ON WEIGHTS SATISFYING PARABOLIC MUCKENHOUPT CONDITIONS

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ABSTRACT. This note collects results related to parabolic Muckenhoupt A_p weights for a doubly nonlinear parabolic partial differential equation. A general approach is proposed, which extends the theory beyond the quadratic growth case. In particular, the natural parabolic geometry for the equation and the unavoidable time lag are incorporated in the definitions and results. The limiting Muckenhoupt conditions of A_∞ type are also discussed and several open questions are posed.

1. INTRODUCTION

In this note, we discuss a theory of parabolic Muckenhoupt weights and functions of bounded mean oscillation related to the doubly nonlinear parabolic equation

$$\frac{\partial(|u|^{p-2}u)}{\partial t} - \operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty. \tag{1.1}$$

The function

$$u(x,t) = t^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}}, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.2}$$

is the Barenblatt solution of (1.1) in the upper half space. When $p = 2$ we have the heat kernel. Observe that $u(x,t) > 0$ for every $x \in \mathbb{R}^n$ and $t > 0$. This indicates infinite speed of propagation of disturbances. The equation is degenerate in the sense that the modulus of ellipticity vanishes when the spatial gradient Du vanishes. The weak solutions are locally Hölder continuous, see [13] and [26].

The main challenge of (1.1) is the double nonlinearity both in time and space variables. Observe that the solutions can be scaled, but constants cannot be added to a solution. If $u(x,t)$ is a solution, so does $u(\lambda x, \lambda^p t)$ with $\lambda > 0$. This suggests that in the natural geometry for (1.1) the time variable scales as the modulus of the space variable raised to power p. Consequently, Euclidean cubes have to be replaced by parabolic rectangles respecting this scaling in all estimates. An extra challenge is given by the time lag appearing in the estimates.

2010 *Mathematics Subject Classification.* 42B25, 42B37, 35K55.

Key words and phrases. weighted norm inequalities, bounded mean oscillation, p-Laplace equation, doubly nonlinear equation, heat equation.

The research is supported by the Academy of Finland and Väisälä Foundation.

These phenomena are also visible in the Barenblatt solution. The main advantage of (1.1) is that a scale and location invariant parabolic Harnack's inequality holds true for nonnegative weak solutions in parabolic rectangles, see [25], [9], [11]. These estimates imply that nonnegative solutions of (1.1) are parabolic Muckenhoupt A_p weights and their logarithms have parabolic bounded mean oscillation (BMO). The parabolic BMO and Muckenhoupt classes were already implictly present in Moser's proof of parabolic Harnack's inequality, see [20] and [21]. Later the parabolic BMO was explicitly defined by Fabes and Garofalo in [6], who also gave a simplified proof for the parabolic John-Nirenberg lemma in [20].

We propose a more general approach, which extends the theory beyond the quadratic growth case and applies to the doubly nonlinear parabolic equation with all parameter values of p. In particular, the parabolic geometry and the time lag is incorporated in the definitions. This work is inspired by the classical interaction between Muckenhoupt weights and the regularity theory for elliptic equations as well as the recent attempts to generalize weighted norm inequalities for one-sided maximal operators to higher dimensions, see [2], [7], [12], [14], [22] and [23]. There is a relatively complete one-dimensional theory, see [15], [16], [17], [18], [19] and [24]. However, in the parabolic case the time dependence and parabolic geometry give several challenges and completely new phenomena that are not visible in the time-independent and one-dimensional cases. For the classical theory for weighted norm inequalities, we refer to [8]. The main results in [12] are characterizations of weighted norm inequalities for parabolic forward in time maximal functions, self-improving phenomena related to parabolic reverse Hölder inqualities, factorization results and a Coifman-Rochberg type characterization of parabolic BMO. In this note, we collect results related to parabolic Muckenhoupt weights, give a brief discussion of Muckenhoupt A_∞ conditions and pose several open questions.

2. Parabolic A_p weights

A generic space-time point is denoted $(x,t) \in \mathbb{R}^{n+1}$. The Lebesgue measure of a set $E \subset \mathbb{R}^{n+1}$ is written as $|E|$. A nonnegative locally integrable function on $\mathbb{R}^{n+1}$ is called a weight. For a weight w, we denote

$$w(E) = \int_E w = \int_E w(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

and

$$w_E = \fint_E w = \frac{1}{|E|}\int_E w, \quad |E| > 0.$$

Various positive constants are denoted by C.

Before the definition of the parabolic Muckenhoupt weights, we introduce the parabolic space-time rectangles in the natural geometry for the doubly nonlinear equation (1.1).

Definition 2.1. Let $Q = Q(x,l) \subset \mathbb{R}^n$ be a cube with center x and sidelength l. Let $\gamma \in [0,1)$ and $t \in \mathbb{R}$. We denote

$$\begin{aligned} R = R(x,t,l) &= Q(x,l) \times (t - l^p, t + l^p), \\ R^+(\gamma) &= Q(x,l) \times (t + \gamma l^p, t + l^p) \quad \text{and} \\ R^-(\gamma) &= Q(x,l) \times (t - l^p, t - \gamma l^p). \end{aligned}$$

We say that R is a parabolic rectangle in $\mathbb{R}^{n+1}$ with center (x,t) and sidelength l. $R^\pm(\gamma)$ are the upper and lower parts of R respectively and γ is called the time lag.

Now we are ready for the definition of the parabolic Muckenhoupt classes, see [12].

Definition 2.2. Let $q > 1$ and $\gamma \in (0,1)$. A weight w belongs to the parabolic Muckenhoupt class $A_q^+(\gamma)$, if

$$\sup_R \left(\fint_{R^-(\gamma)} w \right) \left(\fint_{R^+(\gamma)} w^{1-q'} \right)^{q-1} =: [w]_{A_q^+(\gamma)} < \infty, \tag{2.1}$$

where the supremum is taken over all parabolic rectangles R. If (2.1) is satisfied with the direction of the time axis reversed, we denote $w \in A_q^-(\gamma)$. If γ is clear from the context, or does not play any role, it will be omitted in the notation.

Observe that there is a time lag in the definition for $\gamma > 0$. The definition makes sense also for $\gamma = 0$, but this is not relevant in partial differential equations. The special role of the time variable makes the parabolic Muckenhoupt weights quite different from the classical ones. For example, the doubling property does not hold, but it can be replaced by a weaker forward in time doubling condition.

Remark 2.3. (1) If $w = w(x,t) \in A_q^+(\gamma)$, then $e^t w(x,t) \in A_q^+(\gamma)$. Consequently, a parabolic Muckenhoupt weight may grow exponentially in time.

(2) A trivial extension in time of a standard Muckenhoupt weight is clearly a parabolic Muckenhoupt weight. This implies that our theory is consistent extension of the classical Muckenhoupt theory.

The next proposition is a collection of useful facts about the parabolic Muckenhoupt condition, the most important of which is the property that the size of the lag does not play any role in the theory. This is crucial in our arguments. The same phenomenon occurs later with the parabolic BMO.

Proposition 2.4. (i) *(Nestedness) If $1 < q < r < \infty$, then $A_q^+(\gamma) \subset A_r^+(\gamma)$.*

(ii) *(Duality) Assume that $\sigma = w^{1-q'}$. Then σ is in $A^-_{q'}(\gamma)$ if and only if $w \in A^+_q(\gamma)$.*
(iii) *(Forward in time doubling) Assume that $w \in A^+_q(\gamma)$ and let $S \subset R^+(\gamma)$. Then*
$$w(R^-(\gamma)) \leq C \left(\frac{|R^-(\gamma)|}{|S|} \right)^q w(S).$$
(iv) *(Independence of the lag) If $w \in A^+_q(\gamma)$ with some $\gamma \in (0,1)$, then $w \in A^+_q(\gamma')$ for all $\gamma' \in (0,1)$.*

Proof. See [12]. □

The previous structural properties together with a reverse Hölder type inequality allow us to characterize weighted norm inequalities for the following maximal operator.

Definition 2.5. Let $\gamma \in (0,1)$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ define the parabolic forward in time maximal function
$$M^{\gamma+} f(x,t) = \sup_{R(x,t)} \fint_{R^+(\gamma)} |f|,$$
where the supremum is taken over all parabolic rectangles $R(x,t)$ centered at (x,t). The backward in time operator $M^{\gamma-}$ is defined analogously.

Observe that the point (x,t) does not belong to $R^+(\gamma)$ since $\gamma > 0$. It is remarkable that even if the parabolic maximal operators are not necessary pointwise comparable with different lags, the lag does not play any role the characterization for the weighted norm inequalities. Recall, that the operator $M^{\gamma+}$ is of weighted weak type (q,q), if
$$w(\{x \in \mathbb{R}^{n+1} : M^{\gamma+} f > \lambda\}) \leq \frac{C}{\lambda^q} \int_{R^+} |f|^q w, \quad \lambda > 0,$$
and it is of weighted strong type (q,q), if
$$\int_{\mathbb{R}^{n+1}} (M^{\gamma+} f)^q w \leq C \int_{\mathbb{R}^{n+1}} |f|^q w.$$
It is essential, that the constant C is independent of f.

Theorem 2.6. *The following conditions are equivalent:*
(i) *$w \in A^+_q(\gamma)$ for some $\gamma \in (0,1)$,*
(ii) *$w \in A^+_q(\gamma)$ for all $\gamma \in (0,1)$,*
(ii) *$M^{\gamma+}$ is of weighted weak type (q,q) for every $\gamma \in (0,1)$,*
(iii) *$M^{\gamma+}$ is of weighted strong type (q,q) for every $\gamma \in (0,1)$.*

For the proof, we refer to [12]. The strategy is first to characterize the weak type inequality and then prove a self improving property of weights (see Theorem 2.7). There are several challenges in the argument. First, the parabolic geometry does not have the usual dyadic structure. In the classical Muckenhoupt theory this would not be a serious problem, but here the forwarding in time gives new complications. The proof proceeds via an estimate for level sets, which implies the reverse Hölder property by Cavalieri's principle.

Theorem 2.7. *Assume that $w \in A_q^+$. Then there exist $\delta > 0$ and a constant C such that*

$$\left(\fint_{R^-(0)} w^{\delta+1} \right)^{1/(1+\delta)} \le C \fint_{R^+(0)} w \tag{2.2}$$

for all parabolic rectangles R. Furthermore, there exists $\epsilon > 0$ such that $w \in A_{q-\epsilon}^+$.

Proof. See [12]. □

We conclude this section with an informal remark.

Remark 2.8. In [12] a Muckenhoupt A_1^+ condition was proposed and used to prove a factorization theorem for A_q^+ weights. That was applied to obtain a Coifman-Rochberg type characterization for the parabolic BMO to be discussed in the next section (Theorem 3.4). On the other hand, there is characterization of the strong type inequality for the forward in time maximal operator and a "reverse factorization property", the discussion in [5] shows that the Rubio de Francia extrapolation applies to parabolic Muckenhoupt weights as well.

3. Parabolic BMO

In this section we discuss the connection between parabolic Muckenhoupt weights and the parabolic bounded mean oscillation.

Definition 3.1. Let $\gamma \in (0, 1)$. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ belongs to $\mathrm{PBMO}^+(\gamma)$, if for every parabolic rectangle R there is a constant a_R (possibly depending on R) such that

$$\sup_R \left(\fint_{R^+(\gamma)} (f - a_R)^+ + \fint_{R^-(\gamma)} (a_R - f)^+ \right) < \infty. \tag{3.1}$$

If (3.1) holds with the time axis reversed, then $f \in \mathrm{PBMO}^-(\gamma)$.

This definition has two advantages. First, the trivial extension of a function in the classical BMO obviously belongs to PBMO^+. Second, if (3.1) holds for some $\gamma \in (0, 1)$, then it holds for every such γ. This is a similar phenomenon as in the case of parabolic Muckenhoupt classes, see Proposition 2.4. The fact that $\gamma > 0$ is crucial here. For example, the John-Nirenberg inequality (Lemma 3.2) for the parabolic BMO cannot hold without a time lag. Otherwise Harnack's inequality would hold without a lag, which is physically impossible as shown by the Barenblatt solution. The following version of the John-Nirenberg lemma can be found in [23]. See also [6] and [1].

Lemma 3.2. *Assume that $f \in \mathrm{PBMO}^+(\gamma)$ with $\gamma \in (0,1)$. Then there are constants $A, B > 0$ such that*
$$|R^+(\gamma) \cap \{(f - a_R)^+ > \lambda\}| \le Ae^{-B\lambda}|R|$$
and
$$|R^-(\gamma) \cap \{(a_R - f)^+ > \lambda\}| \le Ae^{-B\lambda}|R|$$
for every parabolic rectangle R and $\lambda > 0$.

The next goal is to characterize PBMO^+ in sense of Coifman and Rochberg [4]. Factorization results analogous to [10] and [3] are available for parabolic Muckenhoupt weights and it remains to prove the connection between the parabolic BMO and Muckenhoupt conditions. The following lemma from [12] characterizes PBMO^+ as logarithms of A_q^+ weights. Note carefully, that $q = \infty$ is excluded in the statement. We do not know whether it can be included or not.

Lemma 3.3. $\mathrm{PBMO}^+ = \{-\lambda \log w : w \in A_q^+(\gamma), \lambda \in (0,\infty)\}$.

The following Coifman-Rochberg type characterization of the parabolic BMO gives us a method to construct functions in PBMO^+, for example, with prescribed singularities.

Theorem 3.4. *If $f \in \mathrm{PBMO}^+$, then there exist constants $\alpha, \beta > 0$, a function $b \in L^\infty(\mathbb{R}^{n+1})$ and nonnegative Borel measures μ and ν such that*
$$f = -\alpha \log M^{\gamma-}\mu + \beta \log M^{\gamma+}\nu + b.$$
Conversely, if any f is of the form above with $\gamma = 0$ and $M^-\mu < \infty$ and $M^+\nu < \infty$, then $f \in \mathrm{PBMO}^+$.

Proof. See [12]. □

4. Parabolic A_∞ weights

In this section we discuss parabolic Muckenhoupt A_∞ conditions. In the one-dimensional case, a complete theory of various equivalent definitions was obtained in [17]. The multidimensional case has turned out to be more unclear. We start by giving a list of conditions that could be taken as possible definitions for a parabolic Muckenhoupt A_∞^+ weight.

(i) (Reverse Jensen inequality) There is $\gamma \in (0,1)$ such that
$$\sup_R \left(\fint_{R^-(\gamma)} w \right) \exp \left(\fint_{R^+(\gamma)} \log w^{-1} \right) =: [w]_{A_\infty^+(\gamma)} < \infty.$$

(ii) (Reverse Hölder inequality) There is $\delta > 0$ and a constant C such that
$$\left(\fint_{R^-(0)} w^{\delta+1} \right)^{1/(1+\delta)} \le C \fint_{R^+(0)} w$$
for all parabolic rectangles R.

(iii) (Measure ratio condition) There are $\delta > 0$ and a constant C such that whenever R is a parabolic rectangle and $E \subset R^-(0)$ a measurable set,

$$\frac{w(E)}{w(R^+(0))} \le C\left(\frac{|E|}{|R^-(0)|}\right)^\delta.$$

(iv) (Fujii-Wilson condition) There is a constant C such that for all parabolic rectangles R,

$$\fint_{R^-(0)} M(1_{R^-(0)}w) \le C w_{R^+(0)}.$$

Again, the definition of the class A_∞^- is obvious.

4.1. **Reverse Jensen inequality and one-sided** BMO**.** In this subsection we assume that A_∞^+ is defined by the reverse Jensen inequality (i) above. This definition is very convenient from the point of view of the following characterization of the A_q^+ weights.

Proposition 4.1. *$w \in A_q^+$ if and only if $w \in A_\infty^+$ and $w^{1-q'} \in A_\infty^-$.*

Proof. Consider the translation τ acting on sets congruent to $R^-(\gamma)$ with $\tau R^-(\gamma) = R^+(\gamma)$. Then

$$\begin{aligned}
&\left(\fint_{R^-(\gamma)} w\right)\left(\fint_{\tau^2 R^-(\gamma)} w^{1-q'}\right)^{q-1} \\
&= \left(\fint_{R^-(\gamma)} w\right)\exp\left(\fint_{\tau R^-(\gamma)} \log w^{-1}\right) \\
&\qquad\qquad \times \exp\left(\fint_{\tau R^-(\gamma)} \log w\right)\left(\fint_{\tau^2 R^-(\gamma)} w^{1-q'}\right)^{q-1} \\
&= \left(\fint_{R^-(\gamma)} w\right)\exp\left(\fint_{\tau R^-(\gamma)} \log w^{-1}\right) \\
&\qquad\qquad \times \left(\exp\left(\fint_{\tau R^-(\gamma)} \log w^{-(1-q')}\right)\left(\fint_{\tau^2 R^-(\gamma)} w^{1-q'}\right)\right)^{q-1} \\
&\le [w]_{A_\infty^+}[w^{1-q'}]_{A_\infty^-}^{q-1}.
\end{aligned}$$

The reverse implication follows directly from Jensen's inequality. □

Remark 4.2. Assume that w satisfies the classical reverse Jensen inequality over cubes and let $u = \log w$. Then

$$[w]_{A_\infty} \ge \fint_Q w \exp\left(\fint_Q \log w^{-1}\right) = \fint_Q \exp(u - u_Q).$$

This implies that

$$\fint_Q |u - u_Q| = 2\fint_Q (u - u_Q)_+ \le 2\log(1 + [w]_{A_\infty})$$

and by the John-Nirenberg inequality we conclude that $w^\epsilon, w^{-\epsilon} \in A_2$ for some $\epsilon > 0$. By the previous proposition, which holds also for elliptic reverse Jensen inequality, we have that $w \in A_q$ with $\epsilon = 1/(q-1)$. This proof of is probably not very standard, but it is instructive in the sense that it uses the symmetry of A_∞ in BMO context: since $u - u_Q$ has zero mean, the BMO condition with the integral of the positive part is equally strong as the one with the integral of the absolute value. In the parabolic context with the time lag the corresponding phenomenon is not as clear.

The previous remark motivates the following definition of one-sided parabolic BMO.

Definition 4.3. Let $\gamma \in (0,1)$. A function $f \in L^1_{loc}(\mathbb{R}^{n+1})$ belongs to $\mathrm{BMO}^+(\gamma)$, if

$$\sup_R \fint_{R^-(\gamma)} (f - f_{R^+(\gamma)})^+ < \infty. \tag{4.1}$$

The class $\mathrm{BMO}^-(\gamma)$ is defined analogously.

This condition is connected to the parabolic BMO. By Proposition 4.1 an A_q^+ weight can be factored into two $A_\infty^\pm$ type conditions and clearly PBMO^- is an intersection of two $\mathrm{BMO}^\pm$ spaces (mind the unfortunate sign convention). Note that $f \in \mathrm{BMO}^+$ corresponds to $e^f \in A_\infty^+$, $f \in \mathrm{BMO}^-$ corresponds to $e^{-f} \in A_\infty^-$ and PBMO^- is a logarithm of A_q^+, see Lemma 3.3. Hence we conclude the BMO analogue of Proposition 4.1.

Proposition 4.4. $\mathrm{PBMO}^- = \mathrm{BMO}^+ \cap (-\mathrm{BMO}^-)$.

Proof. Let τ be the translation that sends $R^-(\gamma)$ to $R^+(\gamma)$. If u satisfies the one-sided conditions of the right hand side, then the PBMO^- condition with sets $R^-(\gamma)$ and $\tau^2 R^-(\gamma)$ is satisfied with $a_R = u_{\tau R^-(\gamma)}$. Equivalence of definitions with different lags takes care of the rest, see [23]. The converse follows from the characterization of PBMO^+ through Muckenhoupt conditions, see Lemma 3.3. □

We do not know if the condition $\mathrm{BMO}^+ \cap (-\mathrm{BMO}^-)$ is optimal, that is, whether $\mathrm{BMO}^+ = (-\mathrm{BMO}^-)$ or not. This equality holds in the one-dimensional case, but it is not clear how to extend the argument to the higher dimensional case.

Question 4.5. Is it true that $\mathrm{BMO}^+ = (-\mathrm{BMO}^-)$ or (and) $A_\infty^+ = \cup_{q>1} A_q^+(\gamma)$?

Note that an affirmative answer to one of the questions would also solve the other question. If the BMO^+ question has a negative answer, it is likely that this can be bootstrapped to a one-sided John-Nirenberg inequality similar to the one in [2] in order to disprove the A_∞^+ question.

On the other hand, a counterexample to one of the questions would probably do for the other question as well.

4.2. **Reverse Hölder inequality and volume ratios.** By Theorem 2.7 every $w \in A_q^+(\gamma)$ satisfies a reverse Hölder inequality. On the other hand, the reverse Hölder inequality (ii) is equivalent to the volume ratio condition (iii). Indeed, let R be a parabolic rectangle and $E \subset R^-(0)$. Then

$$\begin{aligned} w(E) &= \int_{R^-(0)} \chi_E w \\ &\le |E|^{\delta/(\delta+1)} |R^-(0)|^{1/(\delta+1)} \left(\fint_{R^-(0)} w^{\delta+1} \right)^{1/(\delta+1)} \\ &\le C|E|^{\delta/(\delta+1)} |R^-(0)|^{-\delta/(\delta+1)} w(R^+(0)) \\ &\le Cw(R^+(0)) \left(\frac{|E|}{|R^-(0)|} \right)^{\delta/(\delta+1)}, \end{aligned}$$

which is (iii) with δ replaced by $\delta/(\delta+1)$.

Conversely, assume that volume ratio condition is satisfied with $\delta = 1/q$. Let R be a parabolic rectangle. Denote $E_\lambda = R^-(0) \cap \{w > \lambda\}$. By Chebyshev's inequality

$$|E_\lambda| \le \frac{1}{\lambda} w(E_\lambda).$$

This together with volume ratio condition gives

$$|E|^{1/q'} \le C \frac{1}{\lambda} \frac{w(R^+(0))}{|R^-(0)|^{1/q}}.$$

Consequently

$$\begin{aligned} \fint_{R^-(0)} w^{1+\epsilon} &\le |R^-(0)|\gamma^{1+\epsilon} + \int_\gamma^\infty \lambda^\epsilon |E_\lambda| \,\mathrm{d}\lambda \\ &\le |R^-(0)|\gamma^{1+\epsilon} + C(1+\epsilon) \frac{w(R^+(0))^{q'}}{|R^-(0)|^{q'/q}} \int_\gamma^\infty \lambda^{\epsilon - q'} \,\mathrm{d}\lambda \\ &= |R^-(0)|\gamma^{1+\epsilon} + \frac{C(1+\epsilon) w(R^+(\gamma))^{q'}}{(q'-1-\epsilon)|R^-(0)|^{q'/q}} \gamma^{\epsilon - q' + 1}, \end{aligned}$$

where we assume $\epsilon + 1 < q'$. The choice $\gamma = w_{R^+(0)}$ gives the claimed reverse Hölder type inequality.

We point out that the reverse Hölder inequality with separated R^+ and R^- was obtained, roughly speaking, using a reverse Hölder inequality for the pair of sets (R^-, R) and the property $w(R^-) \lesssim w(R^+)$. If we do not assume the latter condition, it is clear that the overlapping reverse Hölder inequality does not necessarily imply any A_q^+ condition. Indeed, this can be seen by taking $w(x,t) = 1 - 1_{\{0<t<1\}}(x,t)$.

Question 4.6. Does the reverse Hölder type inequality (ii) or the volume ratio condition (iii) imply the parabolic Muckenhoupt condition A_q^+ for some q?

4.3. **Further observations.** The reverse Hölder type inequality implies (iv) but not much more can be said in this respect. The proof is simple, just use Hölder's inequality, boundedness of the maximal operator and reverse Hölder inequality to conclude that

$$\begin{aligned}\fint_{R^-(0)} M(1_{R^-(0)}w) &\le \left(\fint_{R^-(0)} M(1_{R^-(0)}w)^\alpha\right)^{1/\alpha} \\ &\le C\left(\fint_{R^-(0)} w^\alpha\right)^{1/\alpha} \le C\fint_{R^+(0)} w\end{aligned}$$

provided α is smaller than the reverse Hölder exponent of w.

We conclude by listing two other possible A_∞^+ conditions that have their analogues in the classical case. Both of these imply a reverse Hölder inequality, but otherwise their role is unclear.

(i) There are $\alpha, \beta \in (0,1)$ such that

$$|R^+(\gamma) \cap \{w > \beta w_{R^-(\gamma)}\}| > \alpha |R^+(\gamma)|$$

for all parabolic rectangles R.

(ii) For all parabolic rectangles R and all $\lambda \ge w_{R^-(\gamma)}$, we have

$$w(R^-(\gamma) \cap \{w > \lambda\}) \le C\lambda |R \cap \{w > \beta\lambda\}|.$$

5. Doubly nonlinear equation

This section focuses on the regularity of nonnegative weak solutions to the doubly nonlinear parabolic equation (1.1). Let $1 < p < \infty$. The Sobolev space $W^{1,p}(\mathbb{R}^n)$ is the completion of $C^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{1,p} = \|u\|_p + \|Du\|_p$. A function belongs to the local Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ if it belongs to $W^{1,p}(\Omega)$ for every $\Omega \Subset \mathbb{R}^n$. We denote by $L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^n))$, the space of functions $u = u(x,t)$ such that for almost every t the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^{n+1}} (|u|^p + |Du|^p) < \infty.$$

Roughly speaking the functions in $L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^n))$ are Sobolev functions in the spatial variable for a fixed moment of time and L^p-functions in the time variable at a fixed point in $\mathbb{R}^n$. The definition for the local parabolic Sobolev space $L^p_{\mathrm{loc}}(\mathbb{R}; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n))$ is clear.

Definition 5.1. A function $u \in L^p_{\mathrm{loc}}(\mathbb{R}; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n))$ is a weak solution to (1.1) in $\mathbb{R}^{n+1}$, if

$$\int_{\mathbb{R}^{n+1}} \left(|Du|^{p-2}Du \cdot D\varphi - |u|^{p-2}u\frac{\partial \varphi}{\partial t}\right) = 0 \tag{5.1}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. Further, we say that u is a supersolution to (1.1), if the integral (5.1) is nonnegative for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\varphi \geq 0$. If this integral is nonpositive, we say that u is a subsolution.

Observe that the time derivative u_t is avoided in the definition and, a priori, the weak solution is not assumed to have the weak derivative in the time direction. The assumption that the function belongs to $L^p_{\mathrm{loc}}(\mathbb{R}; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n))$ guarantees that the integral in (5.1) is well defined.

Remark 5.2. We point out that our theory also applies to a more general class of equations than just (1.1), but we have chosen to focus only on the prototype here. More precisely, our theory covers equations

$$\frac{\partial(|u|^{p-2}u)}{\partial t} - \operatorname{div} A(x,t,u,Du) = 0, \quad 1 < p < \infty,$$

where A satisfies the structural conditions

$$A(x,t,u,Du) \cdot Du \geq C_0 |Du|^p$$

and

$$|A(x,t,u,Du)| \leq C_1 |Du|^{p-1}.$$

See [11] and [23] for more.

We begin with a reformulation of a lemma from [11]. Similar results in different forms can also be found in [20] and [25]. We refer to [11] for all necessary definitions.

Lemma 5.3. *Assume that u is a positive supersolution of the doubly nonlinear equation. Then for every parabolic rectangle R there are constants C, C' and β_R (possibly depending on R) such that*

$$|R^- \cap \{\log u > \lambda + \beta_R + C'\}| \leq \frac{C}{\lambda^{p-1}} |R^-|$$

and

$$|R^+ \cap \{\log u < -\lambda + \beta_R - C'\}| \leq \frac{C}{\lambda^{p-1}} |R^+|$$

for all $\lambda > 0$.

Note that the only dependency on R in the previous estimates is in the constant β_R. Since being a supersolution is a local property, a supersolution in a domain is obviously a supersolution in all of its parabolic subrectangles. Setting first $f = -\log u$, we can use Lemma 5.3 together with Cavalieri's principle to obtain

$$\sup_R \left(\fint_{R^+} (f - a_R)_+^b + \fint_{R^-} (a_R - f)_+^b \right) < \infty$$

with $b = \min\{(p-1)/2, 1\}$, see [23]. Here the supremum is taken over all parabolic rectangles R. The John-Nirenberg machinery developed in [1] together with local-to-global results for parabolic John-Nirenberg inequality in [23] can be used to deduce that this implies

$$\sup_R \left(\fint_{R^+(\gamma)} (f - a_R)_+ + \fint_{R^-(\gamma)} (a_R - f)_+ \right) < \infty,$$

which is exactly the definition of the parabolic BMO, see Definition 3.1. Hence the negative logarithm of a nonnegative supersolution belongs to PBMO^+.

Theorem 5.4. *Assume that u is a positive supersolution of the doubly nonlinear equation. Then $-\log u \in \mathrm{PBMO}^+$.*

Already this result is interesting, but further, it follows from Lemma 3.3 that there is some small power $\epsilon > 0$ such that $u^\epsilon \in A_2^+(\gamma)$, or equivalently,

$$\sup \left(\fint_{R^-(\gamma)} u^\varepsilon \right) \left(\fint_{R^+(\gamma)} u^{-\varepsilon} \right) < \infty,$$

where the supremum is taken over all parabolic rectangles R. To see this, recall that $f = -\log u \in \mathrm{PBMO}^+$. Let $0 < \varepsilon < B$, where B is the constant in Lemma 3.2. We conclude that

$$\begin{aligned}
\fint_{R^+(\gamma)} u^{-\varepsilon} = \fint_{R^+(\gamma)} e^{\varepsilon f} &\le e^{\varepsilon a_R} \fint_{R^+(\gamma)} e^{\varepsilon (f - a_R)_+} \\
&= e^{\varepsilon a_R} \int_0^\infty e^\lambda |R^+(\gamma) \cap \{(f - a_R)_+ > \lambda/\varepsilon\}| \, \mathrm{d}\lambda + e^{\varepsilon a_R} \\
&\le A e^{\varepsilon a_R} |R| \int_0^\infty e^{\lambda(1 - B/\varepsilon)} \, \mathrm{d}\lambda + e^{\varepsilon a_R} \\
&= A e^{\varepsilon a_R} |R| \left(\frac{\varepsilon}{B - \varepsilon} + 1 \right).
\end{aligned}$$

Similarly, we obtain

$$\fint_{R^-(\gamma)} u^{-\varepsilon} \le A e^{-\varepsilon a_R} |R| \left(\frac{\varepsilon}{B - \varepsilon} + 1 \right).$$

The claim follows from these estimates.

That fact, in turn, was used by Moser in his proof of Harnack inequality for parabolic differential equations with quadratic growth. More generally, every nonnegative solution u of the doubly nonlinear equation satisfies the uniform scale and location invariant Harnack's inequality

$$\begin{aligned}
\operatorname{ess\,sup}_{R^-(\gamma)} u &\le \left(\fint_{\tilde{R}^-(\gamma)} u^\varepsilon \right)^{1/\varepsilon} \\
&\le C \left(\fint_{\tilde{R}^+(\gamma)} u^{-\varepsilon} \right)^{-1/\varepsilon} \le C \operatorname{ess\,inf}_{R^+(\gamma)} u,
\end{aligned}$$

see [25] and [11]. Here $\tilde{R}^-(\gamma)$ denotes parabolic dilation expanding $R^-(\gamma)$ backwards in time and in the usual manner in space. Harnack's inequality implies that nonnegative solutions to the doubly nonlinear equation belong to all parabolic Muckenhoupt classes.

Theorem 5.5. *Assume that u is a positive solution of the doubly nonlinear equation. Then $u \in A_q^+(\gamma)$ for every $q > 1$ and $\gamma \in (0,1)$.*

In addition to the Coifman-Rochberg type characterization, this gives us another source of examples of parabolic weights, that is, all positive solutions of the doubly nonlinear equation are parabolic Muckenhoupt weights. For example, the Barenblatt solution in (1.2) satisfies all of these properties in the upper half space.

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