A CHARACTERIZATION OF NEWTONIAN FUNCTIONS WITH ZERO BOUNDARY VALUES

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Abstract. We give an intrinsic characterization of the property that the zero extension of a Newtonian function, defined on an open set in a doubling metric measure space supporting a strong relative isoperimetric inequality, belongs to the Newtonian space on the entire metric space. The theory of functions of bounded variation is used extensively in the argument and we also provide a structure theorem for sets of finite perimeter under the assumption of a strong relative isoperimetric inequality. The characterization is used to prove a strong version of quasicontinuity of Newtonian functions.

1. Introduction

Sobolev spaces with zero boundary values are essential when we want to specify or compare boundary values of Sobolev functions. This is particularly important in connections with boundary value problems in the calculus of variations and partial differential equations and with comparison principles in potential theory. The Sobolev space with zero boundary values is classically defined as a completion of compactly supported smooth functions with respect to the Sobolev space norm. In analysis on metric measure spaces, in the absence of a Poincaré type inequality, this approach seems to be too restrictive and therefore an alternative definition is used instead. Indeed, a function is said to belong to Newtonian space with zero boundary values if it can be extended by zero to the complement so that the extended function belongs to the Newtonian space on the entire metric space, see [KKM00] and [Sha01].

Our goal is to study pointwise characterizations that are related to Lebesgue points of Newtonian functions. In the Euclidean setting this has been studied by Havin [Hav68], Bagby [Bag72], Swanson and Ziemer [SZ99] and Swanson [Swa07]. See also Theorem 9.1.3 in the monograph of Adams and Hedberg [AdHe96]. In particular, we extend

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the Euclidean results of [SZ99] and [Swa07] to Newton-Sobolev functions with zero boundary values in the setting of more general metric measure spaces. The power of the theorem below lies in the fact that it applies to a general open set Ω, and that u is assumed to belong to the Newtonian space only in Ω. Now we state the main result of this paper. The precise definitions are presented later in this work.

Theorem 1.1. *Assume that μ is a doubling Borel regular outer measure on X and that X supports a strong relative isoperimetric inequality. Let $\Omega \subset X$ be an open and bounded set and let $u \in N^{1,p}(\Omega)$ with $1 \le p < \infty$. Then $u \in N_0^{1,p}(\Omega)$ if and only if*

$$\limsup_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u|\, d\mu = 0 \tag{1.2}$$

for p-quasievery $x \in \partial\Omega$.

One of the geometric tools we introduce in this paper is called the strong relative isoperimetric inequality, which is also the main assumption in Theorem 1.1. This condition gives control over the measure of the piece of a Borel set inside a ball in terms of the codimension one Hausdorff measure of the part of the measure theoretic boundary of that set that lies inside the ball. More precisely, we say that X supports a *strong relative isoperimetric inequality*, if there exist positive constants C and λ such that for all balls $B(x,r)$ and for all Borel sets E, we have

$$\min\{\mu(B(x,r)\cap E), \mu(B(x,r)\setminus E)\} \le Cr\,\mathcal{H}(B(x,\lambda r)\cap \partial^* E).$$

The more usual analog of relative isoperimetric inequality found in current literature on BV theory in metric spaces (see [Amb02], [AMP04] and [Mir03]) gives control of the left-hand side in terms of the *perimeter* of the set, that is,

$$\min\{\mu(B(x,r)\cap E), \mu(B(x,r)\setminus E)\} \le Cr\,P(E, B(x,\lambda r)).$$

In the event that the perimeter $P(E, B(x,\lambda r))$ is finite, that is E is *already* known to be of finite perimeter in the enlarged ball $B(x,\lambda r)$, *then* by results of [AMP04] we may replace $P(E, B(x,\lambda r))$ with $\mathcal{H}(B(x,\lambda r)\cap \partial^* E)$ in the standard relative isoperimetric inequality. See Theorem 3.6 below for a statement of this result. Thus the strong relative isoperimetric inequality is a strengthening of the relative isoperimetric inequality, where we have control of the left-hand side of the above inequality in terms of $\mathcal{H}(B(x,\lambda r)\cap \partial^* E)$ *even without knowing* whether E is of finite perimeter in $B(x,\lambda r)$ or not.

In the Euclidean case, a set is of finite perimeter if and only if the codimension one Hausdorff measure of its measure-theoretic boundary is finite; this result is due to Federer [Fed69] (see also [EG92]). In the metric setting such a general structure theorem seems to be unknown.

However, our Theorem 4.6 shows that the Federer type structure theorem holds under the assumption that X supports the strong relative isoperimetric inequality. In many situations, including weighted Euclidean spaces (see [Cam08]), Riemannian manifolds, and more exotic spaces such as Heisenberg groups and Bourdon-Pajot spaces (see [BP99]) the strong relative isoperimetric inequality holds. This can be seen by a pencil of curves type of argument by Semmes [Sem96] and the details will be presented in the forthcoming paper [KKS]. In fact, the authors do not know of an example of a doubling metric measure space supporting a relative isoperimetric inequality that does *not* support a strong relative isoperimetric inequality.

Because of the above mentioned obstacles, our proof diverges from [SZ99] in many respects. Their proof relies heavily on the Euclidean structure, and in particular on a characterization of Sobolev functions and functions of bounded variation with respect to almost every line segments parallel to the coordinate axes. Such a Cartesian characterization is not possible in the metric space setting. For Sobolev functions on metric spaces an analog exists in terms of absolute continuity on modulus almost every rectifiable path; however, no such analog is known for functions of bounded variation. We use alternate, more geometric methods in the proof. We also feel that arguments related to the strong isoperimetric inequality and the structure theorem may be of independent interest.

As an application of our main result we study a strong Lusin type quasicontinuity result, which generalizes the Euclidean results by Michael and Ziemer [MZ82] for $1 < p < \infty$ and Swanson [Swa07] for $p = 1$. See also [BHS02]. An approximation by Hölder continuous functions has been studied, for example in [Mal93], [HK98], [BHS02], and [KT07].

2. Newtonian spaces

We assume that $X = (X, d, \mu)$ is a complete metric measure space equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x, r) = \{y \in X : d(x, y) < r\}$. For $\tau > 0$, we write $\tau B = B(x, \tau r)$. The results in this paper hold true even without the extra assumption of completeness, but for simplicity we assume that X is complete.

We also assume that the measure μ is *doubling*. This means that there exists a constant $c_D \geq 1$, called the *doubling constant* of μ, such that

$$\mu(2B) \leq c_D \mu(B)$$

for all balls B of X. As a complete metric space with a doubling measure, X is proper, that is, closed and bounded sets are compact.

We define Sobolev spaces on X using upper gradients, see [Sha00] and [Hei01]. By replacing X with an open set $\Omega \subset X$, we may define define the corresponding concepts in Ω.

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real valued function u on X if for all paths γ joining points x and y in Ω we have

$$|u(x) - u(y)| \le \int_\gamma g\,ds, \tag{2.2}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. If (2.2) holds for p-almost every path, then g is a *p-weak upper gradient* of u.

In this work, a *path* in X is a rectifiable nonconstant continuous mapping from a compact interval to X. By saying that (2.2) holds for *p-almost every path* with $1 \le p < \infty$, we mean that it fails only for a path family with zero p-modulus, see [Hei01].

Definition 2.3. We say that X supports a *(weak)* $(1,p)$*-Poincaré inequality* if there exist constants $c_P > 0$ and $\lambda \ge 1$ such that for all balls B of X, all locally integrable functions u and for all p-weak upper gradients g of u, we have

$$\fint_B |u - u_B|\,d\mu \le c_P r \Big(\fint_{\lambda B} g^p\,d\mu \Big)^{1/p}, \tag{2.4}$$

where

$$u_B = \fint_B u\,d\mu = \frac{1}{\mu(B)} \int_B u\,d\mu$$

and r is the pre-assigned radius of the ball B.

Definition 2.5. Let $1 \le p < \infty$. If u is a function that is integrable to power p in X, let

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p\,d\mu + \inf_g \int_X g^p\,d\mu \Big)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\!\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Definition 2.6. Let $1 \le p < \infty$. The *p-capacity* of a set $E \subset X$ is the number

$$\operatorname{Cap}_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u = 1$ on E. If there are no functions which satisfy the requirements, then we set $\operatorname{Cap}_p(E) = \infty$.

We point out here that the functions in $N^{1,p}(X)$ are necessarily p-quasicontinuous whenever the measure on X is doubling and X supports a $(1,p)$-Poincaré inequality (see [BBS08]) and thus the above definition of the capacity agrees with the classical definition where the functions are required in addition to satisfy $u = 1$ in a neighbourhood of E (see [FZ73] or [EG92]). The p-quasicontinuity of the function u means that for every $\varepsilon > 0$ there is an open set U with $\mathrm{Cap}_p(U) < \varepsilon$ such that the restriction of u to $X \setminus U$, denoted by $u|_{X\setminus U}$, is continuous.

Definition 2.7. For $E \subset X$, the Newtonian space with zero boundary values is

$$N_0^{1,p}(E) = \{u \in N^{1,p}(X) : u = 0 \text{ in } X \setminus E\}.$$

Note that we obtain the same class of functions as above if we require u to vanish p-quasi everywhere in $X \setminus E$, since Newtonian functions are defined pointwise outside sets of zero capacity.

We begin with the following simple characterization of Newtonian spaces with zero boundary values. In the Euclidean case with $1 < p < \infty$ the previous result has been proved by Havin [Hav68] and Bagby [Bag72] for an open set E. Recently, the case $p = 1$ has been studied by Swanson in [Swa07]. Our result holds true with an arbitrary set E, which is a slight generalization of known results already in the Euclidean case.

Theorem 2.8. *Assume that $u \in N^{1,p}(X)$ with $1 \le p < \infty$ and let E be a subset of X. Then $u \in N_0^{1,p}(E)$ if and only if*

$$\lim_{r\to 0} \fint_{B(x,r)} u \, d\mu = 0$$

for p-quasievery $x \in X \setminus E$.

Proof. If $u \in N_0^{1,p}(E)$, then by the definition $u = 0$ in $X \setminus E$. By Theorem 4.1 in [KL02] for $1 < p < \infty$ and Theorem 4.1 in [KKST08] for $p = 1$, p-quasievery point of X is a Lebesgue point of u. Hence

$$\lim_{r\to 0} \fint_{B(x,r)} u \, d\mu = 0$$

for p-quasievery $x \in X \setminus E$.

Assume then that $u \in N^{1,p}(X)$ and

$$\lim_{r\to 0} \fint_{B(x,r)} u \, d\mu = 0$$

for p-quasievery $x \in X \setminus E$. Since p-quasievery point of X is a Lebesgue point of u, we conclude that $u = 0$ p-quasi everywhere in $X \setminus E$. $\square$

3. Functions of bounded variation and the perimeter measure

We recall the definition and properties of functions of bounded variation on metric measure spaces, see [Mir03].

Definition 3.1. For $u \in L^1_{\mathrm{loc}}(X)$, we define

$$\|Du\|(X) = \inf\Big\{ \liminf_{i\to\infty} \int_X g_{u_i}\,d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(X), u_i \to u \text{ in } L^1_{\mathrm{loc}}(X)\Big\},$$

where g_{u_i} is a 1-weak upper gradient of u_i. We say that a function $u \in L^1(X)$ is of *bounded variation*, $u \in BV(X)$, if $\|Du\|(X) < \infty$. Moreover, a measurable set $E \subset X$ is said to have *finite perimeter* if $\|D\chi_E\|(X) < \infty$. By replacing X with an open set $\Omega \subset X$, we may define $\|Du\|(\Omega)$ and we denote

$$P(E,\Omega) = \|D\chi_E\|(\Omega),$$

the perimeter of E in Ω.

Remark 3.2. Observe that in [Mir03] the functions of bounded variation are defined in terms of the *local Lipschitz constant*

$$\mathrm{Lip}\ u(x) = \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{|u(x)-u(y)|}{d(x,y)},$$

but we may use the 1-weak upper gradient instead. Indeed, if u is a locally Lipschitz continuous function, its local Lipschitz constant is an upper gradient of u. We observe that all results of [Mir03] hold for upper gradients as well.

From Theorem 3.4 in [Mir03], we have that $\|Du\|$ is a Borel regular outer measure.

Theorem 3.3. *Let $u \in BV(X)$. For a set $A \subset X$, we define*

$$\|Du\|(A) = \inf\big\{\|Du\|(\Omega) : \Omega \supset A,\, \Omega \subset X \text{ is open}\big\}.$$

Then $\|Du\|(\cdot)$ is a finite Borel outer measure.

Let E be a set of finite perimeter in X. For every set $A \subset X$, we denote

$$P(E,A) = \|D\chi_E\|(A).$$

The following *coarea formula* will be useful for us, see Proposition 4.2 in [Mir03].

Theorem 3.4 (Coarea formula)**.** *If $u \in L^1_{\mathrm{loc}}(X)$ and $A \subset X$ is open, then*

$$\|Du\|(A) = \int_{-\infty}^{\infty} P(\{u>t\},A)\,dt. \tag{3.5}$$

In particular, if $u \in BV(X)$, then the set $\{u > t\}$ has finite perimeter for almost every $t \in \mathbb{R}$ and formula (3.5) holds for all Borel sets $A \subset X$.

The *restricted spherical Hausdorff content of codimension one* on X is defined as

$$\mathcal{H}_R(E) = \inf\Big\{ \sum_{i\in I} \frac{\mu(B(x_i,r_i))}{r_i} : E \subset \bigcup_{i\in I} B(x_i,r_i),\ r_i \le R\Big\},$$

where $0 < R < \infty$. When $R = \infty$, the infimum is taken over coverings with finite radius. The number $\mathcal{H}_\infty(E)$ is the *Hausdorff content* of E. The *Hausdorff measure of codimension one* of $E \subset X$ is defined as

$$\mathcal{H}(E) = \lim_{R\to 0} \mathcal{H}_R(E).$$

The *measure theoretic boundary* of E, denoted by $\partial^* E$, is the set of points $x \in X$, where both E and its complement have positive density, i.e.

$$\limsup_{r\to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} > 0.$$

A combination of Theorems 4.4. and 4.6 in [AMP04] gives the following equivalence of the perimeter measure and the Hausdorff measure of codimension one for sets with finite perimeter.

Theorem 3.6. *Assume that E is a set of finite perimeter. The measure $P(E,\cdot)$ is concentrated on $\partial^* E$ and*

$$\frac{1}{C} P(E,A) \le \mathcal{H}(\partial^* E \cap A) \le C P(E,A), \tag{3.7}$$

where C depends only on the doubling constant and the Poincaré inequality.

Note carefully that the theorem above does not imply that the Hausdorff measure of $\partial^* E$ would be infinite whenever the perimeter measure of E is infinite. See also Theorem 4.6.

Theorem 5.3 in [AMP04] gives us the following useful decomposition of the perimeter measure.

Theorem 3.8. *Let $u \in BV(X)$. Then we have*

$$\|Du\| = \|D^g u\| + \|D^c u\| + \|D^j u\|.$$

Here $d\|D^g u\| = g\, d\mu$ and g is the density of $\|Du\|$ with respect to μ. The measure $\|D^j u\|$, the "jump" part of $\|Du\|$, is absolutely continuous with respect to the Hausdorff measure of codimension one, and is concentrated on the jump set of u i.e. the set J consisting of all points $x \in X$ where

$$\limsup_{r\to 0} \fint_{B(x,r)} |u-a|\, d\mu > 0 \quad \textit{for all } a \in \mathbb{R}. \tag{3.9}$$

The measure $\|D^c u\|$ is the Cantor part of $\|Du\|$ and it is concentrated inside $X \setminus J$.

The following result is from Theorems 6.2.3 and 6.2.2 in [Cam08].

Theorem 3.10. *We have that $u \in N^{1,1}_{\mathrm{loc}}(\Omega)$ if and only if $u \in BV_{\mathrm{loc}}(\Omega)$ and $\|Du\|$ is absolutely continuous with respect μ. In this case, for every $\Omega' \Subset \Omega$,*

$$\|Du\|(\Omega') = \int_{\Omega'} g_u \, d\mu,$$

where g_u is the minimal upper gradient of u.

4. Strong relative isoperimetric inequality

In this section, we give the definition for the strong isoperimetric inequality and show that this property implies the standard relative isoperimetric inequality. The main result of this section is Theorem 4.6, which shows that Borel sets whose measure theoretic boundary has finite Hausdorff measure of codimension one are of finite perimeter if the space satisfies the strong isoperimetric inequality. This structure theorem will be crucial for us later in this work.

Definition 4.1. We say that X supports a *strong relative isoperimetric inequality* if there exist positive constants C and λ such that for all balls $B \subset X$ and for all Borel sets $E \subset X$, with r denoting the (pre-chosen) radius of B, we have

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq Cr\, \mathcal{H}(\lambda B \cap \partial^* E).$$

First we observe that the strong isoperimetric inequality implies the $(1,1)$-Poincaré inequality. The proof of this result is a rather straightforward modification of Theorem 1.1 and Lemma 3.1 in [BH97]. For the sake of completeness, we present the required modification here.

Theorem 4.2. *If X supports a strong relative isoperimetric inequality, then X supports a $(1,1)$-Poincaré inequality.*

Proof. Fix a ball $B \subset X$ and consider the normalized measure

$$\nu = \frac{1}{\mu(\lambda B)} \mu|_{\lambda B}$$

on λB. The measure ν is then a Radon probability measure. Let

$$\nu^+(A) = \limsup_{\varepsilon \to 0} \frac{\nu\left(\bigcup_{x \in \partial A} B(x,\varepsilon) \cap \lambda B\right)}{\varepsilon}$$

be the *codimension one upper Minkowski content* of ∂A and

$$\mathcal{L}(u) = \fint_B |u - u_B|\, d\mu.$$

A modification of Theorem 1.1 and (1.4) of [BH97] states that for some constant $C > 0$, the following two conditions are equivalent:

(i) For all Borel sets $A \subset B$,

$$\max\{\mathcal{L}(\chi_A), \mathcal{L}(-\chi_A)\} \leq C\nu^+(A).$$

(ii) For all Lipschitz functions u on λB,

$$\mathcal{L}(u) \leq C \int_{\lambda B} \operatorname{Lip} u\, d\nu.$$

The definition of ν^+ used in [BH97] is slightly different from that used above. The only difference is that we need to consider not merely the outer Minkowski content but the entire Minkowski content of the boundary of level sets of Lipschitz functions, and we have to consider the symmetric difference $(u_h - u_{-h})/(2h)$ rather than one-sided difference $(u_h - u)/h$. Here,

$$u_h(x) = \sup_{d(x,y)<h} u(y) \quad \text{and} \quad u_{-h}(x) = \inf_{d(x,y)<h} u(y).$$

Indeed, the modification needed in the proof of Lemma 3.1 of [BH97] is as follows. For a bounded non-negative Lipschitz function u, we denote

$$A_t^{-h} = \{x \in X : u_{-h}(x) > t\} \quad \text{and} \quad A_t^h = \{x \in X : u_h(x) > t\}.$$

Then the inequality (3.4) of [BH97] becomes

$$\int_X \frac{u_h - u_{-h}}{2h}\, d\mu = \int_0^\infty \frac{\mu(A_t^h \setminus A_t^{-h})}{2h}\, dt,$$

and

$$\begin{aligned}
&\limsup_{h\to 0^+} \frac{u_h(x) - u_{-h}(x)}{2h} \\
&\qquad \le 2^{-1}\left(\limsup_{h\to 0^+} \frac{u_h(x) - u(x)}{h} + \limsup_{h\to 0^+} \frac{u(x) - u_{-h}(x)}{h}\right) \\
&\qquad \le \limsup_{y\to x} \frac{|u(y) - u(x)|}{d(x,y)} = \operatorname{Lip} u(x).
\end{aligned}$$

The remaining portions of the proof now go through directly. Hence it suffices to show that the condition (i) holds.

Let $A \subset B$ be a Borel set. Then

$$\mathcal{L}(\chi_A) = 2\frac{\mu(A\cap B)\,\mu(B\setminus A)}{\mu(B)^2} = \mathcal{L}(-\chi_A),$$

and consequently

$$\frac{\min\{\mu(A\cap B), \mu(B\setminus A)\}}{\mu(B)} \le \mathcal{L}(\chi_A) \le 2\frac{\min\{\mu(A\cap B), \mu(B\setminus A)\}}{\mu(B)}.$$

Since $\mu^+(A) = \mu(\lambda B)\,\nu^+(A)$ and the measure is doubling, the condition (i) is equivalent to

$$\min\{\mu(A\cap B), \mu(B\setminus A)\} \le C\,\mu^+(A). \tag{4.3}$$

Here μ^+ is defined in a similar way as ν^+. We will use the strong isoperimetric inequality to show that (4.3) holds. Let $\varepsilon < \lambda/2$ and let $\{B_i\}_{i\in I}$ be a countable covering of ∂A with the balls $B_i = B(x_i, \varepsilon)$ such that for each i, $x_i \in \partial A$ and the balls $\frac{1}{5}B_i$ are disjoint. Such a cover exists by the doubling property of μ; see [Hei01]. Then

$$\partial A \subset \bigcup_{i\in I} B_i \subset \bigcup_{x\in\partial A} B(x,\varepsilon)\cap\lambda B,$$

This implies that

$$\mathcal{H}(\lambda B\cap\partial^* A) \le \mathcal{H}(\partial A) \le \liminf_{\varepsilon\to 0}\sum_{i\in I}\frac{\mu(B_i)}{\varepsilon},$$

where by the doubling property of μ we have

$$\sum_{i\in I}\mu(B_i) \le C\sum_{i\in I}\mu\left(\tfrac{1}{5}B_i\right) = C\mu\Big(\bigcup_{i\in I}\tfrac{1}{5}B_i\Big) \le C\mu\Big(\bigcup_{i\in I}B_i\Big).$$

The middle equality in the above series of inequalities used the fact that the balls $\frac{1}{5}B_i$, $i\in I$, are pairwise disjoint. Thus the strong isoperimetric inequality now implies that

$$\begin{aligned}\min\{\mu(A\cap B),\mu(B\setminus A)\} &\le Cr\mathcal{H}(\lambda B\cap\partial^* A)\\ &\le Cr\liminf_{\varepsilon\to 0}\sum_{i\in I}\frac{\mu(B_i)}{\varepsilon} \le Cr\limsup_{\varepsilon\to 0}\frac{\mu\left(\bigcup_{i\in I}B_i\right)}{\varepsilon}\\ &= Cr\limsup_{\varepsilon\to 0}\frac{\mu\left(\bigcup_{i\in I}B_i\cap\lambda B\right)}{\varepsilon} \le Cr\mu^+(A),\end{aligned}$$

where r is the radius of B. This shows the validity of (4.3) with the constant Cr. It then follows from Theorem 1.1 of [BH97] that for all Lipschitz functions u on X,

$$\int_B |u-u_B|\,d\mu \le Cr\int_{\lambda B}\operatorname{Lip} u\,d\mu.$$

The above is the $(1,1)$-Poincaré inequality for Lipschitz functions together with the local Lipschitz constants. The $(1,1)$-Poincaré inequality for all function-upper gradient pairs now follows from Theorem 1.3.4 of [Kei03]. We note here that [Kei03] assumes that X is a geodesic space. However, as the $(1,1)$-Poincaré inequality for Lipschitz function-local Lipschitz constant pairs implies that X is a quasiconvex space (that is, every pair of points $x,y\in X$ can be connected by a curve with length controlled by $Cd(x,y)$, see Section 6 of [Kei03]), such an assumption is not necessary for our setting. $\square$

Next we recall the definition of the standard relative isoperimetric inequality.

Definition 4.4. We say that X supports a *relative isoperimetric inequality* if there exist positive constants C and λ such that for all balls $B\subset X$ and for all Borel sets $E\subset X$, with r denoting the (pre-chosen) radius of B, we have

$$\min\{\mu(B\cap E),\mu(B\setminus E)\} \le Cr\,P(E,\lambda B).$$

We obtain the following corollary because the $(1,1)$-Poincaré inequality implies the relative isoperimetric inequality, see Theorem 4.5 in [Mir03].

Corollary 4.5. *If X supports a strong relative isoperimetric inequality, then X also supports a relative isoperimetric inequality.*

We point out here that because of Theorem 3.6, the relative isoperimetric inequality and the strong relative isoperimetric inequality are equivalent *provided* that we know E to be of finite perimeter. The following structure theorem is the main result of this section.

Theorem 4.6. *If X supports a strong relative isoperimetric inequality, then all Borel sets E with $\mathcal{H}(\partial^* E) < \infty$ are of finite perimeter in X.*

Proof. It suffices to find a sequence of locally Lipschitz continuous functions u_ε on X such that $u_\varepsilon \to \chi_E$ in $L^1(X)$ as $\varepsilon \to 0$, and

$$\sup_\varepsilon \|Du_\varepsilon\|(X) < \infty,$$

see, for example, Theorem 3.7 in [Mir03].

We use a discrete convolution to find such a sequence (u_ε). Let $\varepsilon > 0$, and let $\{B_i\}_{i \in I}$ be a countable cover of X by balls $B_i = B(x_i, \varepsilon)$ such that the bounded overlap property

$$\sum_{i \in I} \chi_{4\lambda B_i} \le C$$

holds. Let $\{\varphi_i\}_{i \in I}$ be the corresponding partition of unity; that is, $0 \le \varphi_i \le 1$, φ_i is C/ε-Lipschitz continuous, $\operatorname{supp}(\varphi_i) \subset 2B_i$ and

$$\sum_{i \in I} \varphi_i(x) = 1$$

for every $x \in X$. We set

$$u_\varepsilon(x) = \sum_{i \in I} (\chi_E)_{B_i} \varphi_i(x) = \sum_{i \in I} \frac{\mu(B_i \cap E)}{\mu(B_i)}\, \varphi_i(x).$$

First we show that $u_\varepsilon \to \chi_E$ in $L^1(X)$. For a ball B_j, let

$$I_j = \{i \in I : 2B_i \cap B_j \neq \emptyset\}.$$

For $x \in B_j$, we have

$$u_\varepsilon(x) - \chi_E(x) = \sum_{i \in I_j} \left(\frac{\mu(B_i \cap E)}{\mu(B_i)} - \chi_E(x) \right) \varphi_i(x),$$

and hence

$$\begin{aligned}
\int_{B_j} &|u_\varepsilon - \chi_E| \, d\mu \\
&\le \sum_{i\in I_j} \Big[\Big(1 - \frac{\mu(B_i \cap E)}{\mu(B_i)} \Big) \mu(B_j \cap E) + \frac{\mu(B_i \cap E)}{\mu(B_i)} \, \mu(B_j \setminus E) \Big] \\
&\le \sum_{i\in I_j} \frac{\mu(B_j \cap E)\mu(B_i \setminus E) + \mu(B_i \cap E)\mu(B_j \setminus E)}{\mu(B_i)}.
\end{aligned}$$

If $i \in I_j$, then $B_i \subset 4B_j$. Therefore, by the bounded overlap property and by the doubling property of μ,

$$\begin{aligned}
\int_{B_j} |u_\varepsilon - \chi_E| \, d\mu &\le C \frac{\mu(4B_j \cap E)\mu(4B_j \setminus E)}{\mu(4B_j)} \\
&\le C \, \min\{\mu(4B_j \cap E), \mu(4B_j \setminus E)\}.
\end{aligned}$$

An application of the strong isoperimetric inequality now yields

$$\int_{B_j} |u_\varepsilon - \chi_E| \, d\mu \le C\varepsilon \, \mathcal{H}(4\lambda B_j \cap \partial^* E),$$

and another application of the bounded overlap property shows that

$$\int_X |u_\varepsilon - \chi_E| \, d\mu \le \sum_{j\in I} \int_{B_j} |u_\varepsilon - \chi_E| \, d\mu \le C\varepsilon \, \mathcal{H}(\partial^* E).$$

Letting $\varepsilon \to 0$, we see that $u_\varepsilon \to \chi_E$ in $L^1(X)$.

Let $x, y \in B_j$. Then

$$\begin{aligned}
u_\varepsilon(x) - u_\varepsilon(y) &= \sum_{i\in I_j} \frac{\mu(B_i \cap E)}{\mu(B_i)} (\varphi_i(x) - \varphi_i(y)) \\
&= \sum_{i\in I_j} \Big[\frac{\mu(B_i \cap E)}{\mu(B_i)} - \frac{\mu(B_j \cap E)}{\mu(B_j)} \Big] (\varphi_i(x) - \varphi_i(y)).
\end{aligned}$$

For $i \in I_j$, by the doubling property of μ we have

$$\begin{aligned}
\Big| \frac{\mu(B_i \cap E)}{\mu(B_i)} - \frac{\mu(B_j \cap E)}{\mu(B_j)} \Big| &= \Big| \frac{\mu(B_i \cap E)\mu(B_j) - \mu(B_j \cap E)\mu(B_i)}{\mu(B_i)\mu(B_j)} \Big| \\
&= \Big| \frac{\mu(B_i \cap E)\mu(B_j \setminus E) - \mu(B_j \cap E)\mu(B_i \setminus E)}{\mu(B_i)\mu(B_j)} \Big| \\
&\le \frac{\mu(B_i \cap E)\mu(B_j \setminus E) + \mu(B_j \cap E)\mu(B_i \setminus E)}{\mu(B_i)\mu(B_j)} \\
&\le C \, \frac{\mu(4B_j \setminus E)\mu(4B_j \cap E)}{\mu(4B_j)^2} \\
&\le C \, \frac{\min\{\mu(4B_j \setminus E), \mu(4B_j \cap E)\}}{\mu(4B_j)}.
\end{aligned}$$

The C/ε-Lipschitz continuity of φ_i, the bounded overlap property, the above equation, and the strong isoperimetric inequality yield

$$\begin{aligned}|u_\varepsilon(x)-u_\varepsilon(y)| &\le \frac{C}{\varepsilon}d(x,y)\sum_{i\in I_j}\left|\frac{\mu(B_i\cap E)}{\mu(B_i)}-\frac{\mu(B_j\cap E)}{\mu(B_j)}\right|\\ &\le \frac{C}{\varepsilon}d(x,y)\,\frac{\min\{\mu(4B_j\setminus E),\mu(4B_j\cap E)\}}{\mu(4B_j)}\\ &\le Cd(x,y)\,\frac{\mathcal{H}(4\lambda B_j\cap\partial^*E)}{\mu(4B_j)}.\end{aligned}$$

Hence, for $x\in X$,

$$\operatorname{Lip}u_\varepsilon(x)\le C\min\left\{\frac{\mathcal{H}(4\lambda B_k\cap\partial^*E)}{\mu(4B_k)}:x\in B_k\right\}.$$

Thus by the bounded overlap property and the doubling property of μ,

$$\begin{aligned}\|Du_\varepsilon\|(X) &\le \sum_{j\in I}\int_{B_j}\operatorname{Lip}u_\varepsilon\,d\mu\\ &\le C\sum_{j\in I}\mathcal{H}(4\lambda B_j\cap\partial^*E)\le C\,\mathcal{H}(\partial^*E).\end{aligned}$$

This completes the proof. □

5. A Lebesgue point characterization

In this section we prove the main result of this paper, Theorem 1.1. For technical reasons, the theorem is stated only for bounded sets, but since the question is essentially local, the assumption on boundedness can be removed. We start with a technical lemma which is needed in the proof of the main theorem.

Lemma 5.1. *Let $1\le p<q<\infty$ and assume that X supports a $(1,p)$-Poincaré inequality. If $u\in N^{1,p}(X)$ and both u and its p-weak upper gradient belong to $L^q(X)$, then there is a modification of u on a set of measure zero that also belongs to $N^{1,q}(X)$.*

Remark 5.2. If X does not support a $(1,p)$-Poincaré inequality, then the above lemma fails: if X is the Fred Gehring bow-tie (two triangular regions pasted together at one vertex), then the function that takes on the value 1 in one triangle and 0 in the other triangle has 0 as a p-weak upper gradient when $p<2$, and this weak upper gradient is in $L^q(X)$ for all $q>2$, but this function is not in $N^{1,q}(X)$ for any $q>2$, not even locally.

Proof of Lemma 5.1. Let the covering $\{B_i\}_{i\in I}$ and the partition of unity $\{\varphi_i\}_{i\in I}$ be as in the proof of Theorem 4.6. We define the discrete convolution of u at the scale $\varepsilon>0$ as

$$u_\varepsilon(x)=\sum_{i\in I}u_{B_i}\varphi_i(x).$$

By the bounded overlap property of the cover, we have

$$\begin{aligned}\int_X |u_\varepsilon|^q \, d\mu &\le C \sum_{i\in I} \left(\int_{2B_i} |u_{B_i}\varphi_i| \, d\mu \right)^q \\ &\le C \sum_{i\in I} \int_{2B_i} |u_{B_i}|^q \, d\mu \le C \int_X |u|^q \, d\mu.\end{aligned}$$

For $x, y \in B_j$, we have

$$\begin{aligned}|u_\varepsilon(x) - u_\varepsilon(y)| &= \left| \sum_{i\in I} (u_{B_i} - u_{B_j})(\varphi_i(x) - \varphi_i(y)) \right| \\ &\le \sum_{i\in I_j} |u_{B_i} - u_{B_j}||\varphi_i(x) - \varphi_i(y)| \\ &\le \frac{C}{\varepsilon} d(x,y) \sum_{i\in I_j} |u_{B_i} - u_{B_j}| \\ &\le \frac{C}{\varepsilon} d(x,y) \sum_{i\in I_j} \fint_{4B_j} |u - u_{4B_j}| \, d\mu,\end{aligned}$$

where we used the doubling property of μ and that if $i \in I_j$, then $B_i \subset 4B_j$ and $B_j \subset 4B_i$. Here I_j is the same set of indices as in the proof of Theorem 4.6. By the bounded overlap property and the $(1, p)$-Poincaré inequality we obtain

$$\begin{aligned}|u_\varepsilon(x) - u_\varepsilon(y)| &\le \frac{C}{\varepsilon} d(x,y) \fint_{4B_j} |u - u_{4B_j}| \, d\mu \\ &\le C d(x,y) \left(\fint_{4\lambda B_j} g^p \, d\mu \right)^{1/p}\end{aligned}$$

for every p-weak upper gradient g of u. This implies that

$$g_\varepsilon(x) = C \sum_{j\in I} \left(\fint_{4\lambda B_j} g^p \, d\mu \right)^{1/p} \chi_{B_j}(x),$$

is an upper gradient of u_ε. Moreover, by the bounded overlap property of the cover and Hölder's inequality, we have

$$\int_X g_\varepsilon^q \, d\mu \le C \sum_{j\in I} \mu(B_j) \fint_{4\lambda B_j} g^q \, d\mu \le C \int_X g^q \, d\mu.$$

Thus the sequence u_ε is bounded in $N^{1,q}(X)$. As $q > 1$ and $u_\varepsilon \to u$ almost everywhere, by Mazur's lemma a modification of u on a set of measure zero gives a function in $N^{1,q}(X)$. $\square$

Proof of Theorem 1.1. First suppose that $u \in N^{1,p}(\Omega)$ is nonnegative and bounded and that the condition (1.2) holds for every $x \in \partial\Omega$. For

$x \in X$, let

$$u^*(x) = \begin{cases} \lim\limits_{r\to 0} \dfrac{1}{\mu(B(x,r))} \displaystyle\int_{B(x,r)\cap\Omega} u\, d\mu & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

The set of points $x \in \Omega$ for which $u(x) \neq u^*(x)$ is of zero p-capacity, and hence $u^* = u$ on Ω from the point of view of $N^{1,p}(\Omega)$. This is shown for $p > 1$ by Theorem 4.1 in [KL02] and for $p = 1$ by Theorem 4.1 in [KKST08]. By assumption (1.2), $u^*(x) = 0$ for every $x \in X \setminus \Omega$.

Our first goal is to show that $u^* \in BV(X)$. For $t \in \mathbb{R}$, let

$$E_t^* = \{x \in X : u^*(x) > t\}.$$

Whenever

$$\lim_{r\to 0} \fint_{B(x,r)} |u^* - u^*(x)|\, d\mu = 0,$$

we have

$$\lim_{r\to 0} \frac{\mu(E_t^* \cap B(x,r))}{\mu(B(x,r))} = \begin{cases} 1, & \text{if } t < u^*(x), \\ 0, & \text{if } t > u^*(x). \end{cases} \tag{5.3}$$

Let g be a p-weak upper gradient of u (and hence also of u^*) in Ω. Since Ω is bounded, g is also an integrable 1-weak upper gradient of u and u^* in Ω. We have $u^* \in BV(\Omega)$ because $N^{1,p}(\Omega) \subset N^{1,1}(\Omega) \subset BV(\Omega)$. Since by the coarea formula

$$\int_0^\infty P(E_t^*, \Omega)\, dt = \|Du^*\|(\Omega) \leq \int_\Omega g\, d\mu < \infty,$$

we observe that $P(E_t^*, \Omega)$ is finite for almost every $0 < t < \infty$. By Theorem 3.6, $\mathcal{H}(\partial^* E_t^* \cap \Omega)$ is comparable with $P(E_t^*, \Omega)$ for all such values of t. Now by the assumption and (5.3), we have $\partial^* E_t^* \subset \Omega$ for every $t > 0$ and thus

$$\mathcal{H}(\partial^* E_t^* \cap \Omega) = \mathcal{H}(\partial^* E_t^*).$$

Hence Theorem 4.6 implies that $P(E_t^*, X)$ is finite for almost every t. Note that here we need the assumption of strong relative isoperimetric inequality. Using again the coarea formula and Theorem 3.6, we obtain

$$\begin{aligned} \|Du^*\|(X) &= \int_0^\infty P(E_t^*, X)\, dt \leq C \int_0^\infty \mathcal{H}(\partial^* E_t^*)\, dt \\ &= C \int_0^\infty \mathcal{H}(\partial^* E_t^* \cap \Omega)\, dt \leq C \int_0^\infty P(E_t^*, \Omega)\, dt < \infty. \end{aligned}$$

This implies that $\|Du^*\|$ is finite, and hence $u^* \in BV(X)$.

The following step is to show that $u^* \in N^{1,1}(X)$. Since $u^* \in BV(X)$, we have a decomposition of the measure $\|Du^*\|$ into three parts: $\|D^g u^*\|$, $\|D^c u^*\|$ and $\|D^j u^*\|$, see Theorem 3.8. Since $u^* \in$

$N^{1,1}(\Omega)$, the jump part $\|D^j u^*\|$ and the Cantor part $\|D^c u^*\|$ are concentrated on $\partial\Omega$. By the coarea formula and Theorem 3.6,

$$\begin{aligned}\|D^c u^*\|(X) + \|D^j u^*\|(X) =&\|D^c u^*\|(X \setminus \Omega) + \|D^j u^*\|(X \setminus \Omega)\\ \leq&\|Du^*\|(X \setminus \Omega) = \int_0^\infty P(E_t^*, X \setminus \Omega)\, dt\\ \leq& C\int_0^\infty \mathcal{H}(\partial^* E_t^* \setminus \Omega)\, dt = 0.\end{aligned}$$

Hence $\|Du^*\|$ is absolutely continuous with respect to μ. Thus Theorem 3.10 implies that $u \in N^{1,1}(X)$. Since g is a 1-weak upper gradient of u^* in Ω and the zero function is an upper gradient of u^* in $X \setminus \Omega$ we have that the zero extension of g is a 1-weak upper gradient of u^*, see Lemma 4.3 in [Sha00] and the proof of Lemma 7.17 in [Haj03]. Therefore, $u^* \in N^{1,1}(X)$.

Then we consider the case $p > 1$. Now $u^* \in N^{1,1}(X)$ and both u^* and its 1-weak upper gradient g belong to $L^p(X)$. Thus by Lemma 5.1, a modification of u^* on a set of measure zero belongs to $N^{1,p}(X)$. However, we do not have to modify the function at Lebesgue points of the function (see for example [KL02]), and as by the assumption every point in $\partial\Omega$ is a Lebesgue point for u^*, we see that u^* belongs to $N^{1,p}(X)$. As $u^* = 0$ everywhere in $X \setminus \Omega$, we conclude that $u^* \in N_0^{1,p}(\Omega)$.

Now assume that (1.2) holds only outside a set $E \subset \partial\Omega$ with p-capacity zero. Then for every $\varepsilon > 0$, there exists $0 \leq \varphi_\varepsilon \leq 1$ such that $\varphi_\varepsilon = 1$ in a neighbourhood of E and

$$\|\varphi_\varepsilon\|_{N^{1,p}(X)} < \varepsilon.$$

Then define $u_\varepsilon = (1 - \varphi_\varepsilon)u$. As $u_\varepsilon \in N^{1,p}(\Omega)$ and u_ε satisfies the condition (1.2) for all $x \in \partial\Omega$, the proof above shows that $u_\varepsilon \in N_0^{1,p}(\Omega)$. Using the properties of weak upper gradients and the dominated convergence theorem, we have $u_\varepsilon \to u$ in $N^{1,p}(\Omega)$ as $\varepsilon \to 0$. By completeness, this implies that $u \in N_0^{1,p}(\Omega)$. Finally, we remove the assumptions that u is non-negative and bounded. We first remove the assumption that u is non-negative. If $u \in N^{1,p}(\Omega)$ is bounded and satisfies (1.2) for p-quasievery $x \in \partial\Omega$, then also the positive and negative parts u^+ and u^- have the same properties. Hence by the previous arguments, u^+ and u^- are in $N_0^{1,p}(\Omega)$. Thus so does $u = u^+ - u^-$.

If u is unbounded, then we look at its truncations

$$u_k = \max\{\min\{u, k\}, -k\}, \quad k = 1, 2, \dots.$$

The set of points $x \in \partial\Omega$ where (1.2) fails for u_k is a subset of points where it fails for u. Thus by the previous arguments, $u_k \in N_0^{1,p}(\Omega)$ for every $k = 1, 2, \dots$. Now the claim follows since $u_k \to u$ in $N^{1,p}(\Omega)$ as $k \to \infty$ and $N_0^{1,p}(\Omega)$ is a Banach space.

To prove the converse implication, assume that $u \in N_0^{1,p}(\Omega)$. Then $u \in N^{1,p}(X)$ and $u = 0$ in $X \setminus \Omega$. By Theorem 4.1 in [KL02] and

Theorem 4.1 in [KKST08], p-quasievery point of X is a Lebesgue point of u. Hence for p-quasievery $x \in X \setminus \Omega$, we have

$$0 = u(x) = \lim_{r \to 0} \fint_{B(x,r)} u \, d\mu = \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} u \, d\mu. \qquad \square$$

Remark 5.4. The assumption that Ω is open can be replaced with the condition that Ω is Borel measurable in Theorem 1.1. This is a generalization of the known result for open sets already in the Euclidean case. Let us briefly explain, how this extension can be obtained. Since $N_0^{1,p}(\operatorname{int} E) \subset N_0^{1,p}(E)$, we see that by applying Theorem 1.1 to the restriction of u to $\operatorname{int} E$ and noting that $\partial(\operatorname{int} E) \subset \partial E$, we get an extension $f \in N_0^{1,p}(E)$ such that $f = u$ in $\operatorname{int} E$. By the fact that u satisfies inequality (1.2) p-capacity almost everywhere in ∂E and so does f, by Lebesgue differentiation theorem we know that $u = f$ μ-a.e. in E (note that a set of zero p-capacity is necessarily of zero μ-measure). Since the restriction of f to E and the function u agree μ-a.e. in E and both belong to $N^{1,p}(E)$, we have that $u = f$ p-capacity almost everywhere in E (see [Sha00]). It follows that $u \in N_0^{1,p}(E)$.

6. Extendability of sets of finite perimeter

In this section, we assume that X supports a $(1,1)$-Poincaré inequality. We will show that if the Hausdorff measure of codimension one of the boundary is finite, we do not need to assume the validity of a strong relative isoperimetric inequality in order to prove Theorem 1.1. We start with two simple lemmas.

Lemma 6.1. *If $\mathcal{H}(A)$ is finite, then $\mu(A) = 0$.*

Proof. Let $\varepsilon > 0$, and let $\{B_i\}_{i\in I}$, $B_i = B(x_i, r_i)$, be a countable cover of A such that $r_i < \varepsilon$ for all $i \in I$ and

$$\sum_{i\in I} \frac{\mu(B_i)}{r_i} < \mathcal{H}(A) + \varepsilon.$$

Since $A \subset \bigcup_{i\in I} B_i$, it follows that

$$\mu(A) \le \sum_{i\in I} \mu(B_i) \le \varepsilon \sum_{i\in I} \frac{\mu(B_i)}{r_i} \le \varepsilon\big(\mathcal{H}(A) + \varepsilon\big).$$

Letting $\varepsilon \to 0$ gives the desired result. $\square$

Lemma 6.2. *Let $x \in X$ and $r > 0$ such that $X \setminus B(x, 4r)$ is non-empty. Then there exists $\rho \in [r, 2r]$ such that $B(x,\rho)$ is a set of finite perimeter in X with*

$$\frac{1}{C}\frac{\mu(B(x,\rho))}{\rho} \le P(B(x,\rho), X) \le C\frac{\mu(B(x,\rho))}{\rho},$$

where the constant C is independent of x, r and ρ.

Proof. Recall that as X supports a $(1,1)$-Poincaré inequality, for all $x \in X$ and $r > 0$, the sphere $\{y \in X : d(x,y) = r\}$ is non-empty whenever $X \setminus B(x,r)$ is non-empty. Since $X \setminus B(x,4r) \neq \emptyset$, by the remark above there is a point $y \in B(x,2r) \setminus B(x,r)$ such that $d(x,y) = 3r/2$. Hence $B(y,r/2) \subset B(x,2r) \setminus B(x,r)$ and by the doubling property of μ we have

$$\mu(B(x,2r) \setminus B(x,r)) \geq \mu(B(y,r/2)) \geq \frac{1}{C}\,\mu(B(x,r)).$$

By the relative isoperimetric inequality,

$$\begin{aligned}\mu(B(x,r)) &\leq C \min\{\mu(B(x,r)), \mu(B(x,2r) \setminus B(x,r))\}\\ &\leq Cr\, P(B(x,r), B(x,2r)) = Cr\, P(B(x,r), X).\end{aligned}$$

This implies that

$$\frac{\mu(B(x,\rho))}{\rho} \leq C\, P(B(x,\rho), X)$$

for every $\rho \in [r, 2r]$ and the lower bound follows.

To prove the upper bound, let u be the compactly supported Lipschitz function defined by

$$u(y) = \max\left\{0, \min\left\{\frac{2r - d(x,y)}{r}, 1\right\}\right\}.$$

Observe that $g = r^{-1}\chi_{B(x,2r)\setminus B(x,r)}$ is an upper gradient of u, so by the coarea formula

$$\int_0^1 P(B(x,(2-t)r), X)\, dt = \int_0^1 P(\{u > t\}, X)\, dt \leq \int_X g\, d\mu.$$

Hence it follows from the doubling property of μ that

$$\begin{aligned}\int_0^1 P(B(x,(2-t)r), X)\, dt &\leq \frac{C}{r}\mu(B(x,2r) \setminus B(x,r))\\ &\leq \frac{C}{r}\,\mu(B(x,r)) < \infty.\end{aligned}$$

We choose $\rho \in [r, 2r]$ such that

$$P(B(x,\rho), X) \leq \int_0^1 P(B(x,(2-t)r), X)\, dt,$$

from which it follows that

$$P(B(x,\rho), X) \leq \frac{C}{r}\,\mu(B(x,r)) \leq \frac{C}{\rho}\,\mu(B(x,\rho)).$$

This gives the required upper bound. □

Proposition 6.3. *Let $\Omega \subset X$ be an open set such that $\mathcal{H}(\partial\Omega)$ is finite and let $E \subset \Omega$ be a Borel set such that $P(E,\Omega)$ is finite. Then $P(E,X)$ is finite.*

Proof. Let $\varepsilon > 0$. Since $\mathcal{H}(\partial\Omega) < \infty$, there is a countable cover of $\partial\Omega$ with the balls $\widetilde{B}_i = B(x_i, r_i)$, $i \in I$, such that $r_i < \varepsilon$ for every $i \in I$, and

$$\sum_{i\in I} \frac{\mu(\widetilde{B}_i)}{r_i} < \mathcal{H}(\partial\Omega) + \varepsilon.$$

By Lemma 6.2, for each $i \in I$ there is $\rho_i \in [r_i, 2r_i]$ such that with $B_i = B(x_i, \rho_i)$, we have

$$\frac{1}{C}\frac{\mu(B_i)}{\rho_i} \le P(B_i, X) \le C\frac{\mu(B_i)}{\rho_i}.$$

This implies that

$$\begin{aligned}\sum_{i\in I} P(B_i, X) &\le C\sum_{i\in I}\frac{\mu(B_i)}{\rho_i} \\ &\le C\sum_{i\in I}\frac{\mu(\widetilde{B}_i)}{r_i} \le C\big(\mathcal{H}(\partial\Omega) + \varepsilon\big).\end{aligned} \tag{6.4}$$

Let

$$J = \{i \in I : B_i \cap \overline{E} \ne \emptyset\} \quad \text{and} \quad E_\varepsilon = E \cup \bigcup_{i\in J} B_i.$$

By the proof of Lemma 6.1, we conclude that $\chi_{E_\varepsilon} \to \chi_E$ in $L^1(X)$ as $\varepsilon \to 0$. Furthermore, by the properties of the perimeter measure, we have

$$\begin{aligned}P(E_\varepsilon, X) &\le P(E,\Omega) + \sum_{i\in J} P(B_i, X) \\ &\le P(E,\Omega) + C\big(\mathcal{H}(\partial\Omega) + \varepsilon\big) \to P(E,\Omega) + C\,\mathcal{H}(\partial\Omega)\end{aligned}$$

as $\varepsilon \to 0$. Hence by the lower semicontinuity of the BV-norm (see [Mir03]), we obtain

$$P(E, X) \le \liminf_{\varepsilon\to 0} P(E_\varepsilon, X) < \infty.$$

Thus E is of finite perimeter in X. □

Note that, in Theorem 1.1, the strong relative isoperimetric inequality is only needed to show that the level sets E_t^* have finite perimeter not only in Ω but also in X. If $\mathcal{H}(\partial\Omega)$ is finite, this easily follows from Proposition 6.3. In fact, the perimeters are even equal, as the following theorem demonstrates.

Theorem 6.5. *Let $\Omega \subset X$ be an open set such that $\mathcal{H}(\partial\Omega)$ is finite and $E \subset \Omega$ such that $P(E,\Omega)$ is finite and $\mathcal{H}(\partial^* E \cap \partial\Omega) = 0$. Then*

$$P(E, X) = P(E, \Omega).$$

Proof. By Proposition 6.3, we know that $\chi_E \in BV(X)$. By [Mir03], the perimeter measure $P(E, \cdot)$ is a finite Radon measure on X, and by [AMP04], we know that this measure is supported on $\partial^* E$ and is absolutely continuous with respect to $\mathcal{H}\big|_{\partial^* E}$. The conclusion now follows from the fact that $\mathcal{H}(\partial^* E \cap \partial\Omega) = 0$. □

7. A Lusin type theorem

In the Euclidean case Michael and Ziemer [MZ82] for $1 < p < \infty$ and Swanson [Swa07] for $p = 1$ showed that Sobolev functions are quasicontinuous in a strong sense. See also [BHS02]. The difference to the standard quasicontinuity is that the approximating function is a continuous Sobolev function on the entire metric space. Using a discrete convolution and Theorem 1.1, we give an analog for the Newtonian space.

Theorem 7.1. *If X supports the strong relative isoperimetric inequality and $u \in N^{1,p}(X)$, then for every $\varepsilon > 0$ there is an open set E_ε and a continuous function $u_\varepsilon \in N^{1,p}(X)$ such that $\operatorname{Cap}_p(E_\varepsilon) < \varepsilon$, $u = u_\varepsilon$ on $X \setminus E_\varepsilon$, and $\|u - u_\varepsilon\|_{N^{1,p}(X)} < \varepsilon$.*

Proof. A careful study of the proofs of Theorem 4.1 in [KKST08], for $p = 1$, and Theorem 4.1 in [KL02], for $1 < p < \infty$, gives an open set $E_\varepsilon \subset X$ with $\operatorname{Cap}_p(E_\varepsilon) < \varepsilon$ such that the restriction of u to $X \setminus E_\varepsilon$ is continuous and

$$\fint_{B(x,r)} |u - u(x)| \, d\mu \to 0$$

uniformly in $X \setminus E_\varepsilon$ as $r \to 0$.

The next step is to find a continuous extension of $u\big|_{X \setminus E_\varepsilon}$ to E_ε. To do so, we use a Whitney type cover of the open set E_ε; we refer the interested reader to [CW71] and Theorem 3.1 in [BBS07] for explicit construction of such a cover. From this construction, we obtain a countable collection of balls $\{B_i\}_{i \in I}$, where $B_i = B(x_i, r_i)$ with $r_i \approx \operatorname{dist}(x_i, X \setminus E_\varepsilon)$, satisfying

$$E_\varepsilon = \bigcup_{i \in I} B_i = \bigcup_{i \in I} 2\lambda C B_i,$$

where $\lambda \geq 1$ is the scaling constant in the Poincaré inequality and $C \geq 2$ is a contant that depends only on the doubling constant and the constant in the Poincaré inequality. Moreover, the collection $\{2C\lambda B_i\}_{i \in I}$ has a bounded overlap property and if $2B_i \cap 2B_j$ is non-empty, then $B_i \subset CB_j$.

We then fix a Lipschitz partition of unity subordinate to this cover, that is, a collection of functions $\{\varphi_i\}_{i \in I}$ such that $\operatorname{supp}(\varphi_i) \subset 2B_i$, $0 \leq \varphi_i \leq 1$ and φ_i is C/r_i-Lipschitz continuous and

$$\sum_{i \in I} \varphi_i(x) = 1$$

for every $x \in E_\varepsilon$. We define the discrete convolution v_ε on E_ε as

$$v_\varepsilon(x) = \sum_{i \in I} u_{B_i} \varphi_i(x).$$

By the bounded overlap property of the cover, the sum above is locally finite and hence v_ε is locally Lipschitz continuous on the open set E_ε. Moreover, as in the proof of Lemma 5.1, we have

$$\int_{E_\varepsilon} |v_\varepsilon|^p \, d\mu \le C \int_{E_\varepsilon} |u|^p \, d\mu.$$

Furthermore, if $x, y \in B_j$, then by the Lipschitz property of the functions φ_i,

$$\begin{aligned} |v_\varepsilon(x) - v_\varepsilon(y)| &= \left| \sum_{i \in I} (u_{B_i} - u_{B_j})(\varphi_i(x) - \varphi_i(y)) \right| \\ &\le C d(x,y) \sum_{i \in I_j} \frac{1}{r_i} |u_{B_i} - u_{B_j}|, \end{aligned}$$

where I_j is the same set of indices as in the proof of Theorem 4.6. By the properties of the cover given above and the doubling property of μ, we now see that there are at most C number of balls B_i such that $2B_i$ intersect B_j, and all of them have radii comparable to r_j.

Recall that by Theorem 4.2, the strong relative isoperimetric inequality implies the $(1,1)$-Poincaré inequality and consequently also the $(1,p)$-Poincaré inequality. Hence

$$\begin{aligned} |v_\varepsilon(x) - v_\varepsilon(y)| &\le \frac{C}{r_j} d(x,y) \sum_{i \in I_j} |u_{B_i} - u_{B_j}| \\ &\le \frac{C}{r_j} d(x,y) \sum_{i \in I_j} r_i \fint_{CB_j} |u - u_{CB_j}| \, d\mu \\ &\le C d(x,y) \left(\fint_{C\lambda B_j} g_u^p \, d\mu \right)^{1/p}. \end{aligned}$$

It follows that the function

$$g(x) = C \sum_{j \in I} \left(\fint_{C\lambda B_j} g_u^p \, d\mu \right)^{1/p} \chi_{B_j}(x),$$

is a p-weak upper gradient of v_ε in E_ε whenever g_u is a p-weak upper gradient of u. As in the proof of Lemma 5.1, we also have

$$\int_{E_\varepsilon} g^p \, d\mu \le C \int_{E_\varepsilon} g_u^p \, d\mu.$$

It follows from the above estimates that

$$\|v_\varepsilon\|_{N^{1,p}(E_\varepsilon)} \le C \|u\|_{N^{1,p}(E_\varepsilon)},$$

and we can choose E_ε to be small enough so that by the absolute continuity of the integral we have

$$\|v_\varepsilon\|_{N^{1,p}(E_\varepsilon)} < \varepsilon. \tag{7.2}$$

Now we define the function $u_\varepsilon = v_\varepsilon \chi_{E_\varepsilon} + u\chi_{X\setminus E_\varepsilon}$. It is clear that the restrictions of u_ε to E_ε and to $X \setminus E_\varepsilon$ are continuous. We now show that u_ε is continuous on X. To do so, it suffices to show that for all $x \in \partial E_\varepsilon$, we have

$$\lim_{E_\varepsilon \ni y \to x} u_\varepsilon(y) = u_\varepsilon(x) = u(x).$$

For $y \in E_\varepsilon$ we can choose (by the properness of X) $y' \in X \setminus E_\varepsilon$ such that $d(y, y') = \operatorname{dist}(y, X \setminus E_\varepsilon) = \delta(y)$. Then

$$|u_\varepsilon(x) - u_\varepsilon(y)| = |u(x) - v_\varepsilon(y)| \le |u(x) - u(y')| + |u(y') - v_\varepsilon(y)|,$$

and because $d(y, y') \le d(y, x)$, when $y \to x$ we see that $d(y, y') \to 0$ as well and hence $y' \to x$. Because $y' \in X \setminus E_\varepsilon$, by the continuity of the restriction of u to this set we see that $|u(y') - u(x)| \to 0$ as $y \in E_\varepsilon$ with $y \to x$. Hence it suffices now to show that we have $|u(y') - v_\varepsilon(y)| \to 0$ as $y \in E_\varepsilon$ with $y \to x$. To this end, we use the doubling property of μ and the fact that if $y \in 2B_i$ then $r_i \approx \delta(y)$, to see that

$$\begin{aligned}
|u(y') - v_\varepsilon(y)| &\le \sum_{i\in I} |u_{B_i} - u(y')| \varphi_i(y) \\
&\le \sum_{i\in I} \varphi_i(y) \fint_{B_i} |u(z) - u(y')| \, d\mu(z) \\
&\le \sum_{i\in J} \fint_{B_i} |u(z) - u(y')| \, d\mu(z) \\
&\le C \sum_{i\in J} \fint_{B(y,\delta(y))} |u(z) - u(y')| \, d\mu(z),
\end{aligned}$$

where $J = \{i \in I : y \in 2B_i\}$.

By the bounded overlap property of the cover and the doubling property of μ, it follows that

$$\begin{aligned}
|u(y') - v_\varepsilon(y)| &\le C \fint_{B(y,\delta(y))} |u(z) - u(y')| \, d\mu(z) \\
&\le C \fint_{B(y',2\delta(y))} |u(z) - u(y')| \, d\mu(z).
\end{aligned}$$

Because $y \in E_\varepsilon$ and $y \to x$, we have $\delta(y) \to 0$, and by the uniform convergence of the integral average on $X \setminus E_\varepsilon$, we have

$$\lim_{E_\varepsilon \ni y \to x} \fint_{B(y',2\delta(y))} |u(z) - u(y')| \, d\mu(z) = 0,$$

and it follows that $|v_\varepsilon(y) - u(y')| \to 0$ as desired. This implies that u_ε is continuous on X.

Finally we show that $u_\varepsilon \in N^{1,p}(X)$. Observe that $u_\varepsilon - u = 0$ on $X \setminus E_\varepsilon$, and that $u_\varepsilon - u \in N^{1,p}(E_\varepsilon)$ by (7.2). Since u_ε is continuous, every point of X is a Lebesgue point of u_ε. On the other hand, by Theorem 4.1 in [KL02] and Theorem 4.1 in [KKST08], p-quasievery point of X is a Lebesgue point of $u \in N^{1,p}(X)$. This implies that p-quasievery point of X is a Lebesgue point of $u - u_\varepsilon$. Since $u - u_\varepsilon = 0$ on ∂E_ε, we conclude that

$$\lim_{r \to 0} \fint_{B(x,r)} |u_\varepsilon - u| \, d\mu = 0$$

for p-quasievery $x \in \partial E_\varepsilon$. The desired conclusion now follows from Theorem 1.1. It also follows from (7.2) that

$$\|u - u_\varepsilon\|_{N^{1,p}(X)} = \|u - u_\varepsilon\|_{N^{1,p}(E_\varepsilon)} < \varepsilon,$$

completing the proof. □

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