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Semigroups of Impedance Conservative Boundary Control Systems

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Abstract

The external Cayley transform is used for the conversion of linear dynamical systems between scattering and impedance forms.

We study this transform in the formal setting of *colligations* which includes all impedance (energy) conservative boundary control systems. Practically motivated sufficient conditions are given to describe when such colligations have well-defined semigroups. We apply these results to a model problem, namely the transmission line equations.

We explain how all this relates to the *abstract boundary space* construction for symmetric operators.

1 Introduction

This paper deals with linear boundary control/observation systems described by differential equations of the form

$$u(t) = Gz(t), \quad \dot{z}(t) = Lz(t), \quad y(t) = Kz(t), \quad t \in [0, \infty). \tag{1.1}$$

In a typical application L is a partial differential operator on a bounded domain $\Omega \subset \mathbb{R}^n$, and G and K are composed of some boundary trace operators on $\partial\Omega$.

Let us explain the purpose of this paper with an easy example. Suppose we wish to compute the total impedance of an electrical transmission line, described by the coupled first order PDEs

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} I(\xi,t) \\ U(\xi,t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{C(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix} \begin{bmatrix} I(\xi,t) \\ U(\xi,t) \end{bmatrix} \\ \qquad \text{for } (\xi,t) \in [0,1] \times \overline{\mathbb{R}}_+, \\ u(t) = I(0,t) \text{ for } t \in \overline{\mathbb{R}}_+ \text{ and } y(t) = U(0,t) \text{ for } t \in \overline{\mathbb{R}}_+; \\ I(\xi,0) = U(\xi,0) = 0 \text{ for } \xi \in [0,1] \text{ and } I(1,t) = 0 \text{ for } t \in \overline{\mathbb{R}}_+. \end{cases} \tag{1.2}$$

The real-valued, continuously differentiable functions $L(\xi)$ and $C(\xi)$ are *the distributed inductances and the capacitances* of the line at point $\xi \in [0,1]$, and it will be assumed that $L(\xi) \geq \eta$ and $C(\xi) \geq \eta$ for some $\eta > 0$. The functions $I(\xi,t)$ and $U(\xi,t)$ are the current and the voltage at point ξ at time t, respectively. It is thus clear that the *transfer function* of (1.2) represents the impedance of the line provided we can make sense out of (1.2) as an infinite-dimensional state space system.

It is mathematically more convenient to choose the signals in a different way, and instead of (1.2) to study the *scattering system* described by

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} I(\xi,t) \\ U(\xi,t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{C(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix} \begin{bmatrix} I(\xi,t) \\ U(\xi,t) \end{bmatrix} \\ \qquad \text{for } (\xi,t) \in [0,1] \times \overline{\mathbb{R}}_+, \\ u^{(1)}(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} I(0,t) \\ U(0,t) \end{bmatrix} \text{ for } t \in \overline{\mathbb{R}}_+, \text{ and} \\ y^{(1)}(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} I(0,t) \\ U(0,t) \end{bmatrix} \text{ for } t \in \overline{\mathbb{R}}_+; \\ I(\xi,0) = U(\xi,0) = 0 \text{ for } \xi \in [0,1] \text{ and } I(1,t) = 0 \text{ for } t \in \overline{\mathbb{R}}_+. \end{cases} \tag{1.3}$$

Since equations (1.3) do not include energy dissipative resistances, it is physically plausible that they define a *scattering conservative, well-posed boundary control system*. Such systems are described in [Mal05, MS06a, MS06b] that are based on [MSW06]. Indeed, the scattering conservativity of (1.3) can be verified rigorously by applying the techniques of [MS06a] to the *boundary node* $\Xi^{(1)}_{TL} = \left[\begin{smallmatrix} G^{(1)} \\ L \\ K^{(1)} \end{smallmatrix}\right]$ consisting of the operators

$$L := \begin{bmatrix} 0 & -\frac{1}{L(\xi)}\frac{\partial}{\partial \xi} \\ -\frac{1}{C(\xi)}\frac{\partial}{\partial \xi} & 0 \end{bmatrix}, \tag{1.4}$$

$$G^{(1)} := \begin{bmatrix} \frac{1}{\sqrt{2}}\gamma_0 & \frac{1}{\sqrt{2}}\gamma_0 \end{bmatrix} |\mathcal{Z}, \quad \text{and} \quad K^{(1)} := \begin{bmatrix} \frac{1}{\sqrt{2}}\gamma_0 & -\frac{1}{\sqrt{2}}\gamma_0 \end{bmatrix} |\mathcal{Z}$$

where $\gamma_0 f = f(0)$ for $f \in H^1(0,1)$,

$$\mathcal{X} := \begin{bmatrix} L^2(0,1)) \\ L^2(0,1)) \end{bmatrix} \quad \text{and} \quad \mathcal{Z} := \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \begin{bmatrix} H^1(0,1) \\ H^1(0,1) \end{bmatrix} : z_1(1) = 0 \right\}. \qquad (1.5)$$

The state space $\mathcal{X}$ is equipped with the physically motivated *energy norm*

$$\left\| \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\|_{\mathcal{X}}^2 := \int_0^1 \left(|z_1(\xi)|^2 L(\xi) + |z_2(\xi)|^2 C(\xi) \right) d\xi. \qquad (1.6)$$

Note that the first three equations in (1.3) can be rewritten in the form[1]

$$u^{(1)}(t) = G^{(1)} z(t), \quad \dot{z}(t) = Lz(t), \quad y^{(1)}(t) = K^{(1)} z(t), \quad t \in [0, \infty), \quad (1.7)$$

where $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \begin{bmatrix} I(\cdot,t) \\ U(\cdot,t) \end{bmatrix}$ is the joint current/voltage distribution at time $t \geq 0$. Similarly, the first three equations in (1.2) can be written as (1.1) where $G := \frac{1}{\sqrt{2}} \left(G^{(1)} + K^{(1)} \right)$, $K := \frac{1}{\sqrt{2}} \left(G^{(1)} - K^{(1)} \right)$, and the operators $G^{(1)}$, L and $K^{(1)}$ are given by (1.4). This translation from system (1.1) to system (1.7) is known as the *external Cayley transform.* (Also the name *diagonal transform* has been used in the Russian literature; see [Liv73].) The external Cayley transform can be applied to a much larger class of linear infinite-dimensional systems than just the transmission line described above.

Suppose that the triple $\Xi^{(1)} = \begin{bmatrix} G^{(1)} \\ L \\ K^{(1)} \end{bmatrix}$ is the external Cayley transform of the triple $\Xi = \begin{bmatrix} G \\ L \\ K \end{bmatrix}$ as explained above. Even when $\Xi^{(1)}$ is known to be a scattering conservative boundary node, additional difficulties may appear:

- There is no guarantee that the triple Ξ itself is a (forward time) boundary node in the sense of Definition 3 below. In particular, the Cauchy problem (1.1) might fail to have a solution.[2]

- Even if such a triple Ξ were an internally well-posed boundary node, it need not define a *well-posed* linear system.

The purpose of this paper is to treat the first of these problems in an abstract framework that includes boundary control systems. We also discuss connections to the earlier results on the *abstract boundary spaces* as defined in, e.g., [GG91]. The proofs of the results are given in [MS06b].

[1]The meaning of superindeces "(1)" will become clear in Definition 4 below.

[2]But if Ξ is a boundary node, then it is even internally well-posed in the sense that it has a strongly continuous semigroup; see [MS06b] for details.

2 Scattering conservative colligations

We shall assume throughout this paper that the operators G, L, and K in (1.1) give rise to a *colligation* as defined below. Colligations provide us with a framework for linear systems theory that involves energy balances and boundary control/observation.

Definition 1. *A* colligation $\Xi := \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right]; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix} \right] \right)$ *consists of the three Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, and the three linear maps G, L, and K, with the same domain $\mathcal{Z} \subset \mathcal{X}$ and with values in $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively. By the domain $\mathcal{D}om\,(\Xi)$ of Ξ we mean the common domain $\mathcal{Z}$ of G, L, and K. A colligation Ξ is* closed *if* $\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right]$ *is closed as an operator $\mathcal{X} \to \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix} \right]$ with domain $\mathcal{Z}$. A colligation Ξ is* strong *if L is closed with $\mathcal{D}om\,(L) = \mathcal{D}om\,(\Xi)$, and G and K are continuous with respect to the graph norm of L on $\mathcal{D}om\,(\Xi)$.*

It is clear that any strong colligation is closed. We call $\mathcal{U}$ the input space, $\mathcal{X}$ the state space, $\mathcal{Y}$ the output space, $\mathcal{D}om\,(\Xi)$ equipped with the graph norm of $\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right]$ the *solution space*, G the *input boundary operator*, L the *interior operator*, and K the *output boundary operator*.

Definition 2. *The colligation* $\Xi = \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right]; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix} \right] \right)$ *is* scattering energy preserving *if*

(i) $\left[\begin{smallmatrix} G \\ \alpha - L \end{smallmatrix} \right]$ *is surjective for some* $\alpha \in \mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$, *and*

(ii) *for all $z \in \mathcal{D}om\,(\Xi)$ we have*

$$2\mathrm{Re}\,\langle z, Lz \rangle + \|Kz\|^2_{\mathcal{Y}} = \|Gz\|^2_{\mathcal{U}}. \tag{2.1}$$

We call such a colligation Ξ scattering conservative *if, in addition,*

(iii) $\left[\begin{smallmatrix} \gamma - L \\ K \end{smallmatrix} \right]$ *is surjective for some* $\gamma \in \mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$.

We leave it for the reader to check that (1.4) – (1.6) define a scattering conservative colligation $\Xi^{(1)}_{TL}$.

Remark 1. *It is easy to see that the colligation* $\Xi = \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right]; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix} \right] \right)$ *is scattering conservative if and only if both Ξ and its* time-flow inverse $\Xi^{\leftarrow} := \left(\left[\begin{smallmatrix} K \\ -L \\ G \end{smallmatrix} \right]; \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{X} \\ \mathcal{U} \end{smallmatrix} \right] \right)$ *are scattering energy preserving.*

The dynamical equations (1.1) associated to scattering energy-preserving colligations have always unique solutions. This follows from Lemma 1 below, using the *boundary nodes* as defined in [MS06a]:

Definition 3. *By a (forward time)* boundary node *on the Hilbert spaces* $\mathcal{U}$*,* $\mathcal{X}$*, and* $\mathcal{Y}$ *we mean the colligation* $\Xi = \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right] ; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix} \right] \right)$ *satisfying the conditions:*

(i) Ξ *is a closed colligation;*

(ii) G *is surjective and* $\mathcal{N}(G)$ *is dense in* $\mathcal{X}$*;*

(iii) *The operator* $L|\mathcal{N}(G)$ *(interpreted as an operator in* X *with domain* $\mathcal{N}(G)$*) has a nonempty resolvent set.*

This boundary node is internally well-posed *(in the forward time direction) if, in addition,*

(iv) $L|\mathcal{N}(G)$ *generates a* C_0 *semigroup.*

It is not difficult to see that every boundary node with a finite-dimensional input space $\mathcal{U}$ is a strong colligation.

Lemma 1. *Let* $\Xi = \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right] ; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix} \right] \right)$ *be a scattering energy preserving colligation. Then* Ξ *is an internally well-posed boundary node, and condition* (i) *in Definition 2 holds for all* $\alpha \in \mathbb{C}^+$*. If* Ξ *is, in addition, scattering conservative, then condition* (iii) *in Definition 2 holds for all* $\gamma \in \mathbb{C}^-$*, too.*

We conclude that for any scattering energy-preserving colligation Ξ, the corresponding dynamical equation (1.1) has a unique solution for sufficiently smooth input functions u and initial states z_0 compatible with $u(0)$. More precisely, as we show in [MS06a, Section 2.1], for all $z_0 \in \mathcal{X}$ and $u \in C^2(\mathbb{R}^+;\mathcal{U})$ with $Gz_0 = u(0)$, the first and the second of equations (1.1) have a unique solution $z \in C^1(\mathbb{R}^+;\mathcal{X}) \cap C(\mathbb{R}^+;\mathcal{Z})$ with $z(0) = z_0$. Hence we can define $y \in C(\mathbb{R}^+;\mathcal{Y})$ by the third equation in (1.1).

Remark 2. *It follows from Lemma 1 that a scattering energy-preserving (or conservative) colligation* Ξ *is actually scattering energy-preserving (or conservative, respectively) as a* boundary node *in the sense of [Mal05, MS06a]. For details, see [MS06a, Section 1].*

3 Impedance conservative colligations

According to the transmission line example in Section 1, we shall use the *external Cayley transform* to translate impedance type systems to scattering type systems. For this reason we must first assume that *the input and output spaces are the same* for the colligation $\Xi = \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix} \right] ; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix} \right] \right)$. The common input/output space is henceforth denoted by $\mathcal{U}$.

Definition 4. *Let* $\Xi = \left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix} \right)$ *be a colligation. By the* external Cayley transform of Ξ with parameter $\beta \in \mathbb{C}^+$ *we mean the colligation* $\Xi^{(\beta)} = \left(\begin{bmatrix} G^{(\beta)} \\ L \\ K^{(\beta)} \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix} \right)$ *with* $\mathcal{D}om\left(\Xi^{(\beta)}\right) = \mathcal{D}om\left(\Xi\right)$*, where*

$$G^{(\beta)} = \frac{\beta G + K}{\sqrt{2 \mathrm{Re}\, \beta}} \quad \text{and} \quad K^{(\beta)} = \frac{\overline{\beta} G - K}{\sqrt{2 \mathrm{Re}\, \beta}}. \tag{3.1}$$

A colligation Ξ is strong if and only if $\Xi^{(\beta)}$ is strong for some (hence, for all) $\beta \in \mathbb{C}^+$. The interpretation of this transform is the following: the old input u and the old output y in (1.1) are replaced by the new input $u^{(\beta)} = (\beta u + y)/\sqrt{2\mathrm{Re}\, \beta}$ and the new output $y^{(\beta)} = (\overline{\beta} u - y)/\sqrt{2\mathrm{Re}\, \beta}$.

Definition 5. *Let* $\Xi = \left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix} \right)$ *be a colligation. Let* $\Xi^{(\beta)} = \left(\begin{bmatrix} G^{(\beta)} \\ L \\ K^{(\beta)} \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix} \right)$ *be the external Cayley transform of* Ξ *with parameter* β*.*

(i) Ξ *is* impedance energy preserving *if* $\Xi^{(\beta)}$ *is scattering energy preserving for some* $\beta \in \mathbb{C}^+$*.*

(ii) Ξ *is* impedance conservative *if* $\Xi^{(\beta)}$ *is scattering conservative for some* $\beta \in \mathbb{C}^+$*.*

Definition 5 is independent of the parameter β in the sense that if the conditions (i)–(ii) are true for some $\beta \in \mathbb{C}^+$, then they are true for all $\beta \in \mathbb{C}^+$:

Theorem 1. *The two words "some" in Definition 5 can be replaced by the word "all", without changing the meaning of the notions there defined.*

Indeed, the parameter β represents the impedance of the load that is used for the measurement of $I(0, t)$ and $U(0, t)$ in the example of Section 1. It is clear that the internal energy properties (such as the impedance conservativity) of the transmission line cannot depend on this external impedance. For further information on passsive networks theory and the external Cayley transform, see [Woh69, Section 2.2].

Theorem 2. *Let* $\Xi = \left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix} \right)$ *be a colligation.*

(i) Ξ *is impedance energy preserving if and only if the following two conditions hold:*

(a) $\begin{bmatrix} \beta G + K \\ \alpha - L \end{bmatrix}$ *is surjective for some* α, $\beta \in \mathbb{C}^+$*;*

(b) For all $z \in \mathcal{D}om\,(\Xi)$ we have

$$\operatorname{Re}\langle z, Lz\rangle_{\mathcal{X}} = \operatorname{Re}\langle Kz, Gz\rangle_{\mathcal{U}}. \tag{3.2}$$

(ii) Ξ *is impedance conservative if and only if it is impedance energy preserving and, in addition,*

(c) $\left[\begin{smallmatrix}\gamma - L\\ \beta G - K\end{smallmatrix}\right]$ *is surjective for some $\beta \in \mathbb{C}^+$ and $\gamma \in \mathbb{C}^-$.*

For an impedance energy preserving Ξ, condition (a) *holds for all $\alpha, \beta \in \mathbb{C}^+$. For an impedance conservative Ξ, also condition* (c) *holds for all $\beta \in \mathbb{C}^+$ and $\gamma \in \mathbb{C}^-$.*

In Section 1 we associated to the transmission line equations (1.2) the colligation $\Xi_{TL} = \left(\left[\begin{smallmatrix}G\\L\\K\end{smallmatrix}\right];\left[\begin{smallmatrix}\mathbb{C}\\\mathcal{X}\\\mathbb{C}\end{smallmatrix}\right]\right)$ where $L := \begin{bmatrix} 0 & -\frac{1}{L(\xi)}\frac{\partial}{\partial\xi} \\ -\frac{1}{C(\xi)}\frac{\partial}{\partial\xi} & 0\end{bmatrix}$, $G := \begin{bmatrix}\gamma_0 & 0\end{bmatrix}|\mathcal{Z}$, $K := \begin{bmatrix}0 & \gamma_0\end{bmatrix}|\mathcal{Z}$, and the spaces $\mathcal{X}$, $\mathcal{Z} = \mathcal{D}om\,(\Xi_{TL})$ are defined by (1.5). We leave it for the reader to check (using Theorem 2) that the colligation Ξ_{TL} is impedance conservative when $\mathcal{X}$ is equipped with the energy norm (1.6).

Note that Theorem 2 does not imply the existence of a solution of the dynamical equation (1.1) but stronger assumptions (like those in Theorem 3 below) are required.

4 Abstract boundary spaces

We consider next certain extensions of the operator

$$L_0 := L|\mathcal{D}om\,(L_0) \quad \text{with} \quad \mathcal{D}om\,(L_0) := \mathcal{N}\,(G) \cap \mathcal{N}\,(K) \tag{4.1}$$

where L is taken from a impedance conservative colligation $\Xi = \left(\left[\begin{smallmatrix}G\\L\\K\end{smallmatrix}\right];\left[\begin{smallmatrix}\mathcal{U}\\\mathcal{X}\\\mathcal{Y}\end{smallmatrix}\right]\right)$. Such an operator L_0 is called the *minimal operator* (of colligation Ξ, or of L), and it is symmetric by (2.1). It is a classical problem in operator theory to parameterize various extensions of such symmetric operators. The related notion of an *abstract boundary space* appears in works that are predominantly of russian (Soviet) origin. The following definition is from [GG91, Definition 1.4 on p. 155]:

Definition 6. *Let $\mathcal{A}$ be a closed, densely defined symmetric operator on $\mathcal{X}$. The triple $(\mathcal{U}, \Gamma_1, \Gamma_2)$ is* an abstract boundary space *for $\mathcal{A}$ if $\mathcal{U}$ is a Hilbert space and Γ_1, Γ_2 are linear mappings from $\mathcal{D}om\,(\mathcal{A}^*)$ into $\mathcal{U}$ with the following properties:*

(i) *for any* $x, z \in \mathcal{D}om\,(\mathcal{A}^*)$ *we have*

$$\langle \mathcal{A}^* x, z \rangle_{\mathcal{X}} - \langle x, \mathcal{A}^* z \rangle_{\mathcal{X}} = \langle \Gamma_1 x, \Gamma_2 z \rangle_{\mathcal{U}} - \langle \Gamma_2 x, \Gamma_1 z \rangle_{\mathcal{U}}\,;$$

(ii) *the mapping* $\left[\begin{smallmatrix} \Gamma_1 \\ \Gamma_2 \end{smallmatrix}\right]$ *from* $\mathcal{D}om\,(\mathcal{A}^*)$ *into* $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{U} \end{smallmatrix}\right]$ *is surjective.*

The abstract boundary space (H, Γ_1, Γ_2) for $\mathcal{A}^*$ is in fact a special case of impedance conservative *strong* colligations:

Proposition 1. *Let* $\mathcal{A}$ *be a closed, densely defined symmetric operator on* $\mathcal{X}$ *such that the triple* $(\mathcal{U}, \Gamma_1, \Gamma_2)$ *is an abstract boundary space for* $\mathcal{A}$.

Then $\Xi = \left(\left[\begin{smallmatrix} \Gamma_1 \\ i\mathcal{A}^* \\ -i\Gamma_2 \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \right)$ *is an impedance conservative strong colligation with* $\mathcal{D}om\,(\Xi) = \mathcal{D}om\,(\mathcal{A}^*)$.

There is also the following converse result to Proposition 1:

Proposition 2. *Let* $\Xi = \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \right)$ *be an impedance conservative colligation satisfying* $\mathcal{R}\,(G) = \mathcal{R}\,(K) = \mathcal{U}$. *Then any of the following conditions implies the other two:*

(i) *The colligation* Ξ *is strong;*

(ii) *The operator* L_0 *is densely defined and* $\cap_{k \geq 1} \mathcal{D}om\left(L_0^{*k}\right) \subset \mathcal{D}om\,(L)$*;*

(iii) *The operator* L_0 *is densely defined and* $L = -L_0^*$.

Assume, in addition, that $\mathcal{R}\,(\left[\begin{smallmatrix} G \\ K \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{U} \end{smallmatrix}\right]$. *If the above equivalent conditions (i) – (iii) hold, then the triple* $(\mathcal{U}, G, iK)$ *is an abstract boundary space for the closed, densely defined symmetric operator* $\mathcal{A} := iL_0$ *in* $\mathcal{X}$.

5 Internal well-posedness of strong colligations

We consider next the impedance conservative *strong* colligations. A number of such strong colligations is provided by Proposition 1 and examples in [GG91]. However, all impedance conservative colligations are not strong. An example involving an impedance conservative, internally well-posed boundary node with a *non-closed* interior operator L is given in [MS06b, Section 6]. This example is based on the boundary controlled wave equation in $\Omega \subset \mathbb{R}^n$ for $n \geq 2$.

Theorem 3. *Let Ξ be an impedance conservative strong colligation. Then Ξ is an internally well-posed boundary node if and only if its input boundary operator G is surjective. When these equivalent conditions hold, the semigroup of Ξ is unitary; i.e. $A := L|\mathcal{N}(G)$ is maximally dissipative and it satisfies $A^* = -A$.*

Since the colligation Ξ_{TL} defined above is strong, it follows from Theorem 3 that Ξ_{TL} is an internally well-posed boundary node. Thus, the transmission line equations (1.2) are uniquely solvable as explained after Lemma 1. The (impedance) transfer function of Ξ_{TL} is analytic in $\mathbb{C}^+$, and it is given by $\hat{Z}(s) := C\&D \left[\begin{smallmatrix}(s-A_{-1})^{-1}B \\ I\end{smallmatrix}\right]$ where the operators A_{-1}, B, and $C\&D$ are related to G, L, and K as explained in [MS06a, Section 2].

Remark 3. Lemma 1 can be generalized to the larger class of *scattering passive* colligations. Theorems 1, 2, and 3 can be generalized to the larger class of *impedance passive* colligations. All this can be found in [MS06b].

Remark 4. The proof of Theorem 3 is related to the proof of [GG91, Theorem 1.5 on p. 156]. Likewise, the analogous semigroup generation result for Hamiltonian ports in [lGZM05] is a consequence of this result. The generalized impedance passive version of Theorem 3 has not been treated in [GG91].

References

[GG91] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for operator differential equations*, Mathematics and its Applications (Soviet Series), vol. 48, Kluwer Academic Publishers Group, Dordrecht, (1991).

[lGZM05] Y. le Gorrec, H. Zwart, and B. Mashke, *Dirac structures and boundary control systems associated with skew-symmetric differential operators*, SIAM J. Control Opt., **44** (2006), 1864 – 1892.

[Liv73] M. S. Livšic. *Operators, Oscillations, Waves (Open Systems)*, volume 34 of *Translations of Mathematical Monographs*, AMS, Providence, Rhode Island, (1973).

[Mal05] J. Malinen, *Conservativity and Time-Flow Invertiblity of Boundary Control Systems*, Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC'05), Seville, (2005).

[MS06a] J. Malinen and O. J. Staffans, *Conservative boundary control systems*, submitted (2006).

[MS06b] J. Malinen and O. J. Staffans, *Internal Well-posedness of Impedance Passive Boundary Control Systems*, Manuscript (2006).

[MSW06] J. Malinen, O. J. Staffans, and G. Weiss, *When is a linear system conservative?*, Quart. Appl. Math. **10** (2006), 61–91.

[Sta02] O. J. Staffans, *Passive and conservative continuous-time impedance and scattering systems. Part I: Well-posed systems*, Math. Control Signals Systems **15** (2002), 291–315.

[Woh69] M. R. Wohlers, *Lumped and distributed passive networks*, Academic press, (1969).