European Journal of Control (2003)9:605–607

European Journal of Control

Discussion on: “Boundary Control of a Class of Hyperbolic Systems”

Jarmo Malinen*

Institute of Mathematics Box 1100, Fin-02015, Helsinki University of Technology, Finland

We review the example of linear quadratic regulator (LQR) problem given in paper[2] by A. Chapelon and C.-Z. Xu. We discuss three different algebraic Riccati equations that are associated to such LQR problems.

Keywords: Optimal Control; Linear Quadratic Regulator; Operator Algebraic Riccati Equation

1. Introduction

The linear quadratic regulator (LQR) problem for the linear boundary control system, governed by the hyperbolic PDE

$$\partial_t \begin{bmatrix} R^-(x,t) \\ R^+(x,t) \end{bmatrix} = (A(x)\partial_x + K(x)) \begin{bmatrix} R^-(x,t) \\ R^+(x,t) \end{bmatrix},$$
$$R^-(0,t) = D_0 R^+(0,t) + u^-(t),$$
$$R^+(1,t) = D_1 R^-(1,t) + u^+(t),$$
$$R(x,0) = R^0(x), \quad \text{for all } (x,t) \in (0,1) + \mathbb{R}_+; \tag{1}$$

is considered in [2]. Here all the functions $A(x)$, $K(x)$, $R^{\pm}(x,t)$, $u^{\pm}(t)$, and $R^0(x)$ are vector/matrix-valued with compatible dimensions. It is shown in [2] (under some extra assumptions) that (1) defines a stable, exactly controllable, regular well-posed linear system on Hilbert space H, described by the Cauchy problem

$$\begin{aligned} z'(t) &= A_{-1}z(t) + Bu(t), \\ y(t) &= Cz(t) + Du(t), \quad t \geq 0; \\ z(0) &= z_0; \end{aligned} \tag{2}$$

in the sense of e.g. [12–14,18]. Here $C = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ and $A_{-1} : H \to (\mathrm{dom}(A^*))'$ is the usual extension of $A : \mathrm{dom}(A^*) \to H$. The LQR problem is to find the (unique) minimizing input $u_{\min}(z_0, \cdot)$ for the cost functional

$$J(z_0, u) = \int_0^\infty \|y(t)\|^2_{H \times U}\, \mathrm{d}t,$$

and to express the required control law in “feedback form”. The optimal cost is given by the *Riccati operator* $P = P^* \geq 0$ on H in the sense that $J(z_0, u_{\min}) = \langle z_0, Pz_0 \rangle_H$ for any initial state $z_0 \in H$. Such a P can be shown to satisfy several *continuous time algebraic Riccati equations*, henceforth referred as CAREs. The classical reference for matrix CARE is [7], but also [1] is highly recommendable reading. In the infinite-dimensional case, there exists two main “schools” that write down their CAREs in essentially different ways, but we shall not treat these differences here.

The earlier operator CARE results and techniques are motivated by systems governed by concrete PDEs, and they can be found in [3,4,8,9,16]. The later results are given for abstractly defined *regular well-posed linear systems*. These operator CAREs are generalizations of the matrix CARE, and they are derived by spectral factorization techniques. The main references

*E-mail: Jarmo.Malinen@hut.fi

are [14, 20–23], and these papers are the background for [2].

2. Why Should We Care about CAREs?

Because of the mathematical simplicity of the underlying PDE (1), the authors of [2] are able to integrate it explicitly. An explicit formula for the regular transfer function $\widehat{\mathcal{G}}(s) = [(s - A_{-1})^{-1} \quad B \quad I]^{\mathrm{T}}$ of (2) is obtained, as well as for the spectral function

$$\Pi(\mathrm{i}\omega) = \widehat{\mathcal{G}}(\mathrm{i}\omega)^* \widehat{\mathcal{G}}(\mathrm{i}\omega). \tag{3}$$

As $\Pi(\mathrm{i}\omega) \geq I$ for all $\omega \in \mathbb{R}$, the stable outer spectral factor $\Xi \in H^\infty(\mathbb{C}_+)$ exists. Furthermore, Ξ can be given in closed form (see [2, Lemma 5.1]), and it is regular; i.e. the strong limit

$$X = \lim_{s \to \infty} \Xi(s) \quad \text{exists and satisfies } X^{-1} \in \mathcal{L}(U).$$

Under these conditions, all terms have explicit formulas in the CARE

$$PA + A_{-1}^* P + I = (B_s^* P)^* \cdot (X^* X)^{-1} \cdot B_s^* P \quad \text{on dom}(A)$$

given in [14,22]. Here $A_{-1}^* : H \to (\mathrm{dom}(A^*))'$ is the extension of $A^* : \mathrm{dom}\,(A^*) \to H$, $B_s^* P^*$ is the strong Yoshida extension of B^*, and $(B_s^* P)^*$ is the adjoint of $B_s^* P \in \mathcal{L}(\mathrm{dom}(A); U)$. The Riccati operator P maps (A) into $\mathrm{dom}(B_s^*)$, and it is a solution of (4).

But once we have a formula for outer spectral factor Ξ, the LQR problem has already been fully solved. Indeed, the optimal control law can then be expressed in feedback form by using the results for well-posed linear systems, given in [14]. To obtain the generating operators (and their domains) of the (regular) closed loop system, the results of [17] should be used. The Riccati operator P is the observability gramian of the closed-loop system.

The motivation for writing down and solving a CARE (of any kind) is to large extent gone. Note that solving (discrete time) algebraic Riccati equations is typically *equivalent* to computing spectral factors in the associated state space, see [5,6,10,11]. We must conclude that the system described by (1) is too simple to qualify as a relevant test bench for CAREs.

3. What Sort of CAREs Should We Solve?

Let us discuss on a general level, when and why we would like to deal with CAREs.

Possibly the most important *practical* applications of state space techniques are related to numerical computations for linear systems which (unlike (1)) *do not* allow for a concrete and explicit solution formulas. Most relevant PDE problems are of this kind. A control problem is often defined via some particular realization, constituting the data of the problem. The transfer function or the spectral factor are then inaccessible. In order to be practically useful, the form of a CARE is thus subject to many restrictive conditions. These conditions are expected to vary from one problem to another.

The situation looks particularly grim so as to CARE (4), as we need to establish the regularity of Ξ and compute X^*X before being able to write down the equation, let alone solve it. It might be the case that in some LQR problem X^*X could be computed without knowing all of Ξ, but we do not know of any such problem. It is true that in the LQR case, by [15, Corollary 7.2]

$$X^* X = I + \operatorname*{slim}_{\alpha \to \infty} B_s^* P (\alpha - A_{-1})^{-1} B. \tag{5}$$

Formula (5) contains the Riccati operator P to be solved and also the Yoshida extension B_s^* that is difficult to obtain in relevant problems.[1] If a practical description for B_s^* could be given, then (4) together with (5) might perhaps be used for updating purposes in an iterative numerical solver.

It is not an unrelated accident that the discrete time Riccati equation (DARE)

$$\begin{aligned} A_d^* P A_d - P &= -C_d^* C_d + Q_P^* \Lambda_P^{-1} Q_P, \\ Q_P &= -D_d^* C_d - B_d^* P A_d, \\ \Lambda_P &= D_d^* D_d + B_d^* P B_d \end{aligned} \tag{6}$$

contains the solution P in the "indicator" operator Λ_{P} – even in the LQR case when $D_d^* C_d = 0$ and $D_d^* D_d = I$. However, no extensions of B_d^* or any limits are required. It thus seems desirable to solve the spectral factorization problem (3) in "discrete time setting" by using the Cayley transform (7). The resulting DAREs are of form (6) where

$$\begin{aligned} A_d &= (\bar{\alpha} + A)(\alpha - A)^{-1}, \\ B_d &= \sqrt{2\Re\alpha}(\alpha - A_{-1})^{-1} B, \\ C_d &= \sqrt{2\Re\alpha}\, C(\alpha - A)^{-1}, \\ D_d &= \widehat{\mathcal{G}}(\alpha), \end{aligned} \tag{7}$$

[1] If we were able to compute as little as just the domain for the weak Yoshida extension B_w^* for a system Σ, then we would know if (the dual of) Σ was weakly regular or not, see [18, Theorem 5.8]. A nontrivial special case is declared as an open problem in [19].

and $\alpha \in \mathbb{C}_+$ is an arbitrary parameter. Unfortunately, the resolvents of A are involved in the resulting DAREs, and they can only rarely be given explicitly. This restricts the theoretical applicability of such DAREs seriously. Dealing with resolvents of (the discretized versions of) A is numerically very expensive, too.

The Riccati operator P satisfies yet another CARE, given in all its beauty by

$$PA_{-1} + A^*P + I = PBB^*P \quad \text{on } \mathrm{dom}(A_P), \quad (8)$$

where

$$\mathrm{dom}(A_P) = \{x \in H \,|\, (A_{-1} + BB^*P)x \in H\}.$$

Because the output operator in (2) satisfies $C \in \mathcal{L}(H; H \times U)$, it follows that $P \in \mathcal{L}(\mathrm{dom}(A_P); \mathrm{dom}\,(A^*))$, see e.g. [14, Corollary 43]. Hence the difficult term B^*P is well-defined without any extensions. Unfortunately, CARE (8) is posed in $\mathrm{dom}\,(A_P)$ which is *a priori* unknown. It is though conceivable that (8) could be sometimes solved numerically in some finite-dimensional (e.g. finite element) subspaces of $\mathrm{dom}\,(A_P)$

4. Conclusion

We conclude that CARE (4) has a serious attitude problem: it offers eagerly a helping hand, but only when it is needed least. The DAREs obtained from (6) and (7) contain resolvents of A, which makes them unpractical. Finally, CARE (8) provides us an impeccable description about what to do, but not a hint where to do it.

References

1. Bittanti S, Laub AJ, Willems JC (eds). The Riccati equation Springer-Verlag, 1991
2. Chapelon A, Xu C-Z. Boundary control of a class of hyperbolic systems. Eur J Control 2003
3. Flandoli F. Riccati equation arising in a boundary control problem with distributed parameters. SIAM J Control Optim 22(1): 76–86
4. Flandoli F, Lasiecka I, Triggiani R. Algebraic Riccati equations with non-smoothing observation arising in hyperbolic and Euler–Bernoulli boundary control problems. Ann Math Pura Appl 1988; CLIII: 307–382
5. Fuhrmann PA. On the characterization and parameterization of minimal spectral factors. J Math Sys, Estim, Control 1995; 5(4): 383–444
6. Fuhrmann PA, Hoffmann J. Factorization theory for stable discrete-time inner functions. J Math Syst, Estim, Control 1997; 7(4): 383–400
7. Lancaster P, Rodman L. Algebraic Riccati equations. Clarendon Press, Oxford, 1995
8. Lasiecka I, Triggiani R. Differential and algebraic Riccati equations with application to boundary/point control problems: Theory and approximation theory, vol 164. Springer-Verlag, 1991
9. Lasiecka I, Triggiani R. Algebraic Riccati equations arising from systems with unbounded input-solution operators: Applications to boundary control problems for wave and plate equations. Nonlinear Anal, Theory, Meth Appl 1993; 20(6): 659–695
10. Malinen J. Discrete time H^∞ algebraic Riccati equations. Doctoral dissertation, 2000
11. Malinen J. Shift realizations and their algebraic Riccati equations. Mittag–Leffler preprints, 2003
12. Salamon D. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. Trans Amer Math Soc 1987; 300: 383–431
13. Salamon D. Realization theory in Hilbert spaces. Math. Syst Theory 1989; 21: 147–164
14. Staffans OJ. Quadratic optimal control of stable well-posed linear systems. Trans Amer Math Soc 1997; 349: 3679–3715
15. Staffans OJ. Quadratic optimal control of well-posed linear systems. SIAM J Control Optim 1998; 37:131–164
16. Triggiani R. An optimal control problem with unbounded control operator and unbounded observation operator where the algebraic Riccati equations is satisfied as a Lyapunov equation. Appl Math Lett 1997; 10(2): 95–102
17. Weiss G. Regular linear systems with feedback. Math Control, Signals, Syst 1994; 7: 23–57
18. Weiss G. Transfer functions of regular linear systems, Part I: characterizations of regularity. Trans Amer Math Soc 1994; 342(2): 827–854
19. Weiss G. Is the conservative wave equation regular? In: Blondel V, Megretski A (eds). Open problems in mathematical systems and control theory, chapter 74. Princeton University Press, 2003
20. Weiss G, Zwart H. An example in linear quadratic optimal control. Syst Control Lett 1998; 33:339–349
21. Weiss M. Riccati equation theory for Pritchard-Salamon systems: a Popov function approach. IMA J Math Control Information, 1996
22. Weiss M, Weiss G. Optimal control of stable weakly regular linear systems. Math Control, Signals, and systems, 1997; 10: 287–330
23. Zwart HJ. Linear quadratic optimal control for abstract linear systems. In: Malanowski K, Nahorski Z, Peszynska M (eds). Modelling and optimization of distributed parameter systems; Applications to engineering. Chapman & Hall, 1996