

THE DERIVATIVE OF THE MAXIMAL FUNCTION

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ABSTRACT. In this note we show that the local Hardy-Littlewood maximal operator is bounded in the Sobolev space. Thus the maximal function often has partial derivatives. We also show that the maximal operator preserves the zero boundary values in Sobolev's sense.

1. INTRODUCTION

The celebrated maximal operator of Hardy and Littlewood is usually used to estimate the absolute size and so its possible regularity properties are often neglected. An essential phenomenon is that the maximal operator preserves the class of Lipschitz continuous functions. By Rademacher's theorem such functions are differentiable almost everywhere. The question about differentiability in general is a more delicate one. It was shown in [K] by the first author that the globally defined maximal operator preserves the first order Sobolev spaces. The objective of our note is the local case. Having applications to partial differential equations in mind we are keen on having a locally defined maximal function that will do as a test-function in the weak formulation of the equation.

To be more precise, let Ω be an open set in the Euclidean space \mathbf{R}^n . For a locally integrable function $u: \Omega \rightarrow [-\infty, \infty]$ we define the local Hardy-Littlewood maximal function $\mathcal{M}_\Omega u: \Omega \rightarrow [0, \infty]$ as

$$(1.1) \quad \mathcal{M}_\Omega u(x) = \sup \frac{1}{|B(x, r)|} \int_{B(x, r)} |u(y)| dy,$$

where the supremum is taken over all radii r with $0 < r < \text{dist}(x, \partial\Omega)$. In other words, *all open balls $B(x, r)$ centered at x and contained in Ω are admissible*. Hence the local maximal function depends also on the domain. With this definition, $\mathcal{M}_\Omega u = u$ exactly when the non-negative function u is superharmonic in Ω . The theorem of Hardy, Littlewood, and Wiener, see for example [S], asserts that the maximal operator is bounded in $L^p(\Omega)$ for $1 < p \leq \infty$, that is

$$(1.2) \quad \|\mathcal{M}_\Omega u\|_{p, \Omega} \leq c(n, p) \|u\|_{p, \Omega}.$$

The Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, consists of those functions u which, together with their first weak partial derivatives $Du = (D_1u, \dots, D_nu)$, belong to $L^p(\Omega)$.

Our main result is the following.

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1.3. Theorem. *Let $1 < p \leq \infty$. If $u \in W^{1,p}(\Omega)$, then $\mathcal{M}_\Omega u \in W^{1,p}(\Omega)$ and*

$$(1.4) \quad |D\mathcal{M}_\Omega u(x)| \leq 2\mathcal{M}_\Omega |Du|(x),$$

for almost every $x \in \Omega$.

We endow the Sobolev space $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{1,p,\Omega} = \|u\|_{p,\Omega} + \|Du\|_{p,\Omega}.$$

Using the previous theorem together with (1.2), we see that *the local maximal operator is bounded in $W^{1,p}(\Omega)$* provided $1 < p \leq \infty$. This generalizes the main result of [K] to the case $\Omega \neq \mathbf{R}^n$.

The proof of Theorem 1.3 is based on a general principle given in Section 3. It is decisive that the first order Sobolev space is a lattice: if u and v belong to $W^{1,p}(\Omega)$, so does the pointwise maximum $\max(u, v)$. This is also the reason why our results do not hold in the higher order Sobolev spaces. Section 2 contains an expedient technical lemma.

Let us point out a question related to the definition. As we shall see in Section 4, the local maximal function preserves the zero boundary values in Sobolev's sense. More precisely, for every $u \in W_0^{1,p}(\Omega)$ with $1 < p < \infty$, the function $\mathcal{M}_\Omega u$ belongs to $W_0^{1,p}(\Omega)$. Here $W_0^{1,p}(\Omega)$ denotes the Sobolev space defined as the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm. In contrast, the usual way of extending $u \in W_0^{1,p}(\Omega)$ as zero to the complement of Ω and, then, of taking the global maximal function of u will not do for our purposes. The global maximal function $\mathcal{M}_{\mathbf{R}^n}$ belongs to $W^{1,p}(\mathbf{R}^n)$ by [K] but its restriction to Ω does not, in general, possess the the same boundary values as u . In other words, the usual remedy would distort the boundary values. To this we may add that neither does the *uncentered* maximal function, in general, preserve the boundary values. We hope to return to applications in our future work.

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2. A LEMMA

We begin with an auxiliary result, which may be of independent interest. We make a standing assumption that $\Omega \neq \mathbf{R}^n$ so that $\delta(x) = \text{dist}(x, \partial\Omega)$ is finite. The functions $u_t: \Omega \rightarrow [-\infty, \infty]$, $0 < t < 1$, defined by

$$(2.1) \quad u_t(x) = \mathop{\int}\limits_{B(x,t\delta(x))} u(y) dy,$$

will play a crucial rôle in the proof of Theorem 1.3. The ball $B(x, t\delta(x))$ is comprised in Ω . The bar on the integral sign denotes the average, i.e., a division by the volume $\omega_n t^n \delta(x)^n$. We also write

$$\mathop{\int}\limits_{\partial B(x,r)} u(y) d\mathcal{H}^{n-1}(y) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y) d\mathcal{H}^{n-1}(y)$$

for the spherical average.

2.2. Lemma. *Let $1 < p \leq \infty$ and suppose that $u \in W^{1,p}(\Omega)$. Then $u_t \in W^{1,p}(\Omega)$, $0 < t < 1$, and*

$$(2.3) \quad |Du_t(x)| \leq 2\mathcal{M}_\Omega |Du|(x), \quad 0 < t < 1,$$

for almost every $x \in \Omega$.

Proof. Suppose first that $u \in C^\infty(\Omega)$. Let t , $0 < t < 1$, be fixed. According to Rademacher's theorem, as a Lipschitz function δ is differentiable almost everywhere in Ω . Moreover, $|D\delta(x)| = 1$ for almost every $x \in \Omega$. The Leibnitz rule gives

$$\begin{aligned} D_i u_t(x) &= D_i \left(\frac{1}{\omega_n (t\delta(x))^n} \right) \cdot \int_{B(x,t\delta(x))} u(y) dy \\ &\quad + \frac{1}{\omega_n (t\delta(x))^n} \cdot D_i \int_{B(x,t\delta(x))} u(y) dy, \quad i = 1, 2, \dots, n, \end{aligned}$$

for almost every $x \in \Omega$, and by the chain rule

$$\begin{aligned} D_i \int_{B(x,t\delta(x))} u(y) dy &= \int_{B(x,t\delta(x))} D_i u(y) dy \\ &\quad + t \int_{\partial B(x,t\delta(x))} u(y) d\mathcal{H}^{n-1}(y) \cdot D_i \delta(x), \quad i = 1, 2, \dots, n, \end{aligned}$$

for almost every $x \in \Omega$. Here we also used the fact that

$$\frac{\partial}{\partial r} \int_{B(x,r)} u(y) dy = \int_{\partial B(x,r)} u(y) dy.$$

Collecting terms we obtain

$$(2.4) \quad \begin{aligned} Du_t(x) &= n \frac{D\delta(x)}{\delta(x)} \left(\int_{\partial B(x,t\delta(x))} u(y) d\mathcal{H}^{n-1}(y) - \int_{B(x,t\delta(x))} u(y) dy \right) \\ &\quad + \int_{B(x,t\delta(x))} Du(y) dy, \end{aligned}$$

for almost every $x \in \Omega$. This is a vector identity.

In order to estimate the difference of the two integrals in the parenthesis in (2.4) we have to take into account a cancellation effect. To this end, suppose that $B(x, R) \subset \Omega$. We use Green's first identity

$$\int_{\partial B(x,R)} u(y) \frac{\partial v}{\partial \nu}(y) d\mathcal{H}^{n-1}(y) = \int_{B(x,R)} (u(y) \Delta v(y) + Du(y) \cdot Dv(y)) dy,$$

where $\nu(y) = (y - x)/R$ is the unit outer normal of $B(x, R)$ and we choose $v(y) = |y - x|^2/2$. With these choices Green's formula reads

$$\int_{\partial B(x,R)} u(y) d\mathcal{H}^{n-1}(y) - \int_{B(x,R)} u(y) dy = \frac{1}{n} \int_{B(x,R)} Du(y) \cdot (y - x) dy.$$

We estimate the right-hand side of the previous equality by

$$\left| \int_{B(x,R)} Du(y) \cdot (y-x) dy \right| \leq R \int_{B(x,R)} |Du(y)| dy \leq R \mathcal{M}_\Omega |Du|(x)$$

and, finally, we conclude that

$$(2.5) \quad \left| \int_{\partial B(x,R)} u(y) d\mathcal{H}^{n-1}(y) - \int_{B(x,R)} u(y) dy \right| \leq \frac{R}{n} \mathcal{M}_\Omega |Du|(x).$$

Let us multiply the vector identity (2.4) with an arbitrary unit vector $e = (e_1, \dots, e_n)$. Using (2.5) with $R = t\delta(x)$, we have by the Schwarz inequality

$$\begin{aligned} |e \cdot Du_t(x)| &\leq n \frac{|e \cdot D\delta(x)|}{\delta(x)} \cdot \frac{t\delta(x)}{n} \mathcal{M}_\Omega |Du|(x) + \left| \int_{B(x,t\delta(x))} e \cdot Du(y) dy \right| \\ &\leq t \mathcal{M}_\Omega |Du|(x) + \int_{B(x,t\delta(x))} |Du(y)| dy \\ &\leq (t+1) \mathcal{M}_\Omega |Du|(x) \end{aligned}$$

for almost every $x \in \Omega$. Since $t \leq 1$ and e is arbitrary, (2.3) is proved for smooth functions.

The case $u \in W^{1,p}(\Omega)$ with $1 < p < \infty$ follows from an approximation argument. To this end, suppose that $u \in W^{1,p}(\Omega)$ for some p with $1 < p < \infty$. Then there is a sequence $\{\varphi_j\}$ of functions in $W^{1,p}(\Omega) \cap C^\infty(\Omega)$ such that $\varphi_j \rightarrow u$ in $W^{1,p}(\Omega)$ as $j \rightarrow \infty$.

Fix t with $0 < t < 1$. By (2.1) we see that

$$u_t(x) = \lim_{j \rightarrow \infty} (\varphi_j)_t(x),$$

when $x \in \Omega$. It is clear that

$$|(\varphi_j)_t(x)| \leq \int_{B(x,t\delta(x))} |\varphi_j(y)| dy \leq \mathcal{M}_\Omega \varphi_j(x), \quad j = 1, 2, \dots,$$

for every $x \in \Omega$. By (2.3) for smooth functions we have

$$(2.6) \quad |D(\varphi_j)_t(x)| \leq 2 \mathcal{M}_\Omega |D\varphi_j|(x), \quad j = 1, 2, \dots,$$

for almost every $x \in \Omega$. These inequalities and the Hardy-Littlewood-Wiener theorem imply that

$$\begin{aligned} \|(\varphi_j)_t\|_{1,p,\Omega} &= \|(\varphi_j)_t\|_{p,\Omega} + \|D(\varphi_j)_t\|_{p,\Omega} \\ &\leq c(n,p) (\|\varphi_j\|_{p,\Omega} + \|D\varphi_j\|_{p,\Omega}) = c(n,p) \|\varphi_j\|_{1,p,\Omega}. \end{aligned}$$

Thus $\{(\varphi_j)_t\}_{j=1}^\infty$ is a bounded sequence in $W^{1,p}(\Omega)$ and, since it converges to u_t pointwise, we conclude that the Sobolev derivative Du_t exists and that $D(\varphi_j)_t \rightarrow$

Du_t weakly in $L^p(\Omega)$ as $j \rightarrow \infty$. This is a standard argument yielding the desired conclusion that u_t belongs to $W^{1,p}(\Omega)$.

To establish inequality (2.3) we want to proceed to the limit in (2.6) as $j \rightarrow \infty$. By the sublinearity of the maximal function we obtain

$$|\mathcal{M}_\Omega |D\varphi_j|(x) - \mathcal{M}_\Omega |Du|(x)| \leq \mathcal{M}_\Omega (|D\varphi_j| - |Du|)(x)$$

for every $x \in \Omega$ and, using the Hardy-Littlewood-Wiener theorem once more, we arrive at

$$\begin{aligned} \|\mathcal{M}_\Omega |D\varphi_j| - \mathcal{M}_\Omega |Du|\|_{p,\Omega} &\leq \|\mathcal{M}_\Omega (|D\varphi_j| - |Du|)\|_{p,\Omega} \\ &\leq c(n,p) \| |D\varphi_j| - |Du| \|_{p,\Omega}. \end{aligned}$$

Hence $\mathcal{M}_\Omega |D\varphi_j| \rightarrow \mathcal{M}_\Omega |Du|$ (even strongly) in $L^p(\Omega)$ as $j \rightarrow \infty$.

To complete the proof, we notice the following simple proposition: *If $f_j \rightarrow f$ and $g_j \rightarrow g$ weakly in $L^p(\Omega)$ and if $f_j(x) \leq g_j(x)$, $j = 1, 2, \dots$, almost everywhere in Ω , then $f(x) \leq g(x)$ almost everywhere in Ω .* Applying the proposition to (2.6), we obtain the desired inequality (2.3).

Finally we consider the case $p = \infty$. Slightly modifying the the above proof we see that $u_t \in W_{\text{loc}}^{1,p}(\Omega)$ for every p with $1 < p < \infty$ and estimate (2.3) holds for the gradient. The claim follows from the Hardy-Littlewood-Wiener theorem. This completes the proof.

3. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 follows now easily, since the hard work has been done in the proof of Lemma 2.2. Suppose that $u \in W^{1,p}(\Omega)$ for some p , $1 < p < \infty$. Then $|u| \in W^{1,p}(\Omega)$. Consider the auxiliary functions defined by formula (2.1). Let t_j , $j = 1, 2, \dots$, be an enumeration of the rationals between 0 and 1 and denote $u_j = |u|_{t_j}$. By Lemma 2.2 we see that $u_j \in W^{1,p}(\Omega)$ for every $j = 1, 2, \dots$ and (2.3) gives us the estimate

$$|Du_j(x)| \leq 2 \mathcal{M}_\Omega |Du|(x), \quad j = 1, 2, \dots,$$

for almost every $x \in \Omega$. Here we also used the fact that $|Du(x)| = |D|u(x)||$ for almost every $x \in \Omega$. We define $v_k: \Omega \rightarrow [-\infty, \infty]$ as the pointwise maximum

$$v_k(x) = \max_{1 \leq j \leq k} u_j(x), \quad k = 1, 2, \dots$$

Using the fact that the maximum of two Sobolev functions belongs to the Sobolev space, see [GT] Lemma 7.6, we see that $\{v_k\}$ is an increasing sequence of functions in $W^{1,p}(\Omega)$ converging to $\mathcal{M}_\Omega u$ pointwise and

$$\begin{aligned} (3.1) \quad |Dv_k(x)| &= \left| D \max_{1 \leq j \leq k} u_j(x) \right| \\ &\leq \max_{1 \leq j \leq k} |Du_j(x)| \leq 2 \mathcal{M}_\Omega |Du|(x), \quad k = 1, 2, \dots, \end{aligned}$$

for almost every $x \in \Omega$. On the other hand $v_k(x) \leq \mathcal{M}_\Omega u(x)$, $k = 1, 2, \dots$, for every $x \in \Omega$.

The rest of the proof goes along the lines of the final part of the proof for Lemma 2.2. By the Hardy-Littlewood-Wiener theorem we obtain

$$\begin{aligned} \|v_k\|_{1,p,\Omega} &= \|v_k\|_{p,\Omega} + \|Dv_k\|_{p,\Omega} \\ &\leq \|\mathcal{M}_\Omega u\|_{p,\Omega} + 2\|\mathcal{M}_\Omega |Du|\|_{p,\Omega} \leq c(n,p)\|u\|_{1,p,\Omega}. \end{aligned}$$

Hence $\{v_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$ such that $v_k \rightarrow \mathcal{M}_\Omega u$ almost everywhere in Ω as $k \rightarrow \infty$. A weak compactness argument shows that $\mathcal{M}_\Omega u \in W^{1,p}(\Omega)$, $v_k \rightarrow \mathcal{M}_\Omega u$ and $Dv_k \rightarrow D\mathcal{M}_\Omega u$ weakly in $L^p(\Omega)$ as $k \rightarrow \infty$. Again we may proceed to the weak limit in (3.1), using the proposition in the end of the previous section.

Let us then briefly consider the case $p = \infty$. Again using the above argument it is easy to see that $\mathcal{M}_\Omega u \in W_{\text{loc}}^{1,p}(\Omega)$ and the claim follows from the Hardy-Littlewood-Wiener theorem.

4. BOUNDARY VALUES OF THE MAXIMAL FUNCTION

We have shown that the local Hardy-Littlewood maximal operator preseverves the Sobolev spaces $W^{1,p}(\Omega)$ provided $1 < p \leq \infty$. In this section we show that the maximal operator also preserves the zero boundary values in Sobolev's sense.

4.1. Corollary. *Suppose that $u \in W_0^{1,p}(\Omega)$ for some p with $1 < p < \infty$, then $\mathcal{M}_\Omega u \in W_0^{1,p}(\Omega)$.*

Proof. To see this, let $\{\varphi_j\}$ be a sequence of functions in $C_0^\infty(\Omega)$ such that $\varphi_j \rightarrow u$ in $W^{1,p}(\Omega)$ as $j \rightarrow \infty$. Using Theorem 1.3 we see that $\mathcal{M}_\Omega \varphi_j \in W^{1,p}(\Omega)$, $j = 1, 2, \dots$. Moreover, it is easy to see that $\mathcal{M}_\Omega \varphi_j(x) = 0$ whenever $\text{dist}(x, \partial\Omega) < 1/2 \text{dist}(\text{supp } \varphi_j, \partial\Omega)$. This implies that $\mathcal{M}_\Omega \varphi_j \in W_0^{1,p}(\Omega)$, $j = 1, 2, \dots$. The sublinearity of the maximal operator yields

$$|\mathcal{M}_\Omega \varphi_j(x) - \mathcal{M}_\Omega u(x)| \leq \mathcal{M}_\Omega(\varphi_j - u)(x)$$

for every $x \in \Omega$ and hence by the Hardy-Littlewood-Wiener theorem

$$\|\mathcal{M}_\Omega \varphi_j - \mathcal{M}_\Omega u\|_{p,\Omega} \leq \|\mathcal{M}_\Omega(\varphi_j - u)\|_{p,\Omega} \leq c(n,p)\|\varphi_j - u\|_{p,\Omega}.$$

On the other hand, using (1.4) we obtain the derivative estimate

$$\|D\mathcal{M}_\Omega \varphi_j\|_{p,\Omega} \leq 2\|\mathcal{M}_\Omega |D\varphi_j|\|_{p,\Omega} \leq c(n,p)\|D\varphi_j\|_{p,\Omega},$$

which means that $\{\mathcal{M}_\Omega \varphi_j\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$ converging to u in $L^p(\Omega)$. Again, weak compactness implies $\mathcal{M}_\Omega u \in W_0^{1,p}(\Omega)$.

4.2. Remark. Suppose that $u \in W^{1,p}(\Omega)$ for some p with $1 < p < \infty$. A modification of our argument shows that $|u| - \mathcal{M}_\Omega u \in W_0^{1,p}(\Omega)$. In particular, the local maximal function preserves the boundary values of non-negative functions in Sobolev's sense.

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