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EXTERIOR SPACE MODEL FOR AN ACOUSTIC EIGENVALUE PROBLEM

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Summary. A method is introduced for speeding up resonance computations for modelling human speech production. In the method, a reduced order basis is constructed for treating the exterior acoustic domain. This allows considerable reduction in the dimension of the eigenvalue problem and leads to substantial savings in computational time.

1 Introduction

The computation of resonant frequencies of the vocal tract air volume is a central problem in modelling human speech. The resonances are related to a quadratic eigenvalue problem[1] which should be solved for several vocal tract geometries corresponding to different vowels. In general, the exterior acoustic space is assumed not to affect the resonances, and the computational domain is limited only to the vocal tract volume. However, recent Magnetic Resonance Imaging (MRI) measurements and simultaneous voice recordings indicate that this results in an error of 2.1 semitones[2]. Motivated by this observation, we propose an efficient method for including a realistic exterior space model into the resonance computation. The performance of the proposed method is tested in a $2D$ benchmark problem.

For simplicity, we consider the following problem: Let $\Omega \subset \mathbb{R}^2$. Find $(\lambda, u) \in (\mathbb{R}, H_0^1(\Omega))$ such that

$$(\nabla u, \nabla v) = \lambda(u, v) \quad \forall v \in H_0^1(\Omega), \tag{1}$$

We assume that $\Omega = \Omega_1 \cup \Omega_2$ where the distinct domains Ω_1 and Ω_2 correspond to the vocal tract and the exterior domain of the acoustic space surrounding the head, respectively. The interface between Ω_1 and Ω_2 is denoted by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The physically meaningful eigenvalues lie within a known range $[\lambda_{min}, \lambda_{max}]$. We note that when the speed of sound is set to one, the computed eigenvalues are squares of the resonant frequencies.

In a typical simulation, Ω_2 remains unchanged while Ω_1 varies between different vowels and patients. Due to the large number of degrees of freedom related to Ω_2, solving the full eigenvalue problem is too costly. Thus, we propose using a reduced basis for discretisation of the exterior domain. We construct such a basis by solving several Helmholtz equations with different inputs at the interface between the vocal tract and the surrounding space. This basis is computed only once, and it remains unchanged when the vocal tract geometry Ω_1 is varied. Similar domain decomposition methods have been studied[3] in the past, but due to the small size of the interface

Γ they are not adequate for our purposes. In the context of acoustics, dimension reduction on the exterior space has been used, e.g., for optimising loudspeaker horn design[4].

We use the finite element method to discretise the eigenvalue problem and an iterative method to solve it. In order to easily exchange different interior geometries, Nitsche's method is used to glue the interior and exterior finite element meshes together. This allows us to use non-matching grids, which considerably simplifies the mesh generation for the interior domain. See Becker et al.[5] for a description of the Nitsche's method.

2 The Method

Discretisation of problem (1) leads to the following generalised eigenvalue problem: Find $(\lambda, \boldsymbol{x}) \in (\mathbb{R}, \mathbb{R}^N)$ such that

$$A\boldsymbol{x} = \lambda M\boldsymbol{x}, \tag{2}$$

where $A, M \in \mathbb{R}^{N\times N}$. Let $\mathcal{N}_1, \mathcal{N}_2$ be the set of degrees of freedom supported in Ω_1 and Ω_2, respectively. This division allows us to decompose the discrete eigenvalue problem as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \lambda \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \tag{3}$$

in which vector $\boldsymbol{x}_i$ contains the coefficients related to degrees of freedom $\mathcal{N}_i$. Since the domain Ω_2 is fixed over all computations, our aim is to eliminate the unknown $\boldsymbol{x}_2$. This elimination is done only once, and it can be allowed to be computationally costly. From the second row in (3), it follows that

$$(A_{22} - \lambda M_{22})\boldsymbol{x}_2 = -(A_{21} - \lambda M_{21})\boldsymbol{x}_1. \tag{4}$$

Based on this expression, we could eliminate the unknown $\boldsymbol{x}_2$. However, the relation between the two vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ will be a λ-dependent linear mapping

$$B(\lambda) = -(A_{22} - \lambda M_{22})^{-1}(A_{21} - \lambda M_{21}). \tag{5}$$

Thus, the direct elimination of $\boldsymbol{x}_2$ would lead to a non-linear eigenvalue problem which is not desirable. Instead, we form an explicit approximation of $B(\lambda)$, which is accurate in the interval $[\lambda_{min}, \lambda_{max}]$.

The matrix $(A_{21} - \lambda M_{21})$ connects the degrees of freedom in Ω_1 and Ω_2 that are joined at the interface Γ. As such, it has a relatively low rank k. Let $\{\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_k \,\}$ be a set of linearly independent vectors such that $\operatorname{span}\{\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_k \,\} = \operatorname{range}(A_{21} - \lambda M_{21})$, and define $V_\lambda := \operatorname{span}\{\, (A_{22} - \lambda M_{22})^{-1}\boldsymbol{q}_1, \ldots, (A_{22} - \lambda M_{22})^{-1}\boldsymbol{q}_k \,\}$. It follows from equation (4) that $\boldsymbol{x}_2 \in V_\lambda$. Our method relies on finding $\widetilde{\boldsymbol{x}}_2 \in V_{\lambda_i}$ that approximates $\boldsymbol{x}_2$ well when $\lambda_i \approx \lambda$. Motivated by this, we construct a solution space W for the exterior domain Ω_2 as

$$W = \operatorname{span}(V_{\lambda_1}, \ldots, V_{\lambda_n}), \tag{6}$$

in which $\{\, \lambda_i \,\}_{i=1}^n \subset (\lambda_{min}, \lambda_{max})$ is a pre-selected set of samples. In practice, we need to construct a basis for this space. This is done by solving a series of problems

$$(A_{22} - \lambda_i M_{22})\boldsymbol{y}_{ij} = \boldsymbol{q}_j. \tag{7}$$

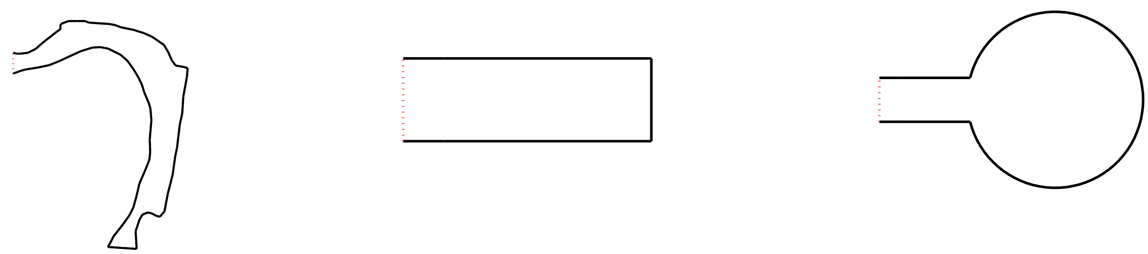

Figure 1: The collection of interior domains used in our computations. From left to right: a midsection of a vocal tract, a $(0,3) \times (0,1)$ rectangle, and union of the aforementioned rectangle and a circle of radius 2, centrepoint $(4, 0.5)$. The interface Γ is drawn in a dotted line.

The sample vectors $\boldsymbol{y}_{ij}$, will be linearly dependent. We form a matrix Y that has the sample vectors $\boldsymbol{y}_{ij}$ as it's columns. An orthormal basis for the space W can be obtained from the columns of U where $U\Sigma V = Y$ is the SVD of Y. We truncate the basis by choosing the columns of U that correspond to singular values σ_k satisfying $\sigma_k > \sigma$ for some truncation lower bound $\sigma > 0$. The corresponding truncated matrix is denoted by $\widetilde{U}$, and any vector in this basis can be written as $\widetilde{U}\alpha$ where $\alpha \in \mathbb{R}^{N_r}$. The original problem (3) can be reduced to the following lower order eigenvalue problem

$$\begin{bmatrix} A_{11} & A_{12}\widetilde{U} \\ \widetilde{U}^* A_{21} & \widetilde{U}^* A_{22}\widetilde{U} \end{bmatrix} \begin{bmatrix} x_1 \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} M_{11} & M_{12}\widetilde{U} \\ \widetilde{U}^* M_{21} & \widetilde{U}^* M_{22}\widetilde{U} \end{bmatrix} \begin{bmatrix} x_1 \\ \alpha \end{bmatrix}. \tag{8}$$

The trick is to keep N_r as small as possible while maintaining accuracy, i.e., without excessively increasing σ.

3 Numerical Experiments

We consider the fixed exterior domain $\Omega_2 = (0, 30) \times (0, 35)$ and three interior domains Ω_1 that are pairwise connected at the same interface $\Gamma = [0, 1]$. The interior domains are shown in Figure 1. The interface is placed on the domain Ω_2 so that it lies in the middle of the longer side. A basis for Ω_2 is computed by using the method presented in Section 2. The sampling points were linearly chosen from the interval $[2, 5]$.

To measure the quality of the eigenvalues computed from the reduced problem, we use the measure

$$\operatorname{dist}(\lambda, \Lambda_h) := \min_{\lambda_h \in \Lambda_h} |\lambda - \lambda_h|, \tag{9}$$

in which λ is a given eigenvalue of the full problem (2) and Λ_h is the set of eigenvalues for the reduced problem. The measure $\operatorname{dist}(\lambda, \Lambda_h)$ is shown in Figure 2 as a function of λ for two different numbers of sample points. We reach excellent accuracy when a sufficiently large number of samples is used. The number of degrees of freedom in Ω_2 is 3084, and the dimension of the reduced basis is 278 with 301 sample points $\{ \lambda_j \}$ and $\sigma = 10^{-6}$. The number of eigenvalues of the full problem on the interval $[2, 5]$ is 200. In our implementation, the computational time for the full eigenvalue problem was 19.4 seconds, whereas the reduced problem only took 6.6 seconds after the basis was formed.

To quantify the effect of the sample points to the accuracy of the method on the interval $[\lambda_{min}, \lambda_{max}]$, we use the measure

$$s^2 = \sum_{\lambda_{min} < \lambda_i < \lambda_{max}} \mathrm{dist}(\lambda_i, \Lambda_h)^2. \tag{10}$$

The measure s is plotted in Figure 2 as a function of the dimension of the reduced basis.

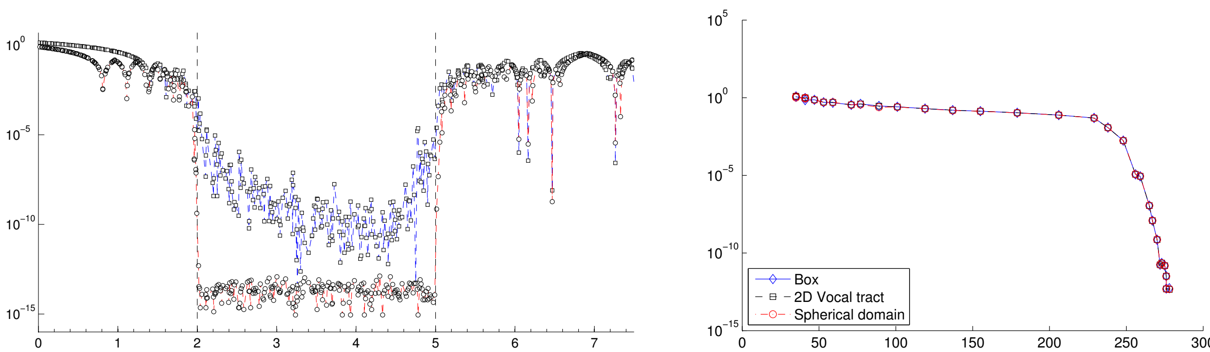

Figure 2: Left: The error measure defined in (9) using the $2D$ vocal tract geometry as Ω_1. The distance between two adjacent sample points was 0.05 and 0.01. Right: The measure s defined in (10) using the three different geometries for Ω_1.

We observe that the method converges rapidly regardless of the interior geometry used. The method was found to be promising but proper mathematical analysis of eigenvalue convergence for the reduced problem is needed. In addition, a better sampling strategy is likely to be required to handle real world geometries.

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