STABILITY FOR DEGENERATE PARABOLIC EQUATIONS

JUHA KINNUNEN AND MIKKO PARVIAINEN

Abstract. We show that an initial and boundary value problem related to the parabolic p-Laplace equation is stable with respect to p if the complement of the cylindrical domain satisfies a uniform capacity density condition. This condition is essentially optimal for our stability results.

1. Introduction

We consider stability of weak solutions to

$$\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = \frac{\partial u}{\partial t} \tag{1.1}$$

in a cylindrical domain. The main question is that do the weak solutions of (1.1) with fixed initial and boundary values converge in any reasonable sense to the solution of the limit problem as p varies. Apart from mathematical interest, the stability questions is motivated by error analysis in applications: It is desirable that solutions remain stable under small perturbations of the measured parameter p.

Equation (1.1) is known as the *p-parabolic equation* or *parabolic p-Laplace equation*. Sometimes it is also called the *non-Newtonian filtration equation* which refers to the fact that the equation models the flow of non-Newtonian fluids. For the regularity theory we refer to DiBenedetto's monograph [4]. See also Chapter 2 of [22]. The equation is singular if $1 < p < 2$ and degenerate if $p \geq 2$. We shall focus on the degenerate case.

The stability turns out to be a rather delicate problem. The main obstruction is that the underlying parabolic Sobolev space changes as p varies and hence the associated energy is not necessarily finite. We give an example of this phenomenon when the lateral boundary of the cylinder is a Cantor type set. In this case, it may also happen that the solutions converge to a solution of a wrong limit problem. These phenomena are already present in the stationary case, see Kilpeläinen-Koskela [8] and Lindqvist [14], but the time dependence offers new challenges.

Our main result shows that solutions with varying exponent converge to the solution of the limit problem in the parabolic Sobolev

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space provided that the lateral boundary of the cylinder is sufficiently regular. A capacity density condition for the complement turns out to be a natural requirement in this context, but the problem is nontrivial already for domains with smooth boundaries. A global higher integrability result, stating that a weak solution belongs to a higher parabolic Sobolev space in the whole cylinder, plays a decisive role in the proof. For this result, we refer to [10], [18], and [19].

Similar stability questions have been studied in the stationary case by Lindqvist in [13] and [14]. See also Li-Martio [12], and Zhikov [23, 24].

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2. Preliminaries

Let Ω be a bounded open set in $\mathbf{R}^n$ with $n \geq 2$ and let $p \geq 2$. The definitions are relevant also for $1 < p < \infty$ and unbounded Ω, but we will work under these more restrictive assumptions throughout the paper. As usual, $W^{1,p}(\Omega)$ denotes the Sobolev space of functions in $L^p(\Omega)$, whose distributional gradient belongs to $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)} .$$

The Sobolev space with zero boundary values, denoted by $W_0^{1,p}(\Omega)$, is a completion of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$.

For $T > 0$, let

$$\Omega_T = \Omega \times (0, T)$$

be a space-time cylinder. We denote the points of the cylinder by $z = (x, t)$ and, for short, we write $\mathrm{d}z = \mathrm{d}x\,\mathrm{d}t$. The parabolic Sobolev space $L^p(0, T; W^{1,p}(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty, \infty]$ such that for almost every $t \in (0, T)$, the function $x \mapsto u(x, t)$ belongs to $W^{1,p}(\Omega)$ and

$$\int_{\Omega_T} (|u|^p + |\nabla u|^p)\,\mathrm{d}z < \infty. \tag{2.1}$$

Analogously, the space $L^p(0, T; W_0^{1,p}(\Omega))$ is a collection of measurable functions such that for almost every $t \in (0, T)$, the function $x \mapsto u(x, t)$ belongs to $W_0^{1,p}(\Omega)$ and (2.1) holds.

Solutions of the p-parabolic equation (1.1) are understood in a weak sense in the parabolic Sobolev space. We recall the definition here.

Definition 2.2. A function $u \in L^p(0, T; W^{1,p}(\Omega))$ is a weak solution to the p-parabolic equation, if

$$\int_{\Omega_T} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - u \frac{\partial \phi}{\partial t} \right) \mathrm{d}z = 0 \tag{2.3}$$

for every $\phi \in C_0^\infty(\Omega_T)$.

In the definition above, it is enough to assume that the function belongs locally to the parabolic Sobolev space, but we are mainly interested in global aspects of the theory in this paper. By parabolic regularity theory, a weak solution belongs for example to $C^{0,\alpha}_{\text{loc}}(\Omega_T)$ for some $\alpha \in (0,1)$, see DiBenedetto [4].

Observe that we do not assume the differentiability of a weak solution in the time direction. A recent observation of Lindqvist in [15] shows that a weak solution also has a weak time derivative, but we do not need this fact here. Instead, we use the traditional convolution

$$f_\sigma(x,t) = \int_{\mathbf{R}} f(x,t-s)\zeta_\sigma(s)\,\mathrm{d}s,$$

where $\zeta_\sigma(s)$ is a standard mollifier in time, whose support is contained in $(-\sigma,\sigma)$. By inserting ϕ_σ into (2.3), changing variables, and applying Fubini's theorem, we obtain

$$\int_{\Omega_T} \left((|\nabla u|^{p-2}\nabla u)_\sigma \cdot \nabla\phi - u_\sigma \frac{\partial \phi}{\partial t} \right) \mathrm{d}z = 0 \tag{2.4}$$

for σ small enough. This form of the equation will be useful for us later. The advantage is that the time derivative of u_σ exists and thus we may integrate by parts.

2.1. **A capacity density condition.** The *p-capacity* of a closed set $E \subset B_r(x)$ with respect to $B_r(x)$ is defined to be

$$\operatorname{cap}_p(E, B_r(x)) = \inf_u \int_{B_r(x)} |\nabla u|^p \,\mathrm{d}y,$$

where the infimum is taken over all the functions $u \in C_0^\infty(B_r(x))$ for which $u \geq 1$ in E. If $p > n$, even a singleton has positive p-capacity. When $p \leq n$, there is are well known connections to the Hausdorff measure: The Hausdorff dimension of a set of zero p-capacity does not exceed $n-p$. On the other hand, if the $(n-p)$-dimensional Hausdorff measure of E is finite, then E is of p-capacity zero. For more details, see [1], [5], and [7]. We shall return to this topic in Section 5.

The following uniform capacity density condition will be important for us. We need this condition for the existence of a solution to the initial and boundary value problem, for a higher integrability property of the gradient and to show that the limit solution takes the correct boundary values. In Section 5, we show that this condition is essentially optimal for our stability result.

Definition 2.5. The set $\mathbf{R}^n \setminus \Omega$ is *uniformly p-thick*, if there exist constants $\mu > 0$ and $r_0 > 0$ such that

$$\operatorname{cap}_p((\mathbf{R}^n \setminus \Omega) \cap \overline{B}_r(x), B_{2r}(x)) \geq \mu \operatorname{cap}_p(\overline{B}_r(x), B_{2r}(x)),$$

for all $x \in \mathbf{R}^n \setminus \Omega$ and $0 < r < r_0$.

Every nonempty $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick for $p > n$, and hence the uniform capacity density condition is nontrivial only when $p \leq n$.

If we replace the capacity with the Lebesgue measure in the definition above, we obtain a stronger measure density condition. Indeed, if the set $\mathbf{R}^n \setminus \Omega$ satisfies the measure density condition, then it is uniformly p-thick for all p. We note that the measure density condition is general enough for many practical purposes.

It is not difficult to see, that if $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick, then it is uniformly q-thick for every $q > p$ as well. The capacity density condition has a deep self-improving property, which is essential in stability questions. This result was shown by Lewis in [11], see also Ancona [2] and Section 8 of Mikkonen [17].

Theorem 2.6. *If $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick, then there exists $q = q(n, p, \mu)$ with $q < p$ for which $\mathbf{R}^n \setminus \Omega$ is uniformly q-thick.*

There is a subtle point related to the boundary values. As examples in Lindqvist [14] show, it may happen that $u \in W^{1,p}(\Omega)$ with $p \leq n$ and $u \in W_0^{1,p-\varepsilon}(\Omega)$ for every $\varepsilon > 0$, but u does not belong to $W_0^{1,p}(\Omega)$. We return to this in Section 5.2. For the next result, see the remark after Corollary 3.5 in Hedberg-Kilpeläinen [6].

Theorem 2.7. *If $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick, then there exists $\varepsilon > 0$ so that $W_0^{1,p-\varepsilon}(\Omega) \cap W^{1,p}(\Omega) = W_0^{1,p}(\Omega)$.*

Rather delicate Theorems 2.6 and 2.7 hold true with relatively easy proofs for smooth domains. We want to point out that our stability results are nontrivial already in this case.

2.2. **Initial and boundary values.** Recall, that throughout the paper we assume that $p \geq 2$ and that Ω is a bounded open set in $\mathbf{R}^n$. We use a Lebesgue-type initial condition and a Sobolev-type boundary condition on the lateral boundary. For expository purposes, we shall only consider initial and boundary values given by a function $\varphi \in C^1(\overline{\Omega}_T)$. The smoothness assumption on φ can be relaxed, but we leave such extensions to the interested reader.

We say that $u \in L^p(0, T; W^{1,p}(\Omega))$ is a weak solution to the p-parabolic equation with the initial and boundary values φ, if u satisfies (2.3),

$$\begin{aligned} &u(\cdot, t) - \varphi(\cdot, t) \in W_0^{1,p}(\Omega) \quad \text{for almost every} \quad t \in (0, T) \\ &\qquad\qquad\qquad \text{and,} \\ &\frac{1}{h} \int_0^h \int_\Omega |u - \varphi|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0 \quad \text{as} \quad h \to 0. \end{aligned} \tag{2.8}$$

For the following existence result, we refer to Theorem 6.5 in Kilpeläinen-Lindqvist [9]. See also Showalter [20].

Theorem 2.9. *Let Ω is a bounded open set in $\mathbf{R}^n$ such that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick. If $\varphi \in C^1(\overline{\Omega}_T)$ is the initial and boundary value function, then there exists a unique solution to the p-parabolic equation with the initial and boundary values φ in the sense of* (2.8).

Moreover, by parabolic regularity theory, $u \in C^{0,\alpha}_{\text{loc}}(\overline{\Omega_T})$. Thus the solution assumes the initial and boundary values also in the pointwise sense.

2.3. **A global higher integrability result.** Next we state the global higher integrability result for the gradients from [18]. The corresponding local result has been previously studied in [10]. These results show that weak solutions belong to a higher Sobolev space than implied by the existence result. This is an important ingredient in the proof of the stability result.

Theorem 2.10. *Let $p \geq 2$ and suppose that Ω is a bounded open set in $\mathbf{R}^n$ such that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick. In addition, assume that φ is an initial and boundary value function as in* (2.8)*. Let u be a solution to the p-parabolic equation with the initial and boundary values φ in Ω_T. Then there exists $\varepsilon = \varepsilon(n, p, r_0, \mu)$ such that $u \in L^{p+\varepsilon}(0, T; W^{1,p+\varepsilon}(\Omega))$.*

This higher integrability result comes with estimates in [10] and [18]. In general, the estimates depend, for example, on the initial and boundary values and constants in the uniform capacity density condition. A careful analysis of the constants in the argument shows that if p varies in a compact subinterval of $[2, \infty)$, then there exists a uniform positive lower bound for $\varepsilon > 0$, which will later be essential.

2.4. **The stability problem.** Let $p_i \geq 2$, $i = 1, 2, \ldots$, and assume that $p_i \to p$ as $i \to \infty$. Suppose that Ω is a bounded open set in $\mathbf{R}^n$ such that $\mathbf{R}^n \setminus \Omega$ is p-thick and let $T > 0$. By Theorem 2.6, the set $\mathbf{R}^n \setminus \Omega$ is p_i-thick for i large enough and by Theorem 2.9 there exists a unique solution

$$u_i \in L^{p_i}(0, T; W^{1,p_i}(\Omega))$$

to the p_i-parabolic equation in Ω_T with the initial and boundary values $\varphi \in C^1(\overline{\Omega}_T)$, that is, u_i satisfies (2.8) with $W_0^{1,p}(\Omega)$ replaced by $W_0^{1,p_i}(\Omega)$.

Note carefully, that the parabolic Sobolev spaces vary but the set Ω_T as well as the initial and boundary function φ are fixed. We are concerned with the question whether the solutions u_i converge in any reasonable sense to the solution of the limit problem with the exponent p. The uniform estimate given by Theorem 2.10 will be crucial for us since it gives us a uniform function space $L^{p+\varepsilon}(0, T; W^{1,p+\varepsilon}(\Omega))$ for some small $\varepsilon > 0$, which applies to all large enough i.

3. Convergence results

We begin with proving a Caccioppoli type estimate. It will be essential that the constants in the estimates are independent of i, because we need uniform bounds to obtain a convergence. Observe that the time derivative of φ appears in the estimate.

Lemma 3.1. *Let p_i, u_i, Ω_T, and φ be as in Section 2.4. Then there exists a constant $c > 0$, independent of i, such that*

$$\int_{\Omega_T} |\nabla u_i|^{p_i} \, \mathrm{d}z \le c \int_{\Omega_T} \left| \frac{\partial \varphi}{\partial t} \right|^{p_i/(p_i-1)} \mathrm{d}z + c \int_{\Omega_T} |\nabla \varphi|^{p_i} \, \mathrm{d}z + \varepsilon \int_{\Omega_T} |u_i - \varphi|^{p_i} \, \mathrm{d}z,$$

where $\varepsilon > 0$ and $c = c(\varepsilon, p_i)$. Moreover,

$$\sup_i c(\varepsilon, p_i) < \infty$$

for each $0 < \varepsilon < 1$.

Proof. Let $T > 0$ and $0 < 4h < T$. We define a cutoff function $\chi^h_{0,T}$ as

$$\chi^h_{0,T}(t) = \begin{cases} 0, & t \le h, \\ (t-h)/h, & h < t < 2h, \\ 1, & 2h < t < T - 2h, \\ (T-h-t)/h, & T - 2h < t < T - h, \\ 0, & t \ge T - h. \end{cases}$$

This function is piecewise linear, continuous, and compactly supported on $(0, T)$.

We insert formally the test function

$$\phi(x,t) = (u_i(x,t) - \varphi(x,t))\chi^h_{0,T}(t)$$

in (2.3) and have

$$\int_{\Omega_T} u_i \frac{\partial((u_i - \varphi)\chi^h_{0,T})}{\partial t} \, \mathrm{d}z = \int_{\Omega_T} \left(|\nabla u_i|^{p_i} - |\nabla u_i|^{p_i-2} \nabla u_i \cdot \nabla \varphi \right) \chi^h_{0,T} \, \mathrm{d}z.$$

Stricly speaking, the function ϕ is not an admissible test function, but a standard smoothing argument applies here, see (2.4). Next we estimate the second term on the right hand side by Young's inequality and get

$$\int_{\Omega_T} |\nabla u_i|^{p_i} \chi^h_{0,T} \, \mathrm{d}z \le c \int_{\Omega_T} u_i \frac{\partial((u_i - \varphi)\chi^h_{0,T})}{\partial t} \, \mathrm{d}z + c \int_{\Omega_T} |\nabla \varphi|^{p_i} \chi^h_{0,T} \, \mathrm{d}z.$$

Notice that we also absorbed a term from the right hand side into the left side by using Young's ε-inequality. Since $p_i \to p$ as $i \to \infty$, we have $2 \le \sup_i p_i < \infty$ and a careful examination of the constants in Young's inequality shows that the constant in the previous estimate can be chosen to be independent of i.

The final estimate should be free of the time derivatives of u_i. Therefore, we add and subtract φ, integrate by parts, and end up with

$$\begin{aligned}
&\int_{\Omega_T} u_i \frac{\partial((u_i-\varphi)\chi_{0,T}^h)}{\partial t}\,\mathrm{d}z \\
&= \frac{1}{2}\int_{\Omega_T}(u_i-\varphi)^2\frac{\partial \chi_{0,T}^h}{\partial t}\,\mathrm{d}z - \int_{\Omega_T}\frac{\partial\varphi}{\partial t}(u_i-\varphi)\chi_{0,T}^h\,\mathrm{d}z \\
&= -\frac{1}{h}\int_{T-2h}^{T-h}\int_\Omega (u_i-\varphi)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{h}\int_h^{2h}\int_\Omega (u_i-\varphi)^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\Omega_T}\frac{\partial\varphi}{\partial t}(u_i-\varphi)\chi_{0,T}^h\,\mathrm{d}z.
\end{aligned}$$

The first term on the right hand side is nonpositive and the second term tends to zero by the initial condition as $h \to 0$. By letting $h \to 0$, we conclude that

$$\int_{\Omega_T}|\nabla u_i|^{p_i}\,\mathrm{d}z \le c\left|\int_{\Omega_T}\frac{\partial\varphi}{\partial t}(u_i-\varphi)\,\mathrm{d}z\right| + c\int_{\Omega_T}|\nabla\varphi|^{p_i}\,\mathrm{d}z$$

and the claim follows from Young's ε-inequality. An examination of the constants in Young's inequality shows that $\sup_i c(\varepsilon, p_i) < \infty$ for each $0 < \varepsilon < 1$. □

The Caccioppoli type inequality in Lemma 3.1 gives us the following uniform integrability estimate.

Corollary 3.2. *Let p_i, u_i, Ω_T and φ be as in Section 2.4. Then*

$$\sup_i\left(\int_{\Omega_T}|u_i|^{p_i}\,\mathrm{d}z + \int_{\Omega_T}|\nabla u_i|^{p_i}\,\mathrm{d}z\right) < \infty.$$

Proof. We estimate the right hand side of the Caccioppoli inequality in Lemma 3.1 with a Sobolev type inequality. To this end, for every $t \in (0,T)$ we extend $u(\cdot,t) - \varphi(\cdot,t)$ by zero to $\mathbf{R}^n \setminus \Omega$. The standard Sobolev type inequality implies

$$\varepsilon\int_{\Omega_T}|u_i-\varphi|^{p_i}\,\mathrm{d}z \le \varepsilon c\,\mathrm{diam}(\Omega)^{p_i}\int_{\Omega_T}|\nabla u_i - \nabla\varphi|^{p_i}\,\mathrm{d}z.$$

According to Lemma 3.1, we end up with

$$\begin{aligned}\int_{\Omega_T} |\nabla u_i|^{p_i}\,\mathrm{d}z \\ &\le c\,\mathrm{diam}(\Omega)^{p_i}\left(\varepsilon\int_{\Omega_T}|\nabla u_i|^{p_i}\,\mathrm{d}z+\varepsilon\int_{\Omega_T}|\nabla\varphi|^{p_i}\,\mathrm{d}z\right)\\ &\quad+c\int_{\Omega_T}\left|\frac{\partial\varphi}{\partial t}\right|^{p_i/(p_i-1)}\mathrm{d}z+c\int_{\Omega_T}|\nabla\varphi|^{p_i}\,\mathrm{d}z.\end{aligned}$$

Then we choose $\varepsilon > 0$ small enough and absorb the first term on the right hand side into the left. This proves the gradient estimate.

The estimate for the solution follows from the gradient estimate by using a Sobolev type inequality again. We have

$$\begin{aligned}\int_{\Omega_T}|u_i|^{p_i}\,\mathrm{d}z &\le c\int_{\Omega_T}|u_i-\varphi|^{p_i}\,\mathrm{d}z+c\int_{\Omega_T}|\varphi|^{p_i}\,\mathrm{d}z\\ &\le c\,\mathrm{diam}(\Omega)^{p_i}\int_{\Omega_T}|\nabla u_i-\nabla\varphi|^{p_i}\,\mathrm{d}z+c\int_{\Omega_T}|\varphi|^{p_i}\,\mathrm{d}z,\end{aligned}$$

from which the claim follows. □

The uniform bound given by Corollary 3.2 together with the global higher integrability result in Theorem 2.10 gives us the following preliminary convergence result.

Lemma 3.3. *Let p_i, u_i, Ω_T and φ be as in Section 2.4. There exist $\varepsilon > 0$, a subsequence of (u_i) and a function $u \in L^{p+\varepsilon}(0,T;W^{1,p+\varepsilon}(\Omega))$ such that*

$$u_i \to u \quad \textit{in} \quad L^{p+\varepsilon}(\Omega_T)$$

and

$$\nabla u_i \to \nabla u \quad \textit{weakly in} \quad L^{p+\varepsilon}(\Omega_T),$$

as $i \to \infty$.

Proof. First, we show that there exists $\varepsilon > 0$ such that $u_i, \nabla u_i \in L^{p+\varepsilon}(\Omega_T)$ for i large enough. This follows from the remark after Theorem 2.10 using the uniform estimate given by Corollary 3.2. Indeed, since $p_i \to p$ as $i \to \infty$, there exists a uniform positive lower bound for ε in Theorem 2.10 for i large enough. Hence

$$\sup_i \int_{\Omega_T} |\nabla u_i|^{p+\varepsilon}\,\mathrm{d}z < \infty,$$

when i is large enough. The fact that

$$\sup_i \int_{\Omega_T} |u_i|^{p+\varepsilon}\,\mathrm{d}z < \infty,$$

when i is large enough, follows by a Sobolev inequality in the same way as in the end of the proof of Corollary 3.2.

Then we consider the convergence. The discussion above implies that subsequences of (u_i) and (∇u_i) are bounded in $L^{p+\varepsilon}(\Omega_T)$. Hence there exist a subsequence, denoted again by (u_i), and a function $u \in L^{p+\varepsilon}(\Omega_T)$ with $\nabla u \in L^{p+\varepsilon}(\Omega_T)$ such that $u_i \to u$ and $\nabla u_i \to \nabla u$ weakly in $L^{p+\varepsilon}(\Omega_T)$ as $i \to \infty$.

The strong convergence of (u_i) in $L^{p+\varepsilon}(\Omega_T)$ is a consequence of the Rellich-Kondrachov compactness result for Sobolev spaces and Theorem 5 in [21]. See also page 106 in [20]. Here we also need the estimate

$$
\begin{aligned}
&\left| \int_{\Omega_T} \frac{\partial \phi}{\partial t} u_i \,\mathrm{d}z \right| \\
&\qquad \le c \left(\int_{\Omega_T} |\nabla u_i|^{p_i} \,\mathrm{d}z \right)^{(p_i-1)/p_i} \left(\int_{\Omega_T} |\nabla \phi|^{p+\varepsilon} \,\mathrm{d}z \right)^{1/(p+\varepsilon)} \qquad (3.4) \\
&\qquad \le c \, ||\phi||_{L^{p+\varepsilon}(0,T;W_0^{1,p+\varepsilon}(\Omega))} ,
\end{aligned}
$$

where we used (2.3), Hölder's inequality and Corollary 3.2. □

Next we pass to a stronger convergence. The main problem in the proof is to show that, after passing to subsequence if necessary, the gradients also converge strongly in $L^{p+\varepsilon}(\Omega_T)$. The general idea in the proof is to show that the gradients form a Cauchy sequence. Thus we avoid testing the equation with the limit itself, which is not known to be a solution at this point. There are three steps in our argument: First, we carefully show that the test function is admissible, and, in particular, that it has zero boundary values in the right Sobolev space. Second, we show that gradients converge in $L^p(\Omega_T)$ by employing the equation together with elementary inequalities, and, third, we extend the convergence result to $L^{p+\varepsilon}(\Omega_T)$ by using the global higher integrability of the gradient.

The double limit procedure with respect to a sequence of solutions and the regularization parameter is delicate. Therefore, we carefully write down the regularizations.

Theorem 3.5. *Let p_i, u_i, Ω_T and φ be as in Section 2.4. Then there exist $\varepsilon > 0$, a subsequence (u_i) and a function $u \in L^{p+\varepsilon}(0, T; W^{1,p+\varepsilon}(\Omega))$ such that*

$$
u_i \to u \quad \textit{in} \quad L^{p+\varepsilon}(0, T; W^{1,p+\varepsilon}(\Omega)),
$$

as $i \to \infty$.

Proof. By passing to a subsequence, if necessary, Lemma 3.3 provides the weak convergence in $L^{p+\varepsilon}(\Omega_T)$ for the gradients and convergence in $L^{p+\varepsilon}(\Omega_T)$ for the solutions, when $\varepsilon > 0$ is small enough. Therefore we can focus our attention to the convergence of the gradients in $L^{p+\varepsilon}(\Omega_T)$.

To establish this, let u_j and u_k be two solutions in the sequence. Since both u_j and u_k satisfy the mollified equation (2.4), by subtracting, we obtain

$$
\begin{aligned}
-\int_{\Omega_T} &(u_j - u_k)_\sigma \frac{\partial \phi}{\partial t}\,\mathrm{d}z \\
&+ \int_{\Omega_T} (|\nabla u_j|^{p_j-2}\nabla u_j - |\nabla u_k|^{p_k-2}\nabla u_k)_\sigma \cdot \nabla\phi\,\mathrm{d}z = 0
\end{aligned}
\tag{3.6}
$$

for every $\phi \in C_0^\infty(\Omega_T)$. We would like to apply the test function

$$\phi(x,t) = \chi^h_{0,T}(t)(u_j(x,t) - u_k(x,t))_\sigma,$$

where $\chi^h_{0,T}(t)$ is the same cutoff function as in the proof of Lemma 3.1.

Step 1: It is not immediately clear that ϕ is an admissible test function in (3.6). There are two problems: The first technical problem is that the cutoff function is not smooth, but the standard smoothing argument as in (2.4) takes care of this. The second problem about the lateral boundary values is more serious. Indeed, we have to assure that the test function takes the zero boundary values in the right Sobolev space. By the boundary value condition, see Section 2.4, we have

$$u_j(\cdot,t) - \varphi(\cdot,t) \in W_0^{1,p_j}(\Omega) \quad \text{and} \quad u_k(\cdot,t) - \varphi(\cdot,t) \in W_0^{1,p_k}(\Omega)$$

for almost every $t \in (0,T)$. Observe, that the Sobolev spaces with zero boundary values depend on the parameter. Let $\varepsilon' > 0$. By choosing j and k large enough, we see that

$$u_j(\cdot,t) - u_k(\cdot,t) \in W_0^{1,p-\varepsilon'}(\Omega)$$

and by Lemma 3.3 we have

$$u_j(\cdot,t) - u_k(\cdot,t) \in W^{1,p+\varepsilon}(\Omega)$$

for almost every $t \in (0,T)$. According to Theorem 2.7, there exists $\varepsilon > 0$ such that

$$u_j(\cdot,t) - u_k(\cdot,t) \in W_0^{1,p+\varepsilon}(\Omega)$$

for almost every $t \in (0,T)$, when j and k are large enough. Note that we apply Theorem 2.7 twice in our argument. First we choose $\varepsilon' > 0$ small enough to reach $W_0^{1,p}(\Omega)$ and then $\varepsilon > 0$ small enough to reach $W_0^{1,p+\varepsilon}(\Omega)$. Thus we may use ϕ as a test function in (3.6) when j and k are large enough.

Step 2: We estimate the first term on the left hand side of (3.6). A substitution of the test function and an integration by parts give

$$-\int_{\Omega_T} (u_j - u_k)_\sigma \frac{\partial(\chi^h_{0,T}(u_j - u_k)_\sigma)}{\partial t}\,\mathrm{d}z = -\frac{1}{2}\int_{\Omega_T} (u_j - u_k)^2_\sigma \frac{\partial \chi^h_{0,T}}{\partial t}\,\mathrm{d}z.$$

This estimate is free of the time derivatives of the functions u_j and u_k, which is essential for us in the passage to the limit with σ. Now, letting

$\sigma \to 0$, we conclude that

$$\begin{aligned}\int_{\Omega_T} &(|\nabla u_j|^{p_j-2}\nabla u_j - |\nabla u_k|^{p_k-2}\nabla u_k)\cdot(\nabla u_j - \nabla u_k)\chi^h_{0,T}\,\mathrm{d}z\\ &\le -\frac{1}{h}\int_{T-2h}^{T-h}\int_\Omega (u_j-u_k)^2\,\mathrm{d}z + \frac{1}{h}\int_h^{2h}\int_\Omega (u_j-u_k)^2\,\mathrm{d}z.\end{aligned}$$

Observe that the first term on the right hand side is nonpositive. Moreover, the second term on the right hand side tends to zero by the initial condition as $h \to 0$. Thus

$$\int_{\Omega_T} (|\nabla u_j|^{p_j-2}\nabla u_j - |\nabla u_k|^{p_k-2}\nabla u_k)\cdot(\nabla u_j - \nabla u_k)\,\mathrm{d}z \le 0. \tag{3.7}$$

We divide the left hand side in three parts as

$$\begin{aligned}\int_{\Omega_T} &(|\nabla u_j|^{p_j-2}\nabla u_j - |\nabla u_k|^{p_k-2}\nabla u_k)\cdot(\nabla u_j - \nabla u_k)\,\mathrm{d}z\\ &= \int_{\Omega_T} (|\nabla u_j|^{p-2}\nabla u_j - |\nabla u_k|^{p-2}\nabla u_k)\cdot(\nabla u_j - \nabla u_k)\,\mathrm{d}z\\ &\quad + \int_{\Omega_T} (|\nabla u_j|^{p_j-2} - |\nabla u_j|^{p-2})\nabla u_j\cdot(\nabla u_j - \nabla u_k)\,\mathrm{d}z\\ &\quad + \int_{\Omega_T} (|\nabla u_k|^{p-2} - |\nabla u_k|^{p_k-2})\nabla u_k\cdot(\nabla u_j - \nabla u_k)\,\mathrm{d}z\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{3.8}$$

First, we concentrate on I_2 and I_3. A straightforward calculation shows that

$$\begin{aligned}\big|\,|\zeta|^a - |\zeta|^b\,\big| &= |\exp(a\log|\zeta|) - \exp(b\log(|\zeta|)|\\ &\le \max_{s\in[a,b]}\left|\frac{\partial \exp(s\log|\zeta|)}{\partial s}\right||a-b|\\ &\le |\log|\zeta||\,(|\zeta|^a + |\zeta|^b)\,|a-b|\,,\end{aligned}$$

where $\zeta \in \mathbf{R}^n$ and $a,\, b \ge 0$. If $|\zeta| \ge 1$, then

$$|\log|\zeta||\,(|\zeta|^a + |\zeta|^b) \le \frac{1}{\varepsilon}\,|\zeta|^{\max(a,b)+\varepsilon}\,,$$

and if $|\zeta| \le 1$, then

$$|\log|\zeta||\,(|\zeta|^a + |\zeta|^b) \le \frac{1}{e}\left(\frac{1}{a}+\frac{1}{b}\right).$$

This leads to

$$||\zeta|^a - |\zeta|^b| \le \left(\frac{1}{\varepsilon}\,|\zeta|^{\max(a,b)+\varepsilon} + \frac{1}{e}\left(\frac{1}{a}+\frac{1}{b}\right)\right)|a-b| \tag{3.9}$$

for every $\zeta \in \mathbf{R}^n$ and $a, \ge 0$.

Next we apply (3.9) with $\zeta = \nabla u_j$, $a = p_j - 2$, and $b = p - 2$. This implies

$$|I_2| \le c\,|p_j - p| \int_{\Omega_T} (1 + |\nabla u_j|^{\max(p_j-2,p-2)+\varepsilon})\,|\nabla u_j|\,|\nabla u_j - \nabla u_k| \,\mathrm{d}z.$$

In the same way as in the proof of Lemma 3.3, we deduce from Corollary 3.2 and Theorem 2.10 that the integral on the right hand side is uniformly bounded. Consequently, $I_2 \to 0$, as $j \to \infty$. A similar reasoning implies that I_3 tends to zero as $k \to \infty$. From the elementary inequality

$$2^{2-p}\,|a-b|^p \le (|a|^{p-2}\,a - |b|^{p-2}\,b)\cdot(a-b),$$

we conclude that

$$\int_{\Omega_T} |\nabla u_j - \nabla u_k|^p \,\mathrm{d}z \le cI_1.$$

This fact together with (3.7) and (3.8) implies

$$\int_{\Omega_T} |\nabla u_j - \nabla u_k|^p \,\mathrm{d}z \le c(|I_2| + |I_3|).$$

The right hand side can be made arbitrary small by choosing j and k large enough. This shows that (∇u_i) is a Cauchy sequence in $L^p(\Omega_T)$, and thus it converges. Since $u_i \to u$ in $L^p(\Omega_T)$, we conclude that $\nabla u_i \to \nabla u$ in $L^p(\Omega_T)$ as $i \to \infty$.

Step 3: We complete the proof by showing that (∇u_i) converges in $L^{p+\varepsilon}(\Omega_T)$ for some $\varepsilon > 0$. The proof of this fact is rather standard. First we remark that as (∇u_i) converges in $L^p(\Omega_T)$, there exists a subsequence that converges almost everywhere in Ω_T. According to Lemma 3.3, there exists $M < \infty$ such that

$$\sup_i \int_{\Omega_T} |\nabla u_i|^{p+\varepsilon} \,\mathrm{d}z \le M.$$

This implies that $\nabla u_i \to \nabla u$ in $L^q(\Omega_T)$ whenever $q \in [p, p+\varepsilon)$. Indeed, we have

$$\begin{aligned}\int_{\Omega_T} |\nabla u - \nabla u_i|^q \,\mathrm{d}z = &\int_{\{|\nabla u_i - \nabla u| > k\}} |\nabla u - \nabla u_i|^q \,\mathrm{d}z \\ &+ \int_{\{|\nabla u_i - \nabla u| \le k\}} |\nabla u - \nabla u_i|^q \,\mathrm{d}z\end{aligned}$$

for every $k = 1, 2, \dots$ By the Cavalieri principle, the first term on the right hand side reads

$$\begin{aligned}\int_{\{|\nabla u_i - \nabla u| > k\}} |\nabla u - \nabla u_i|^q \,\mathrm{d}z = &\, q\int_k^\infty \lambda^{q-1}\,|\{|\nabla u_i - \nabla u| > \lambda\}| \,\mathrm{d}\lambda \\ &+ k^q\,|\{|\nabla u_i - \nabla u| > k\}|\,.\end{aligned}$$

A standard weak type estimate implies

$$|\{|\nabla u_i - \nabla u| > \lambda\}| \le \lambda^{-(p+\varepsilon)} \int_{\Omega_T} |\nabla u_i - \nabla u|^{p+\varepsilon} \, \mathrm{d}z$$
$$\le 2M\lambda^{-(p+\varepsilon)}$$

and

$$|\{|\nabla u_i - \nabla u| > k\}| \le 2Mk^{-(p+\varepsilon)}.$$

Consequently, as $q < p + \varepsilon$, we deduce

$$\int_{\{|\nabla u_i - \nabla u| > k\}} |\nabla u - \nabla u_i|^q \, \mathrm{d}z \le \frac{p+\varepsilon}{p+\varepsilon-q} 2Mk^{q-(p+\varepsilon)}.$$

This quantity can be made as small as we please by choosing k large enough.

On the other hand, since

$$\chi_{\{|\nabla u_i - \nabla u| \le k\}} |\nabla u - \nabla u_i|^q \le k^q$$

and Ω_T is bounded, by Lebesgue's dominated convergence theorem, we may choose i large enough, so that

$$\int_{\{|\nabla u_i - \nabla u| \le k\}} |\nabla u - \nabla u_i|^q \, \mathrm{d}z$$

is as small as we please. This completes the proof. □

4. Properties of the limit function

In this section, we study the properties of the limit function u constructed in the previous section. It immediately follows from the strong convergence that the function is a weak solution to the limiting equation.

Theorem 4.1. *The limit function u given by Theorem 3.5 is a weak solution to the p-parabolic equation in Ω_T.*

Proof. By Theorem 3.5, $u_i \to u$ in $L^{p+\varepsilon}(0, T; W^{1,p+\varepsilon}(\Omega))$ as $i \to \infty$. By Hölder's inequality, we have

$$\int_{\Omega_T} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - u \frac{\partial \phi}{\partial t} \right) \mathrm{d}z$$
$$= \lim_{i \to \infty} \int_{\Omega_T} \left(|\nabla u_i|^{p_i-2} \nabla u_i \cdot \nabla \phi - u_i \frac{\partial \phi}{\partial t} \right) \mathrm{d}z = 0$$

for every $\phi \in C_0^\infty(\Omega_T)$. This proves the claim. □

Next we show that the limit function takes the correct boundary and initial values.

Theorem 4.2. *The limit function u given by Theorem 3.5 is a weak solution to the p-parabolic equation with the boundary and initial conditions* (2.8).

Proof. Since $u_i \to u$ in $L^p(0,T;W^{1,p}(\Omega))$ as $i \to \infty$ and

$$u_i(\cdot,t) - \varphi(\cdot,t) \in W_0^{1,p}(\Omega) \tag{4.3}$$

for almost every $t \in (0,T)$, for i large enough, it follows that u takes the correct lateral boundary values.

Next we show that u takes the correct initial values. To accomplish this, we pass to limits in a particular order and, therefore, define a two-parameter cutoff function by

$$\chi_{0,t_2}^{h,k}(t) = \begin{cases} 0, & t \le h, \\ (t-h)/h, & h < t < 2h, \\ 1, & 2h < t < t_2 - k, \\ (t_2 - t)/k, & t_2 - k < t < t_2, \\ 0, & t \ge t_2, \end{cases}$$

where $2h < t_2 - k$, $t_2 < T$, and $h, k > 0$. Insert the test function

$$\chi_{0,t_2}^{h,k}(u_i - \varphi)$$

into the definition of the weak solution. Similarly as in the proof of Lemma 3.1, we obtain

$$\begin{aligned} &\left| \frac{1}{k} \int_{t_2-k}^{t_2} \int_\Omega (u_i - \varphi)^2 \, dx \, \mathrm{d}t - \frac{1}{h} \int_h^{2h} \int_\Omega (u_i - \varphi)^2 \, dx \, \mathrm{d}t \right| \\ &\le c \int_0^{t_2} \int_\Omega |\nabla u_i|^{p_i} \, \mathrm{d}x \, \mathrm{d}t + c \int_0^{t_2} \int_\Omega |u_i - \varphi|^{p_i} \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + c \int_0^{t_2} \int_\Omega \left| \frac{\partial \varphi}{\partial t} \right|^{p_i/(p_i-1)} \mathrm{d}z + c \int_0^{t_2} \int_\Omega |\nabla \varphi|^{p_i} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.4}$$

By the initial condition (2.8), the second term on the left hand side tends to zero as $h \to 0$. The convergence of the sequence (u_i) in $L^p(\Omega_T)$ implies that the first term on the left hand side of (4.4) converges to

$$\frac{1}{k} \int_{t_2-k}^{t_2} \int_\Omega (u - \varphi)^2 \, \mathrm{d}x \, \mathrm{d}t$$

as $i \to \infty$. Estimate (3.4) and Proposition 1.2 in [20] imply that $u \in C((0,T);L^2(\Omega))$. From this it follows that, for every $t_2 \in (0,T)$, we have

$$\frac{1}{k} \int_{t_2-k}^{t_2} \int_\Omega (u - \varphi)^2 \, \mathrm{d}x \, \mathrm{d}t \to \int_\Omega |u(x,t_2) - \varphi(x,t_2)|^2 \, \mathrm{d}x$$

as $k \to 0$. Furthermore, the terms on the right hand side of (4.4) are uniformly bounded with respect to i due to Corollary 3.2. Combining the facts, we obtain

$$\int_\Omega |u(x,t_2) - \varphi(x,t_2)|^2 \, dx \to 0,$$

as $t_2 \to 0$. This proves that u satisfies the initial condition in (2.8).

The solution to the p-parabolic equation with the boundary and initial conditions (2.8) is unique, and thus subsequences in Theorems 3.5 and 4.2 converge to this unique limit function u. Since every subsequence of the original sequence contains such a converging subsequence, it follows that also the original sequence (u_i) converges to u. □

Remark 4.5. By parabolic regularity theory u_i, $i = 1, 2, \dots$, belong to $C^{1,\alpha}_{\text{loc}}(\Omega_T)$, see [4]. A thorough analysis of constants in the regularity theory shows that the sequences (u_i) and (∇u_i) are locally equicontinuous and, in addition, locally uniformly bounded. Hence by Ascoli's theorem, there exists a subsequence such that $u_i \to u$ and $\nabla u_i \to \nabla u$ locally uniformly in Ω_T as $i \to \infty$. However, we do not need this improvement here.

5. Optimality

Next we show that the uniform capacity density condition is essentially the weakest possible condition for our stability results. Indeed, it may happen that we do not have the higher integrability property of the gradient or that the sequence converges to a wrong solution if the complement does not satisfy the uniform capacity density condition.

5.1. **The failure of higher integrability.** We modify the elliptic example given in Remark 3.3 of [8] to show that a weakening of the capacity density condition implies the existence of a weak solution to the p-parabolic equation such that

$$\int_{\Omega_T} |\nabla u|^{p+\varepsilon} \, \mathrm{d}z = \infty$$

for every $\varepsilon > 0$. This is in strict contrast with our Lemma 3.3.

The idea is first to remove a carefully chosen small set of a ball in $\mathbf{R}^n$. The removed set is thick in the Wiener sense but not uniformly p-thick. The solution for the p-parabolic equation will be a constant-in-time extension of the elliptic solution for p-Laplace equation in the remaining set. Now, the proof is based on the contradiction: If the integral above would be finite, then we could extend u as a solution to the whole space-time cylinder. But as the extension would have zero lateral boundary values, a comparison with the Barenblatt solution provides a contradiction.

To this end, let $2 \le p \le n$ and denote $B = B_1(0)$. Then there exists a compact Cantor type set $K \subset B$ such that

$$\operatorname{cap}_p(K, B) > 0$$

and that the Hausdorff dimension of K is $n - p$. In particular, this implies that

$$\operatorname{cap}_q(K, B) = 0$$

for every $q < p$. The set K cannot be uniformly p-thick, because this would contradict the self-improving property of the uniform density condition stated in Theorem 2.6. However, the set K can be constructed so that the Wiener integral diverges at every point, that is,

$$\int_0^1 \left(\frac{\operatorname{cap}_p(K \cap \overline{B}_r(x), B_{2r}(x))}{r^{n-p}} \right)^{1/(p-1)} \frac{\mathrm{d}r}{r} = \infty$$

for every $x \in K$. Such a set can be constructed using a Cantor type construction and the scaling properties for the capacity, see, for example, Adams-Hedberg [1] and also (ii) in Theorem 2 of Section 4.7 in Evans-Gariepy [5]. A similar set has also been considered in [3] and [8].

Set $\Omega = B \setminus K$ and let $u \in W^{1,p}(\Omega)$ be a weak solution to the elliptic p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

in Ω with the boundary values zero on ∂B and one on K. Since the Wiener integral diverges, every $x \in K$ is a regular point for the Dirichlet problem and hence $u(y) \to 1$ as $y \to x$ with $y \in \Omega$. On the other hand, every boundary point $x \in \partial B$ is regular as well, and hence $u(y) \to 0$ as $y \to x$ with $y \in \Omega$. In particular, this implies that $0 < u < 1$ in Ω and u is not identically zero.

Since the Lebesgue measure of K is zero, the function u can be extended to B so that the extension, still denoted by u, belongs to $W^{1,p}(B)$. Next we take the trivial extension in time and set $u(x,t) = u(x)$ for every $t \in (0,T)$. Clearly, the extended function u is a solution of the p-parabolic equation in Ω_T.

Striving for a contradiction, suppose that for some $\varepsilon > 0$ we have

$$\int_{\Omega_T} |\nabla u|^{p+\varepsilon} \, \mathrm{d}z < \infty. \tag{5.1}$$

We claim that, in this case, u is a weak solution to the p-parabolic equation in the whole of $B_T = B \times (0,T)$. Since $\operatorname{cap}_q(K,B) = 0$ for every $q < p$, we may choose a sequence $\phi_i \in C_0^\infty(B)$, $0 \le \phi_i \le 1$, $\phi_i = 1$ in K such that

$$\int_B |\nabla \phi_i|^q \, \mathrm{d}x \to 0 \tag{5.2}$$

as $i \to \infty$. Again, we take the trivial extension of ϕ_i in time. Let $\phi \in C_0^\infty(B_T)$ and observe that $(1-\phi_i)\phi \in C_0^\infty(\Omega_T)$. Since u does not depend on time, we have

$$\int_{B_T} u \frac{\partial \phi}{\partial t} \, \mathrm{d}z = 0.$$

Furthermore, since $u(\cdot,t)$ is a solution to the elliptic p-Laplace equation in Ω for every $t \in (0,T)$, we conclude that

$$\int_{\Omega_T} |\nabla u|^{p-2}\,\nabla u \cdot \nabla((1-\phi_i)\phi)\,\mathrm{d}z = 0$$

for every $i = 1,2,\ldots$ Consequently,

$$\begin{aligned}
-\int_{B_T} u\frac{\partial\phi}{\partial t}\,\mathrm{d}z &+ \int_{B_T} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\phi\,\mathrm{d}z = \int_{B_T} |\nabla u|^{p-2}\,\nabla u\cdot\nabla\phi\,\mathrm{d}z \\
&= \int_{\Omega_T} |\nabla u|^{p-2}\,\nabla u\cdot\nabla((1-\phi_i)\phi)\,\mathrm{d}z + \int_{B_T} |\nabla u|^{p-2}\,\nabla u\cdot\nabla(\phi_i\phi)\,\mathrm{d}z \\
&= \int_{B_T} |\nabla u|^{p-2}\,\nabla u\cdot\nabla(\phi_i\phi)\,\mathrm{d}z.
\end{aligned}$$

Next we show that the last integral equals zero and hence u is a solution to the p-parabolic equation in B_T. By Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_{B_T} |\nabla u|^{p-2}\,\nabla u\cdot\nabla(\phi_i\phi)\,\mathrm{d}z\right| \\
&= \left(\int_{B_T} |\nabla u|^{p+\varepsilon}\,\mathrm{d}z\right)^{(p-1)/(p+\varepsilon)} \left(\int_{B_T} |\nabla(\phi_i\phi)|^{(p+\varepsilon)/(1+\varepsilon)}\,\mathrm{d}z\right)^{(1+\varepsilon)/(p+\varepsilon)}.
\end{aligned}$$

The first term on the right hand side is finite by assumption (5.1). It follows by (5.2) that the right hand side tends to zero since $(p+\varepsilon)/(1+\varepsilon) < p$. Thus, u is a nonzero solution to parabolic p-Laplace equation in B_T.

Recall the Barenblatt solution $\mathcal{B}_p : \mathbf{R}^{n+1}_+ \to [0,\infty)$,

$$\mathcal{B}_p(x,t) = t^{-n/\lambda}\left(c - \frac{p-2}{p}\lambda^{1/(1-p)}\left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)},$$

where $\lambda = n(p-2)+p$, $p>2$, and $c>0$ is fixed below. By choosing c, $t_0>0$ large enough, we see that

$$\mathcal{B}_p(x,t_0+t) \ge u(x,t)$$

whenever (x,t) belongs to the parabolic boundary

$$(\overline{B}\times\{0\}) \cup (\partial B\times[0,T])$$

of B_T. By the comparison principle, we have

$$u(x,t) \le \mathcal{B}_p(x,t_0+t) \le c\,t^{-n/\lambda} \tag{5.3}$$

for every $t\in(0,T)$. This provides a contradiction since clearly u does not satisfy this decay rate. Thus (5.1) cannot be true.

5.2. **The wrong limit function.** Next we assume that $p_i \to p-$ as $i \to \infty$ and modify the elliptic example in Section 7 of Lindqvist [14] related to the theory of nonlinear eigenfunctions. Let K, B and Ω_T be as above. Choose $\phi \in C_0^\infty(B)$ with $\phi = 1$ in K. Again, we consider the trivial extension of ϕ in time. Let $u_i^{\Omega_T}$ and $u_i^{B_T}$ be solutions to the p_i-parabolic equation with boundary and initial values ϕ in Ω_T and B_T, respectively. Since $\operatorname{cap}_q(K, \Omega) = 0$ for every $q < p$, it follows that $u_i^{\Omega_T} = u_i^{B_T}$ in Ω_T for every $i = 1, 2, \dots$

Somewhat unexpectedly, the corresponding solutions u^{Ω_T} and u^{B_T} for the parabolic p-Laplace equation are different functions in Ω_T. Indeed, due to the decay estimate (5.3), the function u^{B_T} tends to zero as t grows, but clearly u^{Ω_T} does not. The cylinder B_T satisfies the uniform capacity density condition and thus $u_i^{B_T} \to u^{B_T}$ in $L^{p+\varepsilon}(0, T; W^{1,p+\varepsilon}(B))$ as $i \to \infty$ by our Theorem 3.5. But now we have

$$u_i^{\Omega_T} = u_i^{B_T} \to u^{B_T} \neq u^{\Omega_T}$$

in $L^{p+\varepsilon}(0, T; W^{1,p+\varepsilon}(B))$ as $i \to \infty$. This shows that Ω_T does not enjoy the stability property.

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J.K., Institute of Mathematics, P.O. Box 1100, FI-02015 Helsinki University of Technology, Finland
juha.kinnunen@tkk.fi

M.P., Institute of Mathematics, P.O. Box 1100, FI-02015 Helsinki University of Technology, Finland
mikko.parviainen@tkk.fi