6 Bézier Curves II

6.1 Computational Considerations. The de Casteljau Algorithm

The de Casteljau algorithm can be used to evaluate the point on a Bézier curve corresponding to a parameter value $t \in [0, 1]$. For a quadratic Bézier curve $\mathcal{C}(t)$ with control points $\mathbf{b}_0$, $\mathbf{b}_1$, and $\mathbf{b}_2$, and for a parameter value $t \in [0, 1]$, we may write

$$\begin{aligned}\mathcal{C}(t) &= (1-t)^2\mathbf{b}_0 + 2(1-t)t\mathbf{b}_1 + t^2\mathbf{b}_2 \\ &= (1-t)[(1-t)\mathbf{b}_0 + t\mathbf{b}_1] + t[(1-t)\mathbf{b}_1 + t\mathbf{b}_2]\end{aligned}$$

so that a point in the curve can be computed by two linear interpolations of the control polygon.

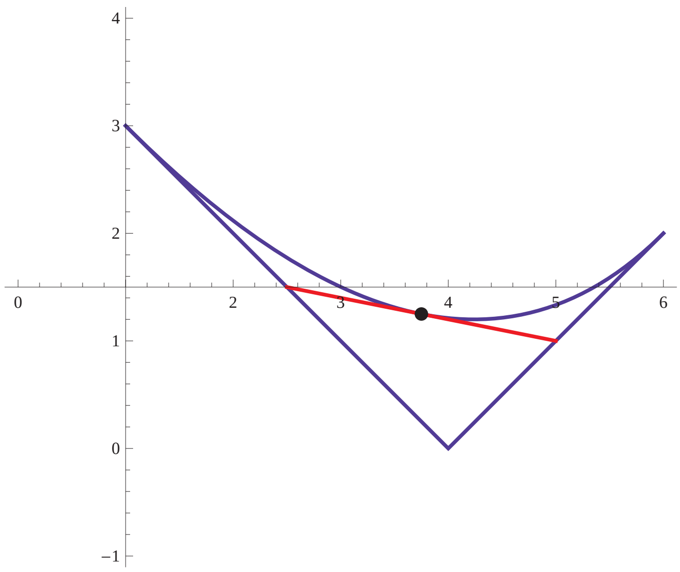

Figure 22: The de Casteljau algorithm for computing a point on quadratic Bézier curve.

For a cubic Bézier curve $\mathcal{C}(t)$ with the control points $\mathbf{b}_0$, $\mathbf{b}_1$,$\mathbf{b}_2$, and $\mathbf{b}_3$, we may write

$$\begin{aligned}\mathcal{C}(t) &= (1-t)^3\mathbf{b}_0 + 3(1-t)^2t\mathbf{b}_1 + 3(1-t)t^2\mathbf{b}_2 + t^3\mathbf{b}3 \\ &= (1-t)[(1-t)^2\mathbf{b}_0 + 2(1-t)t\mathbf{b}_1 + t^2\mathbf{b}_2] + t[(1-t)^2\mathbf{b}_1 + 2(1-t)t\mathbf{b}_2 + t^2\mathbf{b}3]\end{aligned}$$

so that the point $\mathcal{C}(t)$ can be obtained by linear interpolation of the points on the quadratic Bézier curves with the control points $\{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. These in turn can be obtained by the two linear interpolations of the control polygons. The general recursive algorithm is as follows. Let $\mathcal{C}(t)$ be a Bézier curve given by control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$, and let $t \in [0, 1]$ be any parameter value. Then $\mathcal{C}(t) = \mathbf{b}_0^n$, where $\mathbf{b}_i^0 = \mathbf{b}_i$ and

$$\mathbf{b}_i^j = \mathbf{b}_i^{j-1}(1-t) + \mathbf{b}_{i+1}^{j-1}t.$$

The formula generates a triangular set of values

$$
\begin{array}{cccccc}
\mathbf{b}_0^0 & \mathbf{b}_1^0 & \mathbf{b}_2^0 & \mathbf{b}_3^0 & \cdots & \mathbf{b}_n^0 \\
\mathbf{b}_0^1 & \mathbf{b}_1^1 & \mathbf{b}_2^1 & \cdots & \mathbf{b}_{n-1}^1 & \\
\vdots & \vdots & & & & \\
\mathbf{b}_0^{n-1} & \mathbf{b}_1^{n-1} & & & & \\
\mathbf{b}_0^n & & & & &
\end{array}
$$

By using the recursion property, we may write a Bézier curve of degree n as

$$
\begin{aligned}
\mathcal{C}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t) &= \sum_{i=0}^{n} \mathbf{b}_i [(1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)] \\
&= \sum_{i=0}^{n-1} \mathbf{b}_i (1-t) B_{i,n-1}(t) + \sum_{i=1}^{n} \mathbf{b}_i t B_{i-1,n-1}(t) \\
&= \sum_{i=0}^{n-1} \mathbf{b}_i (1-t) B_{i,n-1}(t) + \sum_{i=0}^{n-1} \mathbf{b}_{i+1} t B_{i,n-1}(t) \\
&= \sum_{i=0}^{n-1} [\mathbf{b}_i (1-t) + \mathbf{b}_{i+1} t] B_{i,n-1}(t) \\
&= \sum_{i=0}^{n-1} \mathbf{b}_i^1 B_{i,n-1}(t)
\end{aligned}
$$

where $\mathbf{b}_i^1 = \mathbf{b}_i(1-t) + \mathbf{b}_{i+1}t = \mathbf{b}_i^0(1-t) + \mathbf{b}_{i+1}^0 t$ for $i = 0, \ldots, n-1$. Application of the same argument to the Bézier curve of degree $n-1$

$$\mathcal{C}(t) = \sum_{i=0}^{n-1} \mathbf{b}_i^1 B_{i,n-1}(t)$$

yields

$$\mathcal{C}(t) = \sum_{i=0}^{n-2} \mathbf{b}_i^2 B_{i,n-2}(t)$$

where $\mathbf{b}_i^2 = \mathbf{b}_i^1(1-t) + \mathbf{b}_{i+1}^1 t$ for $i = 0, \ldots, n-2$. In general,

$$\mathcal{C}(t) = \sum_{i=0}^{n-j} \mathbf{b}_i^j B_{i,n-j}(t)$$

where $\mathbf{b}_i^j = \mathbf{b}_i^{j-1}(1-t) + \mathbf{b}_{i+1}^{j-1} t$ for $i = 0, \ldots, n-j$. Taking $j = n$ yields

$$\mathcal{C}(t) = \sum_{i=0}^{0} \mathbf{b}_i^j B_{i,n-n}(t) = \mathbf{b}_0^n.$$

6.2 Spatial Bézier Curves

A spatial Bézier curve

$$\mathcal{C}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t)$$

is obtained when the control points are specified in three-dimensional space. Spatial Bézier curves satisfy the properties of planar curves discussed in the previous sections:

- The end point interpolation and tangency conditions
- Invariance under affine transformations
- The convex hull property
- The variation diminishing property

The convex hull property implies that if the control points are coplanar then also the Bézier curve is contained in a plane. The variation diminishing property in the spatial case means that a plane intersects a Bézier curve less than or equal to the number of times that plane intersects the control polygon.

6.3 Derivatives of Bézier Curves

Derivatives of curves are needed to determine tangents and normals. Derivatives of Bézier curves are obtained by differentiating the Bernstein basis functions.

Example 6.1. The derivatives of the cubic Bernstein basis polynomials $B_{0,3} = (1-t)^3$, $B_{1,3} = 3(1-t)^2 t$, $B_{2,3} = 3(1-t)t^2$, and $B_{3,3} = t^3$ are

$$\begin{aligned}
B'_{0,3} &= -3(1-t)^2 = -3B_{0,2}(t) \\
B'_{1,3} &= 3(1-t)^2 - 6(1-t)t = 3B_{0,2}(t) - 3B_{1,2}(t) \\
B'_{2,3} &= -3t^2 + 6(1-t)t = 3B_{1,2}(t) - 3B_{2,2}(t) \\
B'_{3,3} &= 3t^2 = 3B_{2,2}(t)
\end{aligned}$$

Consequently, the derivative of a cubic Bézier curve

$$\mathcal{C}(t) = \sum_{i=0}^{3} \mathbf{b}_i B_{i,3}(t)$$

is

$$\begin{aligned}
\mathcal{C}(t) &= -3\mathbf{b}_0 B_{0,2}(t) + 3\mathbf{b}_1(B_{0,2}(t) - B_{1,2}(t)) + 3\mathbf{b}_2(B_{1,2}(t) - B_{2,2}(t)) + 3\mathbf{b}_3 B_{2,2}(t) \\
&= 3(\mathbf{b}_1 - \mathbf{b}_0)B_{0,2}(t) + 3(\mathbf{b}_2 - \mathbf{b}_1)B_{1,2}(t) + 3(\mathbf{b}_3 - \mathbf{b}_2)B_{2,2}(t).
\end{aligned}$$

The generalization of the above results to Bernstein polynomials and Bézier curves of degree n are expressed in the following theorems.

Theorem 4. *The derivatives of the Bernstein basis functions $B_{i,n}(t)$ of degree n satisfy the recursion formula*

$$B'_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t)).$$

Proof. The claim follows by differentiation:

$$\begin{aligned}
\frac{d}{dt}\frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i &= -(n-i)\frac{n!}{(n-i)!i!}(1-t)^{n-i-1}t^i + i\frac{n!}{(n-i)!i!}(1-t)^{n-i}t^{i-1} \\
&= -n\frac{(n-1)!}{(n-1-i)!i!}(1-t)^{n-1-i}t^i + n\frac{(n-1)!}{(n-i)!(i-1)!}(1-t)^{n-i}t^i \\
&= -nB_{i,n-1}(t) + nB_{i-1,n-1}(t).
\end{aligned}$$

□

Theorem 5. *The first derivative of a Bézier curve of degree n is*

$$\mathcal{C}'(t) = \sum_{i=0}^{n-1} \mathbf{b}_i^{(1)} B_{i,n-1}(t), \tag{23}$$

where $\mathbf{b}_i^{(1)} = n(\mathbf{b}_{i+1} - \mathbf{b}_i)$.

Proof. Because $B'_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t))$ and $B_{-1,n-1}(t) = B_{n,n-1}(t) = 0$, we have

$$\begin{aligned}
\mathcal{C}'(t) &= \sum_{i=0}^{n} \mathbf{b}_i B'_{i,n}(t) \\
&= \sum_{i=0}^{n} \mathbf{b}_i n(B_{i-1,n-1}(t) - B_{i,n-1}(t)) \\
&= \sum_{i=1}^{n} n\mathbf{b}_i B_{i-1,n-1}(t) - \sum_{i=0}^{n-1} n\mathbf{b}_i B_{i,n-1}(t)
\end{aligned}$$

Change of summation index in the first term yields

$$\mathcal{C}'(t) = \sum_{i=0}^{n-1} n\mathbf{b}_{i+1} B_{i,n-1}(t) - \sum_{i=0}^{n-1} n\mathbf{b}_i B_{i,n-1}(t) = \sum_{i=0}^{n-1} n(\mathbf{b}_{i+1} - \mathbf{b}_i) B_{i,n-1}(t).$$

□

Higher order derivatives of $\mathcal{C}(t)$ can be calculated by repeated application of the formula (23). For instance, the second derivative can be written as

$$\mathcal{C}''(t) = \sum_{i=0}^{n-2} \mathbf{b}_i^{(2)} B_{i,n-2}(t),$$

where

$$\mathbf{b}_i^{(2)} = (n-1)(\mathbf{b}_{i+1}^{(1)} - \mathbf{b}_i^{(1)}) = (n-1)n(\mathbf{b}_{i+2} - 2\mathbf{b}_{i+1} + \mathbf{b}_i).$$

Example 6.2. Consider a cubic Bézier curve defined by control points $(1, 1)$, $(3, 1)$, $(4, 2)$, and $(6, 3)$. The differences of the control points are

$$(3,1) - (1,1) = (2,0), \quad (4,2) - (3,1) = (1,1), \quad (6,3) - (4,2) = (2,1).$$

Multiplication by three yields the control points of the first derivative

$$\mathbf{b}_0^{(1)} = (6,0), \quad \mathbf{b}_1^{(1)} = (3,3), \quad \mathbf{b}_2^{(1)} = (6,3).$$

The derivative can be expressed as the quadratic Bézier curve

$$(6,0)(1-t)^2 + (3,3)2(1-t)t + (6,3)t^2.$$

To determine the control points of the second derivative we compute the differences

$$(3,3) - (6,0) = (-3,3), \quad (6,3) - (3,3) = (3,0)$$

and multiply by two to get $\mathbf{b}_0^{(2)} = (-6, 6)$ and $\mathbf{b}_1^{(2)} = (6, 0)$. The second derivative of the cubic Bézier curve can be expressed as the linear curve

$$(-6,6)(1-t) + (6,0)t.$$

To obtain the tangent vector at, for instance, $t = 0.5$, we make a substitution $t = 0.5$ in the first derivative and get

$$\mathcal{C}'(t) = (4.5, 2.25).$$

6.4 Piecewise Bézier Curves

As the degree of a Bézier curve increases, the relationship between the the shape of the curve and the control polygon becomes weaker. The evaluation of high-degree polynomials requires many floating point operations and is therefore prone to numerical round-off errors. Bézier curves of low degree are more reliable but feature a limited set of curve shapes. However, more diverse shapes can be obtained by joining a number of Bézier curves from end to end to form a single curve called *piecewise Bézier curve.*

Let us start by noting that a Bézier curve of degree n with control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$ can be defined over an arbitrary interval $I = [a, b]$ by

$$\mathcal{C}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}\left(\frac{t-a}{b-a}\right).$$

Notice that the definition is a reparametrization of the ordinary Bézier curve

$$\hat{\mathcal{C}}(\hat{t}) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(\hat{t}), \quad \hat{t} \in [0,1].$$

The curve $\hat{\mathcal{C}}(t)$ is called the normalization of $\mathcal{C}(t)$.

Definition 7. A curve $\mathcal{B}(t)$ is said to be a piecewise Bézier curve on an interval $I = [a, b]$ if the exist

$$a = t_0 < t_1 < \cdots t_{r-1} < t_r = b,$$

and Bézier curves $\mathcal{C}_j(t)$ defined on $[t_j, t_{j+1}]$, $j = 0, \ldots, r-1$ such that

1. $\mathcal{B}(t) = \mathcal{C}_j(t)$ for $t \in (t_j, t_{j+1})$
2. $\mathcal{B}(t_j) = \mathcal{C}_{j-1}(t_j)$ or $\mathcal{B}(t_j) = \mathcal{C}_j(t_j)$

The parameter values t_j are referred to as *breakpoints*. If the largest degree of the curves $\mathcal{C}_j(t)$ is n, then the piecewise Bézier curve $\mathcal{B}(t)$ is said to have degree n.

Notice that the definition guarantees that $\mathcal{B}(t)$ is single-valued on the interval $[a, b]$, but in practice a piecewise Bézier curve is often thought of as the union of Bézier curves and $\mathcal{B}(t)$ has the two values $\mathcal{C}_{j-1}(t_j)$ and $\mathcal{C}_j(t_j)$ at the breakpoints. Usually only continuous curves are considered so that $\mathcal{B}(t_j) = \mathcal{C}_{j-1}(t_j) = \mathcal{C}_j(t_j)$.

Example 6.3. Consider three Bézier curves $\mathcal{C}_0(t)$, $t \in [-2, 0]$, $\mathcal{C}_1(t)$, $t \in [0, 2]$, and $\mathcal{C}_2(t)$, $t \in [2, 5]$ with the control points

$$\begin{aligned} \mathcal{C}_0 : &\quad (1, -1),\ (3, 2),\ (4, -1),\ (3, -1) \\ \mathcal{C}_1 : &\quad (3, -1),\ (2, -1),\ (2, -3),\ (4, -2) \\ \mathcal{C}_2 : &\quad (4, -2),\ (6, 0),\ (4, 1),\ (6, 2) \end{aligned}$$

and let $\mathcal{B}(t)$ be the piecewise Bézier curve defined on the interval $[-2, 5]$ as

$$\mathcal{B}(t) = \begin{cases} \mathcal{C}_0(t), & -2 \le t < 0 \\ \mathcal{C}_1(t), & 0 \le t < 2 \\ \mathcal{C}_2(t), & 2 \le t < 5 \end{cases}$$

The breakpoints are $t_0 = -2$, $t_1 = 0$, $t_2 = 2$, and $t_3 = 5$ and $\mathcal{B}(t)$ is shown in Fig. 23. Since the parametrization has no effect on the trace of a curve, $\mathcal{B}(t)$ can be plotted by rendering the normalizations.

Recall that a parametric curve is said to be C^k(-continuous) when all of the coordinate functions are C^1. Since a polynomial function is C^∞, a piecewise polynomial function is C^∞ everywhere expect at the parameter values corresponding to the joins of the individual functions. If two polynomial functions (curves) of degrees p and q are joint, the piecewise function (curve) is at most C^k, where $k = \min(p, q)$.

Example 6.4. Consider the curve parametrized as $\mathcal{C}(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t^2 + 1, & t \le 1 \\ t + 1, & t > 1 \end{cases}$$

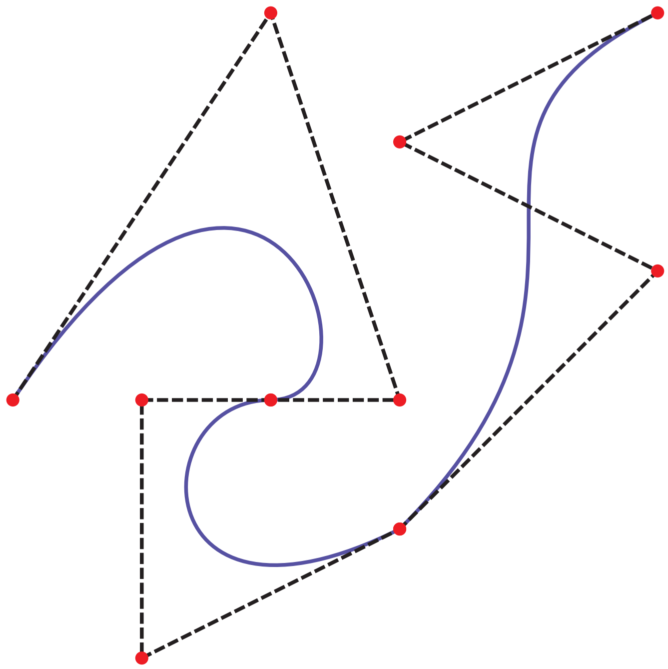

Figure 23: Piecewise Bézier curve.

It is easy to see that $f(t)$ is C^0. The derivative is

$$f'(t) = \begin{cases} 2t, & t \leq 1 \\ 1, & t > 1 \end{cases}$$

and has a jump at $t = 1$. Hence $\mathcal{C}(t)$ is C^0 but not C^1, see Fig. 24.

Suppose that $\mathcal{B}(t)$, $t \in [a, b]$ is a piecewise Bézier curve. Since the coordinate functions are piecewise polynomials, $\mathcal{B}(t)$ is C^∞ for all $t \in [a, b]$ except at the breakpoints. If $\mathcal{C}_j(t)$ has degree n_j and control points $\mathbf{b}_0^j, \ldots, \mathbf{b}_{n_j}^j$, then $\mathcal{B}(t)$ is continuous at $t = t_j$ if and only if

$$\lim_{t \to t_j^+} \mathcal{B}(t) = \lim_{t \to t_j^-} \mathcal{B}(t) \quad \Leftrightarrow \quad \lim_{t \to t_j^+} \mathcal{C}_j(t) = \lim_{t \to t_j^-} \mathcal{C}_{j-1}(t) \quad \Leftrightarrow \quad \mathbf{b}_0^j = \mathbf{b}_{n_{j-1}}^{j-1}$$

Similarly, one can establish that the condition for C^1-continuity is

$$\mathbf{b}_0^j = \mathbf{b}_{n_{j-1}}^{j-1} \quad \text{and} \quad n_j(\mathbf{b}_1^j - \mathbf{b}_0^j) = n_{j-1}(\mathbf{b}_{n_{j-1}}^{j-1} - \mathbf{b}_{n_{j-1}-1}^{j-1})$$

If the tangent vectors from both side of a breakpoint have the same direction but may have different magnitudes, the curve is said to be *visually tangent continuous*. This is the case provided that

$$\mathbf{b}_0^j = \mathbf{b}_{n_{j-1}}^{j-1} \quad \text{and} \quad \mathbf{b}_1^j - \mathbf{b}_0^j = \lambda_j(\mathbf{b}_{n_{j-1}}^{j-1} - \mathbf{b}_{n_{j-1}-1}^{j-1})$$

for some $\lambda_j \neq 0$.

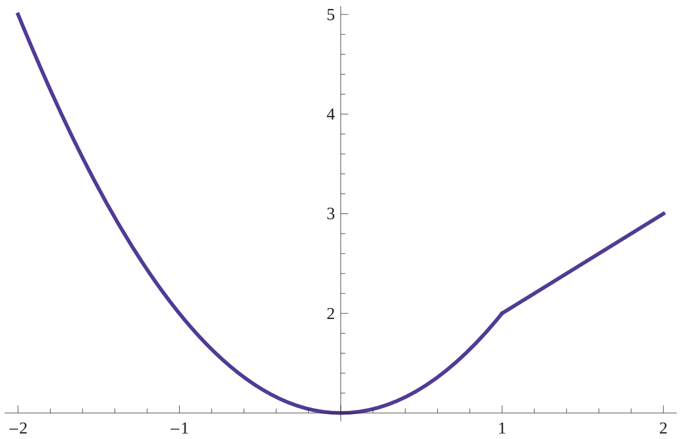

Figure 24: An example of a curve which is C^0 but not C^1.

6.5 Rational Bézier Curves

In Section 4 we showed that conic sections are parametrized by rational functions in general. Thus quadratic Bézier curves with polynomial coordinate functions cannot represent such curves exactly and it become necessary to introduce rational Bézier curves.

Definition 8. A rational Bézier curve of degree n with control points $\mathbf{b}_0, \dots, \mathbf{b}_n$ and scalar weights $w_0, \dots, w_n$ is defined as

$$\mathcal{C}(t) = \frac{\sum_{i=0}^{n} w_i \mathbf{b}_i B_{i,n}(t)}{\sum_{i=0}^{n} w_i B_{i,n}(t)}, \quad t \in [0,1],$$

where we normally assume that $w_i > 0$ for all i.

Since

$$W(t) = \sum_{i=0}^{n} w_i B_{i,n}(t)$$

is a common denominator function, a rational Bézier curve can also be written in the basis form

$$\mathcal{C}(t) = \sum_{i=0}^{n} \mathbf{b}_i R_{i,n}(t),$$

where

$$R_{i,n}(t) = \frac{w_i B_{i,n}(t)}{\sum_{j=0}^{n} w_j B_{j,n}(t)}$$

are the rational basis functions. Notice that when $w_0 = \cdots = w_n$, non-rational, or *integral*, Bézier curve is obtained.

Let $\mathbf{b}_i = (x_i, y_i, z_i)$ and defined the homogeneous control points $\mathbf{b}_i^w$ by

$$\begin{cases} \mathbf{b}_i^w = (w_i x, w_i y, w_i z, w_i), & \text{when } w_i \neq 0 \\ \mathbf{b}_i^w = (x, y, z, 0), & \text{when } w_i = 0 \end{cases}$$

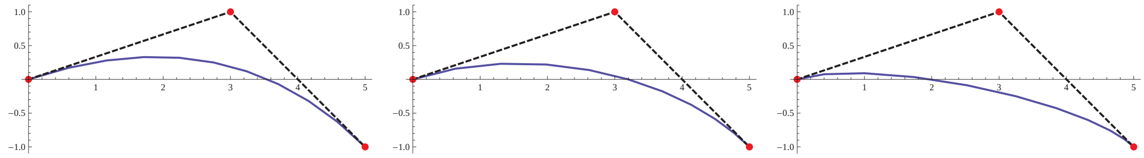

Figure 25: Quadratic Bézier curves with weights $\{1, 1, 1\}$, $\{1, 1, 2\}$, and $\{1, 0.5, 2\}$.

In homogeneous coordinates, the rational Bézier curve becomes an integral curve

$$\mathcal{C}(t) = \sum_{i=0}^{n} \mathbf{b}_i^w B_{i,n}(t).$$

Example 6.5. The rational quadratic with control points $(0, 0)$, $(3, 1)$ and $(5, -1)$ and with weights $\{w_0, w_1, w_2\}$ specified as $\{1, 1, 1\}$, $\{1, 1, 2\}$, and $\{1, 0.5, 2\}$ are shown in Fig. 25.

Example 6.6. The unit quarter circle in the first quadrant can be represented as a quadratic rational Bézier curve with control points $\mathbf{b}_0 = (1, 0)$, $\mathbf{b}_1 = (1, 1)$, and $\mathbf{b}_2 = (0, 1)$ with weights $w_0 = w_1 = 1$, and $w_2 = 2$. Indeed,

$$\begin{aligned} w_0\mathbf{b}_0(1-t)^2 + w_1\mathbf{b}_1 2(1-t)t + w_2\mathbf{b}_2 t^2 &= (1,0)(1-t)^2 + (1,1)2t(1-t) + (0,1)2t^2 \\ &= (1-t^2, 2t) \\ w_0(1-t)^2 + w_1 2(1-t)t + w_2 t^2 &= 1 + t^2 \end{aligned}$$

which yields the familiar parametrization

$$\mathcal{C}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$

Properties of Rational Bézier Curves

Rational Bézier Curves inherit the basic properties of integral Bézier curves:

- Endpoint interpolation: $\mathcal{C}(0) = \mathbf{b}_0$ and $\mathcal{C}(1) = \mathbf{b}_n$
- Endpoint tangency: $\mathcal{C}'(0) = n\frac{w_1}{w_0}(\mathbf{b}_1 - \mathbf{b}_0)$ and $\mathcal{C}'(1) = n\frac{w_{n-1}}{w_n}(\mathbf{b}_n - \mathbf{b}_{n-1})$
- Convex hull property provided that $w_i > 0$ for all $i = 0, \dots, n$
- Invariance under affine transformation: If T is affine transformation, then

$$\mathsf{T}\left(\frac{\sum_i w_i \mathbf{b}_i B_{i,n}(t)}{\sum_i w_i B_{i,n}(t)} \right) = \frac{\sum_i w_i \mathsf{T}\left(\mathbf{b}_i\right) B_{i,n}(t)}{\sum_i w_i B_{i,n}(t)}$$

- Variation diminishing property provided that $w_i > 0$ for all $i = 0, \dots, n$

6.6 Exercises

1. Apply the de Casteljau algorithm with $t = 0.25$ to the spatial cubic Bézier curve $\mathcal{C}(t)$ with control points $(1, 0)$, $(3, 3)$, $(5, 5)$, and $(7, 2)$. Check the result by substituting $t = 0.25$ into the Bernstein basis representation of the curve. Visualize the points derived in the de Casteljau algorithm.

2. Apply the de Casteljau algorithm to the quartic Bézier curve $\mathcal{C}(t)$ with control points $(3, 3)$, $(4, 2)$, $(-1, 0)$, $(6, 1)$, and $(8, 5)$, and evaluate $\mathcal{C}(0.6)$.

3. Determine the first and second derivatives of the cubic Bézier curve with control points $(6, 3)$, $(4, 3)$, $(1, 2)$, and $(-1, 2)$. Compute the tangent vector corresponding to the parameter value $t = 0.5$.

4. Determine the first and second derivatives ot the quartic Bézier curve with control points $(1, 1)$, $(1, 3)$, $(5, 6)$, $(6, 2)$, and $(4, -1)$. Compute the tangent vector corresponding to the parameter value $t = 0.5$.

5. Consider the piecewise curve consisting of two cubic Bézier curves $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ with control points

$$\begin{aligned} \mathcal{C}_1 : &\quad \mathbf{a}_0 = (2, 1),\ \mathbf{a}_1 = (4, 2),\ \mathbf{a}_2 = (5, 4),\ \mathbf{a}_3 = (3, 6) \\ \mathcal{C}_2 : &\quad \mathbf{b}_0 = (3, 6),\ \mathbf{b}_1 = (2, 7),\ \mathbf{b}_2 = (0, 5),\ \mathbf{b}_3 = (0, 3) \end{aligned}$$

   Show that the curves have visual tangent continuity. Plot the curves and their control polygons. Alter the point $\mathbf{a}_2$ so that the curves join with C^1-continuity.

6. The circular arc, radius r, centered at the origin with endpoints $(r, 0)$ and $(r\cos\varphi, r\sin\varphi)$, $\varphi \in [-\pi, \pi]$ has a quadratic Bézier representation given by the control points

$$\mathbf{b}_0 = (r, 0), \quad \mathbf{b}_1 = (r, r\tan\tfrac{\varphi}{2}), \quad \mathbf{b}_2 = (r\cos\varphi, r\sin\varphi)$$

   and weights $w_0 = w_2 = 1$ and $w_1 = \cos\frac{\varphi}{2}$. Express $x(t)$ and $y(t)$ as quadratic rational functions and verify that $x(t)^2 + y(t)^2 = 1$.

7. The unit quarter circle in the first quadrant is obtained by taking $r = 1$, $\varphi = \frac{\pi}{2}$ in the previous exercise. What parametrization does this give? Compute the arclength function of the parametrization and compare with the unit speed arclength function $s(t) = \frac{\pi}{2}t$ and the arclength function of the parametrization of Example 6.6.