to complicated zero–pole cancellations that may appear in the backward steps, it is not at all clear how the McMillan degree $\deg F_{g,\underline{r}}$ is related to that of the free parameter g; apart from the completely trivial estimate

$$\deg F_{g,\underline{r}} \leq \deg g + d.$$

Likewise, constructing an interpolant satisfying $\deg F_{g,\underline{r}} < d$ is not an easy exercise.

In the following theorem, we give a geometric characterization for the McMillan degree of the interpolant for the Carathéodory problem. The geometry of this characterization explains the computational and theoretic difficulties in finding low degree rational interpolants.

Theorem 5.1. *Make the same assumptions and use the same notation as in Theorem 2.5. Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be any conservative realization for the free parameter (whose state space is* X*, and both input and output spaces equal to* U*). If* $\dim U < \infty$*, then the McMillan degree of the corresponding Carathéodory interpolant is given by* $\deg \widehat{\mathcal{D}}_{\tilde{\phi}(\phi,\underline{R})} = \dim X_0$*, where*

$$X_0 := \tilde{X} \ominus \left(\begin{smallmatrix} \{0\}^{d+1} \\ \oplus \\ \ker(I-L_A) \end{smallmatrix} \cap \tilde{X} \right) \subset \begin{smallmatrix} U^{d+1} \\ \oplus \\ X \end{smallmatrix}$$

and

$$\tilde{X} := \begin{bmatrix} D_1 & C_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} I_U & 0 & 0 \\ 0 & D_2 & C_2 \\ 0 & B_2 & A_2 \end{bmatrix} \cdots \tag{40}$$
$$\cdots \begin{bmatrix} I_{U^{d-1}} & 0 & 0 \\ 0 & D_d & C_d \\ 0 & B_d & A_d \end{bmatrix} \begin{bmatrix} I_{U^d} & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \frac{\begin{smallmatrix} U^{d+1} \\ \oplus \end{smallmatrix}}{\mathrm{range}(I-L_{A^*})}$$

and each DLS $\left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right)$ *for* $j = 1, \ldots, d$ *is given by* (35).

Proof. The unobservable subspace of $\tilde{\phi}(\phi, \underline{R})$ is $\ker\left(\mathcal{C}_{\tilde{\phi}(\phi,\underline{R})}\right) = \begin{smallmatrix} \{0\}^{d+1} \\ \oplus \\ \ker(I-L_A) \end{smallmatrix}$ by (29). So as to the controllable subspace of $\tilde{\phi}(\phi, \underline{R})$, we first note that the product of block matrices (denoted by Z henceforth) in (40) is unitary, by the conservativity of ϕ and Proposition 2.4. Because a unitary mapping maps orthogonal complements onto orthogonal complements, we have

$$\overline{\mathrm{range}\left(\mathcal{B}_{\tilde{\phi}(\phi,\underline{R})}\right)} = Z \left(\begin{smallmatrix} U^{d+1} \\ \oplus \\ X \end{smallmatrix} \ominus \begin{smallmatrix} \{0\}^{d+1} \\ \oplus \\ \ker(I-L_{A^*}) \end{smallmatrix} \right) = Z \left(\frac{\begin{smallmatrix} U^{d+1} \\ \oplus \end{smallmatrix}}{\mathrm{range}(I-L_{A^*})} \right) = \tilde{X}$$

by (34) and (46). Now the claim follows from Proposition A.1. $\square$

We leave the statement of the analogous result for the Nevanlinna – Pick and Hermite - Fejér problems to the interested reader.

A On conservative systems

In this appendix, we review some basic facts from linear discrete time system theory. Our emphasis is on the conservative discrete time systems, equivalent to *operator colligations* in the language of [5]. Their continuous time counterparts are sometimes called *operator nodes*, see [2, 18]. See also the related classical references [4, 24, 25].

Let U, X and Y[3] be separable Hilbert spaces. A *discrete time linear system* (DLS) is a quadruple $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of linear operators A, B, C and D, such that the 2×2 block matrix, called the *system matrix*,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{smallmatrix} X \\ \oplus \\ U \end{smallmatrix} \to \begin{smallmatrix} X \\ \oplus \\ Y \end{smallmatrix}.$$

defines a bounded operator between the indicated spaces. We call U the input, Y the output and X the state space of ϕ. The DLS ϕ defines the system of difference equations

$$\begin{cases} x_{j+1} & = Ax_j + Bu_j \\ y_j & = Cx_j + Du_j, \quad j \in \mathbb{Z}, \end{cases} \tag{41}$$

where the sequences $\tilde{u} := \{u_j\}_{j\in\mathbb{Z}} \subset U$, $\{x_j\}_{j\in\mathbb{Z}} \subset X$, $\tilde{y} := \{y_j\}_{j\in Z} \subset Y$. For the solvability of the difference equation, we assume that the input sequence $\tilde{u}$ has only finitely many nonzero elements u_j for $j < 0$, and the initial state is set by $x_J = 0$ for some J negative enough. As usual, the *controllability* and *observability* maps of DLS ϕ are defined by

$$\mathcal{B}_\phi \tilde{u} := \sum_{j>1} A^j B u_{-j} \in X, \tag{42}$$

$$\mathcal{C}_\phi x := \{CA^j x\}_{j\geq 0} \subset Y, \quad x \in X,$$

where $\tilde{u} := \{u_j\}_{j<0} \subset U$ again has only finitely many nonzero elements, in order to have the sum well-defined. Roughly speaking, $\mathcal{B}_\phi$ maps past inputs into present states, and $\mathcal{C}_\phi$ maps present states into future outputs. The *transfer function* of ϕ is defined by $\widehat{\mathcal{D}}_\phi(z) := D + zC(I - zA)^{-1}B$, for all $z^{-1} \notin \sigma(A)$. As is well known, the *input-output mapping* $\mathcal{D}_\phi : \tilde{u} \mapsto \tilde{y}$ of ϕ (for z-transformable input sequences $\tilde{u}$) can be represented by a multiplication by $\widehat{\mathcal{D}}_\phi$.

It is easy to see that the block matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \begin{smallmatrix} X \\ \oplus \\ U \end{smallmatrix} \to \begin{smallmatrix} X \\ \oplus \\ Y \end{smallmatrix}$ is *isometric* if and only if the energy balance equations

$$\|x_{j+1}\|_X^2 - \|x_j\|_X^2 = \|u_j\|_U^2 - \|y_j\|_Y^2 \tag{43}$$

[3] In this paper, we apply always these with the additional assumption that $U = Y$.

hold for the solution of (41), with any initial value $x_0 \in X$, input $\{u_j\} \in \ell^2(\mathbb{Z}_+;U)$ and time $j \geq 0$. In this case, the DLS ϕ itself is called *energy-preserving.* A DLS is by definition *conservative*, if it is energy-preserving together with its *dual DLS*, defined by $\phi^d := \left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix} \right)$. Equivalently, a DLS ϕ is conservative, if and only if its system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$ is unitary, if and only if

$$\begin{aligned} A^*A + C^*C &= I, \quad & B^*A + D^*C &= 0, \quad & B^*B + D^*D &= I, \\ AA^* + BB^* &= I, \quad & CA^* + DB^* &= 0, \quad & CC^* + DD^* &= I. \end{aligned}$$

Not all of these six equations are independent. In [18], the reduction problem of the corresponding equations is considered in the continuous time setting, giving results that are applicable for certain PDE's.

If follows quite easily that for energy-preserving DLSs

$$\begin{aligned} &\|\mathcal{B}_\phi \tilde{u}\|_X \leq \|\tilde{u}\|_{\ell^2(\mathbb{Z}_-;U)}, \quad \tilde{u} \in \ell^2(\mathbb{Z}_-;U) \\ &\|\mathcal{C}_\phi x\|_{\ell^2(\mathbb{Z}_+;Y)} \leq \|x\|_X, \quad x \in X. \end{aligned}$$

Hence, both the operators $\mathcal{B}_\phi$ and $\mathcal{C}_\phi$ are bounded (in fact, contractions) between the indicated Hilbert spaces[4]. Such DLSs are called both *input stable* and *output stable.* Any energy-preserving (hence, conservative) DLS satisfies also

$$\|\mathcal{D}_\phi \tilde{u}\|_{\ell^2(\mathbb{Z}_+;Y)} \leq \|\tilde{u}\|_{\ell^2(\mathbb{Z}_+;U)}.$$

Essentially by Parsevals identity, this implies that the corresponding transfer function satisfies $\widehat{\mathcal{D}}_\phi \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U;Y))$.

For input and output stable DLS ϕ, we define the *unobservable subspace* $\ker(\mathcal{C}_\phi)$ and the *uncontrollable subspace* $\operatorname{range}(\mathcal{B}_\phi)^\perp = \ker\left(\mathcal{C}_{\phi^d}\right)$[5]. If $\ker(\mathcal{C}_\phi) = \{0\}$, then ϕ is called *approximately observable*; and if $\operatorname{range}(\mathcal{B}_\phi)^\perp = \{0\}$, then ϕ is called *approximately controllable.* Any DLS ϕ is said to be *minimal* if it is both approximately controllable and approximately observable. The state space X of any DLS ϕ can be reduced, so as to obtain a minimal DLS:

Proposition A.1. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an input stable and output stable DLS. Decompose the state space X to the orthogonal direct sum*

$$X = \left(\ker(\mathcal{C}_\phi) \cap \overline{\operatorname{range}(\mathcal{B}_\phi)} \right) \oplus X_0 \oplus \operatorname{range}(\mathcal{B}_\phi)^\perp,$$

where

$$X_0 := \overline{\operatorname{range}(\mathcal{B}_\phi)} \ominus \left(\ker(\mathcal{C}_\phi) \cap \overline{\operatorname{range}(\mathcal{B}_\phi)} \right).$$

[4] Note that $\mathcal{B}_\phi$ has been extended by continuity to all of $\ell^2(\mathbb{Z}_-;U)$.

[5] By $\operatorname{range}(\mathcal{B}_\phi)$ we shall denote all the vectors $x \in X$ that are obtained from inputs $\tilde{u}$ having only finitely many nonzero elements.

Then the operators A, B and C, when decomposed to block matrices accordingly, are of form

$$A = \begin{bmatrix} * & * & * \\ 0 & A_0 & * \\ 0 & 0 & * \end{bmatrix}, \quad B = \begin{bmatrix} * \\ B_0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C_0 & * \end{bmatrix}; \tag{44}$$

here $*$ denotes an irrelevant, generally nonzero term. The reduced DLS $\phi^r := \left(\begin{smallmatrix} A_0 & B_0 \\ C_0 & D \end{smallmatrix} \right)$, with state space X_0, has the same transfer function as ϕ. Moreover, ϕ^r is approximately controllable and observable: $\overline{\mathrm{range}\,(\mathcal{B}_{\phi^r})} = X_0$ and $\ker\,(\mathcal{C}_{\phi^r}) = \{0\}$.

Proof. This result is, of course, classical and can be found in e.g. [15]. The form of the decomposition (44) follows from the easily verified invariance conditions $A\,\overline{\mathrm{range}\,(\mathcal{B}_\phi)} \subset \overline{\mathrm{range}\,(\mathcal{B}_\phi)}$ and $A \ker\,(\mathcal{C}_\phi) \subset \ker\,(\mathcal{C}_\phi)$. Because $CA^jB = C_0 {A_0}^j B_0$ for all $j \geq 0$, we have the equality of transfer functions, as claimed. □

If $\dim U = \dim Y < \infty$ and $\widehat{\mathcal{D}}_\phi$ is a rational function, then its *McMillan degree* equals $\dim X_0$ by definition. In the case of conservative DLSs, the energy preserving property is generally lost when such a reduction of X is carried out.

We now describe the geometry of the state space for a conservative DLS.

Definition A.2. For any contraction $A \in \mathcal{L}\,(X)$, the operators

$$L_A := \operatorname*{slim}_{n\to\infty} A^{*n}A^n, \quad L_{A^*} := \operatorname*{slim}_{n\to\infty} A^nA^{*n}$$

are called the residual cost operators of A and A^*, respectively[6].

Such residual cost operators (in a more general context) play an important role in operator Riccati equations, see [17]. Because the decreasing sequences of self-adjoint nonnegative operators have a lower bound, both L_A and L_{A^*} exist for any contractive A. Moreover, they both are self-adjoint and nonnegative. Note that L_A is not generally a projection, as it may have spectrum in $(0, 1)$. If A is normal, then clearly $L_A = L_{A^*}$, and it is an orthogonal projection.

The unobservable and uncontrollable subspaces of a conservative DLS have a particularly simple characterization.

[6]For a sequence of bounded operators $T_n \in \mathcal{L}\,(X)$, the *strong limit* operator $(\operatorname{slim}_{n\to\infty} T_n) \in \mathcal{L}\,(X)$ is defined by $(\operatorname{slim}_{n\to\infty} T_n)\,x := \lim_{n\to\infty} T_n x$ for all $x \in X$.

Proposition A.3. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a conservative DLS. Then*

$$\begin{aligned} &\ker(\mathcal{C}_\phi) = \ker(I - L_A) \\ &= \left\{x \in X \quad | \quad \|A^j x\|_X = \|x\|_X \text{ for all } j \geq 1\right\} \end{aligned} \tag{45}$$

and dually

$$\begin{aligned} &\operatorname{range}(\mathcal{B}_\phi)^\perp = \ker(I - L_{A^*}) \\ &= \left\{x \in X \quad | \quad \|A^{*j} x\|_X = \|x\|_X \text{ for all } j \geq 1\right\}. \end{aligned} \tag{46}$$

Proof. By considering the observability Gramian, we have for any $x \in X$

$$\mathcal{C}_\phi^* \mathcal{C}_\phi x = \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* C A^j x = \lim_{n\to\infty} (x - A^{*n} A^n x) = x - L_A x,$$

where we have used the Lyapunov equation $A^*A + C^*C = I$. This proves the first equality sign in (45).

Let us proceed to prove the second equality in (45). As A is a contraction, the sequence $\{\langle A^{*j} A^j x, x\rangle_X\}_{j\geq 0}$ is non-increasing for any $x \in X$. If for some $x \in X$ and $j \geq 0$ we have

$$\|A^j x\|_X^2 = \left\langle A^{*j} A^j x, x\right\rangle_X > \left\langle A^{*(j+1)} A^{j+1} x, x\right\rangle_X = \|A^{j+1} x\|_X^2$$

then

$$\langle x, x\rangle_X > \lim_{j\to\infty} \left\langle A^{*j} A^j x, x\right\rangle_X$$

and hence $x \notin \ker(I - L_A)$. We have now proved

$$\ker(I - L_A) \subset \left\{x \in X \quad | \quad \|A^j x\|_X = \|x\|_X \text{ for all } j \geq 1\right\}.$$

For the converse inclusion, assume that $\|A^j x\|_X = \|x\|_X$ for all $j \geq 0$. Then $\langle (I - A^{*j} A^j)\, x, x\rangle_X = 0$ for all $j \geq 0$, and by the contractivity of A, we get

$$\| \left(I - A^{*j} A^j\right)^{1/2} x\|_X = 0 \Leftrightarrow \left(I - A^{*j} A^j\right)^{1/2} x = 0 \Leftrightarrow A^{*j} A^j x = x$$

for all $j \geq 0$. Taking the strong limit gives $L_A x = x$.

The dual claim (46) follows by considering the dual DLS instead. □

Note that $L_S = I$ but $L_{S^*} = 0$, where S denotes the forward shift on $\ell^2(\mathbb{Z}_+)$. Hence, there is no general relation between the observable and controllable subspaces of a conservative linear system (as any contraction can appear as a main operator for some conservative DLS).

Conservative DLS called *simple*, if its operator A is completely nonunitary (shortly: c.n.u.); i.e. there is no reducing subspace $X_u \neq \{0\}$, given by

$$X_u = \{x \in X \quad | \quad \|A^j x\| = \|x\| = \|A^{*j} x\| \quad \text{for all} \quad j \in \mathbb{Z}_+\} \subset X,$$

where A operates unitarily. By Proposition A.3, the unitary subspace X_u in fact equals $\operatorname{range}(\mathcal{B}_\phi)^\perp \cap \ker(\mathcal{C}_\phi)$ for any conservative DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Hence, a minimal conservative DLS is always simple, but the converse claim does not hold. It is well known that any contraction A can appear as the main operator for a conservative DLS. This gives us a restatement of Proposition A.3 in terms of A, A^*, L_A and L_{A^*}:

Corollary A.4. *Let $A \in \mathcal{L}(X)$ be a contraction. Then the second equalities in* (45) *and* (46) *hold. Moreover, $A \ker(I - L_A) \subset \ker(I - L_A)$ and $A^* \ker(I - L_{A^*}) \subset \ker(I - L_{A^*})$. The unitary subspace of A satisfies $X_u = \ker(I - L_A) \cap \ker(I - L_{A^*})$.*

The stability of the main operator A has a natural connection to the transfer function of the system. We say that $\widehat{\mathcal{D}}_\phi \in \mathcal{S}(\mathbb{D}; \mathcal{L}(U;Y))$ is *inner from the left (resp. right)*, meaning that the nontangential boundary trace $\widehat{\mathcal{D}}_\phi(e^{i\theta}) \in \mathcal{L}(U;Y)$ is an isometry (co-isometry) for almost all $e^{i\theta} \in \mathbb{T}$.

Proposition A.5. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a conservative DLS. Then $\widehat{\mathcal{D}}_\phi$ is inner from the left if and only if*

$$A^j x \to 0 \quad \textit{for all} \quad x \in \overline{\operatorname{range}(\mathcal{B}_\phi)} \tag{47}$$

if and only if $\operatorname{range}(I - L_{A^*}) \subset \ker(L_A)$. *Dually, $\widehat{\mathcal{D}}_\phi$ is inner from the right if and only if*

$$A^{*j} x \to 0 \quad \textit{for all} \quad x \in \ker(\mathcal{C}_\phi)^\perp$$

if and only if $\operatorname{range}(I - L_A) \subset \ker(L_{A^*})$.

Proof. Let $\tilde{u} = \{u_j\}_{j \geq 0} \in \ell^2(\mathbb{Z}_+; U)$ be such that $u_j = 0$ for all $j \geq n$, for any given $n \geq 1$. Then for all $k \geq 0$ we have $x_{n+k} = A^k x_n$ for the solution of (41) with initial condition $x_0 = 0$. Rewriting now the energy balance (43) we obtain

$$\sum_{j=0}^{n-1} \|u_j\|_U^2 - \sum_{j=0}^{n+k-1} \|u_j\|_U^2 = \|x_{n+k}\|_X^2 = \|A^k x_n\|_X^2.$$

Assuming (47) and letting $k \to +\infty$, we now obtain for such $\tilde{u}$

$$\|\mathcal{D}_\phi \tilde{u}\|_{\ell^2(\mathbb{Z}_+;Y)} = \|\tilde{u}\|_{\ell^2(\mathbb{Z}_+;U)} \tag{48}$$

because $x_n \in \operatorname{range}(\mathcal{B}_\phi)$. By using the contractivity of $\mathcal{D}_\phi : \ell^2(\mathbb{Z}_+;U) \to \ell^2(\mathbb{Z}_+;Y)$ and density of such $\tilde{u}$ in $\ell^2(\mathbb{Z}_+;U)$, we extend (48) to all of $\ell^2(\mathbb{Z}_+;U)$. By the Fourier representation, this is equivalent to $\widehat{\mathcal{D}}_\phi$ to be inner from the left. Reading the above argument in converse direction, we see that assuming (48) we get $A^j x \to 0$ for all $x \in \operatorname{range}(\mathcal{B}_\phi)$. Because the family $\{A^j\}_{j\geq 0}$ is uniformly bounded, the same holds for all $x \in \overline{\operatorname{range}(\mathcal{B}_\phi)}$. The second equivalence in the chain of equivalences follows from (46) noting that $\ker(I - L_{A^*})^\perp = \overline{\operatorname{range}(I - L_{A^*})}$ by self-adjointness. The latter claim involving A^* follows by considering the dual DLS ϕ^d. $\square$

Note that if A is normal in previous proposition, then $L_A = L_A^2 = L_{A^*} = L_{A^*}$ and $\operatorname{range}(I - L_{A^*}) = \ker(L_A)$ follows. Hence, the corresponding transfer function $\widehat{\mathcal{D}}_\phi$ is inner from both sides.

We moreover emphasize that the main operator A of a conservative DLS ϕ must be "rich" in some ergodic sense, so as to make it possible for $\widehat{\mathcal{D}}_\phi$ not to be inner. All conservative DLSs with finite dimensional state space X have inner transfer functions; the possible eigenvalues $\sigma(A) \cap \mathbb{T}$ correspond to eigenvectors that belong to X_u. Indeed, if $Ax = x$ then $\|A^j x\|_X = \|x\|_X$ for all $j \geq 0$. Moreover, $\|Ax\|_X = \|x\|_X$ implies $\langle (I - A^*A)x, x\rangle_X = 0$ and hence $x - A^*x = (I - A^*A)x = 0$, by the contractivity of A. We conclude that $\|A^{*j}x\|_X = \|x\|_X$ for all $j \geq 0$, and thus $x \in X_u$. The same argument holds without change if A is compact and $\dim X = \infty$.

If we know that the Hankel operator $\bar{\pi}_+ \mathcal{D}_\phi \pi_-$ of $\widehat{\mathcal{D}}_\phi$ has closed range, then some stronger conclusions can be drawn for *minimal* conservative DLSs[7]. For rational functions $\mathcal{L}(U)$-valued functions $\dim U < \infty$ this is always the case, by Kroneckers theorem. Moreover, the Hankel range for any inner (from both sides) analytic function (rational or not) is closed, as any such operator is a partial isometry.

Proposition A.6. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a minimal, conservative DLS with* $U = Y$ *and* $\dim U < \infty$*. Assume that* $\operatorname{range}(\bar{\pi}_+ \mathcal{D}_\phi \pi_-) \subset \ell^2(\mathbb{Z}_+;U)$ *is closed.*

Then ϕ *is exactly observable and exactly controllable, both* A *and* A^* *are strongly stable, and* $\widehat{\mathcal{D}}_\phi$ *is inner from both sides.*

Proof. By the usual theory for DLSs, the Hankel operator has the factorization $\bar{\pi}_+ \mathcal{D}_\phi \pi_- = \mathcal{C}_\phi \mathcal{B}_\phi$ on all of $\ell^2(\mathbb{Z}_-;U)$, where $\mathcal{B}_\phi$ has been extended to all

[7]Here $\bar{\pi}_+$, π_- denote the natural orthogonal projections in $\ell^2(\mathbb{Z};U)$ onto $\ell^2(\mathbb{Z}_+;U)$, $\ell^2(\mathbb{Z}_-;U)$, respectively.

of $\ell^2(\mathbb{Z}_-;U)$ by continuity[8]. We get

$$\begin{aligned}\operatorname{range}(\bar\pi_+\mathcal{D}_\phi\pi_-) &= \mathcal{C}_\phi\left(\mathcal{B}_\phi\ell^2(\mathbb{Z}_-;U)\right) \subset \mathcal{C}_\phi\overline{(\mathcal{B}_\phi\ell^2(\mathbb{Z}_-;U))}\\ &\subset \overline{\mathcal{C}_\phi\left(\mathcal{B}_\phi\ell^2(\mathbb{Z}_-;U)\right)} = \overline{\operatorname{range}(\bar\pi_+\mathcal{D}_\phi\pi_-)},\end{aligned}$$

where the last inclusion sign is by the continuity of $\mathcal{C}_\phi$. Because $\operatorname{range}(\bar\pi_+\mathcal{D}_\phi\pi_-)$ is closed and $\mathcal{B}_\phi\ell^2(\mathbb{Z}_-;U)$ is dense in X, we conclude that so is $\operatorname{range}(\mathcal{C}_\phi) = \mathcal{C}_\phi X$. As $\ker(\mathcal{C}_\phi) = \{0\}$, we conclude that $\mathcal{C}_\phi$ is coercive; i.e. ϕ is exactly observable. For all $x \in X$ we have (by a basic property for DLSs) $\mathcal{C}_\phi A^j x = S^{*j}\mathcal{C}_\phi x \to 0$ as $j \to \infty$, whence $A^j x \to 0$ as $j \to \infty$ by coercivity.

By Proposition A.5, we conclude that $\widehat{\mathcal{D}}_\phi$ is inner from the left, and it is also inner from the right because $U = Y$ and $\dim U < \infty$ (just look at the nontangential boundary traces of such H^∞ matrix functions, and note that the nontangential limits converge in matrix norm, by finite dimensionality). Again by Proposition A.5, we conclude that $A^{*j}x \to 0$ for all $x \in \ker(\mathcal{C}_\phi)^\perp = X$. The exact controllability follows by dualizing. □

We remark that there exists a conservative DLS with $\dim U = 0$ and $\dim Y = 1$. Moreover, it is not so difficult to construct a $\mathbb{C}^{1\times 2}$-valued function that is inner from the left, not inner from the right, and whose Hankel operator is infinite-dimensional and compact. Hence, the assumption $U = Y$ in previous proposition cannot be removed.

We conclude this section by recalling a fundamental realization result for Schur functions.

Proposition A.7. *Let U, Y be separable Hilbert spaces. Then the following holds*

(i) The set of transfer functions for simple conservative DLSs (with input space U and output space Y) is exactly the Schur class $\mathcal{S}(\mathbb{D};\mathcal{L}(U;Y))$.

(ii) The state spaces of two simple conservative DLSs $\phi_1 = \left(\begin{smallmatrix} A_1 & C_1\\ B_1 & D_1\end{smallmatrix}\right)$ and $\phi_2 = \left(\begin{smallmatrix} A_2 & C_2\\ B_2 & D_2\end{smallmatrix}\right)$ are unitarily equivalent, i.e. for some $U \in \mathcal{L}(X)$, $U^*U = UU^* = I$,

$$A_2 = U^*A_1U, \quad B_2 = U^*B_1, \qquad C_2 = C_1U,$$

if and only if $\widehat{\mathcal{D}}_{\phi_1} = \widehat{\mathcal{D}}_{\phi_2}$.

Proof. A good classical reference to these results is [5]. □

[8] For clarity, we write $\mathcal{B}_\phi\ell^2(\mathbb{Z}_-;U)$ for the range of this extension, rather than $\operatorname{range}(\mathcal{B}_\phi)$ that has a different meaning.

References

[1] N. I. Akhiezer. *The Classical Moment Problem.* Oliver and Boyd Ltd, 1965.

[2] D. Z. Arov and M. A. Nudelman. Passive linear stationary dynamical scattering systems with continuous time. *Integral equations and operator theory*, 24:1–45, 1996.

[3] J. Ball and J. W. Helton. A Beurling-Lax theorem for the Lie group $u(m, n)$ which contains most classical interpolation theory. *Journal of Operator Theory*, 9:107–142, 1983.

[4] M. S. Brodskiĭ. *Triangular and Jordan Representations of Linear Operators*, volume 32 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, Rhode Island, 1971.

[5] M. S. Brodskĭ. Unitary operator colligations and their characteristic functions. *Russian Math. Surveys*, 33(4):159–191, 1978.

[6] C. I. Byrnes, T. T. Georgiou, and A. Lindquist. A New Approach to Spectral Estimation: A Tunable High-Resolution Spectral Estimator. *IEEE Trans. Signal Processing*, 48(11):3189–3205, November 2000.

[7] C. I. Byrnes, T. T. Georgiou, and A. Lindquist. A generalized entropy criterion for Nevanlinna-Pick interpolation with degree constraint. *IEEE Trans. Automat. Control*, 46(6):822–839, June 2001.

[8] C. Carathéodory. Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen. *Math. Ann.*, 64:95–115, 1907.

[9] C. Carathéodory. Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. *Rend. di Palermo*, 32:193–217, 1911.

[10] Ph. Delsarte, Y. Genin, and Y. Kamp. Schur Parametrization of Positive Definite Block-Toeplitz Systems. *SIAM J. Appl. and Math.*, 36:34–46, Feb 1979.

[11] Ph. Delsarte, Y. Genin, and Y. Kamp. On the role of the Nevanlinna-Pick problem in circuit and system theory. *Circuit Theory and Applications*, 9:177–187, 1981.

[12] C. Foias and A. E. Frazho. *The Commutant-Lifting Approach to Interpolation Problems*. Birkhäuser, Boston, 1990.

[13] T. T. Georgiou. *Partial Realization of Covariance Sequences*. PhD thesis, University of Florida, Gainesville, 1983.

[14] R. Kalman. Realization of covariance sequences. In I. Gohberg, editor, *Toeplitz Centennial*, volume 4 of *Operator Theory: Advances and Applications*, pages 331–342. Birkhäuser, 1981.

[15] R. E. Kalman, P. L. Falb, and M. A. Arbib. *Topics in Mathematical System Theory*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Company, 1969.

[16] H. Kimura. Robust stabilizability for a class of transfer functions. *IEEE Trans. Automat. Control*, 29(10):788–793, October 1984.

[17] J. Malinen. *Discrete Time H^∞ Algebraic Riccati Equations*. PhD thesis, Institute of Mathematics, Helsinki University of Technology, 2000.

[18] J. Malinen, O. Staffans, and G. Weiss. When is a linear system conservative? *Mittag–Leffler preprints 2003/46; Submitted*, 2003.

[19] R. Nagamune. *Robust control with complexity constraint: A Nevanlinna-Pick interpolation approach*. PhD thesis, Department of Mathematics, Royal Institute of Technology, 2002.

[20] M. Rosenblum and J. Rovnyak. *Hardy Classes and Operator Theory*. Oxford University Press, 1985.

[21] W. Rudin. *Functional Analysis*. McGraw-Hill Book Company, New York, TMH edition, 1990.

[22] I. Schur. On power series which are bounded in the interior of the unit circle I. *J. für die Reine und Angewandte Mathematik*, 147:205–232, 1917.

[23] I. Schur. On power series which are bounded in the interior of the unit circle II. *J. für die Reine und Angewandte Mathematik*, 148:122–145, 1918.

[24] M. S. Livšic. *Operators, Oscillations, Waves (Open Systems)*, volume 34 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1973.

[25] B. Sz.-Nagy and C. Foias. *Harmonic Analysis of Operators on Hilbert space.* North-Holland Publishing Company, Amsterdam, London, 1970.

[26] A. Tannenbaum. Feedback stabilization of linear dynamical plants with uncertainty in the gain factor. *Int. J. Control*, 32(1):1–16, 1980.

[27] A. Tannenbaum. Modified Nevanlinna-Pick interpolation and feedback stabilization of linear plants with uncertainty in the gain factor. *Int. J. Control*, 36(2):331–336, 1982.

[28] J. L. Walsh. *Interpolation and Approximation by Rational Functions in the Complex Domain.* American Mathematical Society, Colloquium Publications, Providence, R. I., 1956.

[29] D. C. Youla and M. Saito. Interpolation with positive-real functions. *J. Franklin Institute*, 284:77–108, 1967.